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On Stability Equilibrium Analysis of Endemic Malaria

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Abstract:*This paper considers the stability equilibrium of malaria in a varying population. We established the Disease free equilibrium and the endemic equilibrium and carried out the stability analysis of the Disease free equilibrium state (the state of complete eradication of malaria from the population).*
Keywords: *malaria, mathematical model, disease-free equilibrium, endemic equilibrium, basic reproduction number.*

I. Introduction

Several insects are known to be vectors of human disease. Mosquitoes in particular enjoy the questionable honour of having been the first insects to be associated with the transmission of a disease.

Malaria is an infectious disease caused by the Plasmodium parasite and transmitted between mosquitos.In 1898, the Italian Zoologist G.B. Grassi and his co-workers first described the complete cycle of the human malaria parasites and pointed to a species of the genus Anopheles as responsible of malaria transmission. Of more than 480 species of anopheles, only about 50 species transmit malaria. The habits of most of the anopheles mosquitoes have been characterized as anthropophagic (prefer human blood meal), endothermic (bite indoors), and nocturnal (bite at night) with peak biting at midnight, between 11pm and 2 am. The epidemiology of malaria in a given environment is the result of a complex interplay among man, plasmodia, and anopheline mosquitoes. These three elements have to be present for malaria transmission to occur in nature. The incidence of malaria has been growing recently due to increasing parasite drug- resistance and mosquito insecticide resistance.

It remains one of the most difficult epidemiological, pharmacological and immunological challenges in sub-Sahara Africa. Its influence has been profound and sustained over the centuries and malaria remains a major cause of morbidity and mortality. Moreover, the epidemiological situation appears to be changing for the worse in many countries (Offosu, Amaah, 1991).

An estimated 40% of the world's populations live in malaria endemic areas. Each year 300 to 500 million people develop malaria. It kills about 700,000 to 2.7 million people a year 75% of whom are Africa children

Malaria is the ninth largest cause of death and disability globally.Malaria episodes lead to direct cost associated with health care and health care – related transport and indirect costs associated with lost income for patients, lost income for careers and school days missed by children.

In areas where malaria transmissions are high, the highest number of cases is concentrated among young children. However, malaria illness affects all age groups which leads to a higher loss of income and decreased productivity [8].

We develop a mathematical model to better understand the transmission and spread of malaria. This model is used to determine which factors are most responsible for the spread of malaria.

II. Model Formulation

We formulate a model similar to that of [11,12] describing the transmission of malaria.The equations for model is as described below

$$\frac{dS_h}{dt} = P + \varepsilon R_h - (\delta_h + \mu_h)S_h$$

$$\frac{dE_h}{dt} = \delta_h S_h - (\mu_h + i_h)E_h$$

$$\frac{dI_h}{dt} = i_h E_h - (\mu_h + \eta_h)I_h - qI_h$$

$$\frac{dR_h}{dt} = qI_h - (\mu_h + \varepsilon)R_h$$

$$\frac{dS_v}{dt} = V - (\psi_v + \delta_v)S_v$$
$$\frac{dE_v}{dt} = \delta_v S_v - (\psi_v + i_v)E_v \qquad 1.0$$
$$\frac{dI_v}{dt} = i_v E_v - \psi_v I_v$$

Table 1: The state variables for the malaria model equation 1.0

Parameter		Description
$S_h(t)$	-	Number of susceptible human hosts at time t
$E_h(t)$	-	Number of exposed human hosts at time t
$I_h(t)$	-	Number of infections human hosts at time t
$R_h(t)$	-	Number of recovered human with temporary immunity of time t
$S_v(t)$	-	Number of susceptible mosquito vectors at time t
$E_v(t)$	-	Number of exposed mosquito vectors at time t
$N_h(t)$	-	Total human population
$N_v(t)$	-	Total mosquito population

Table II: Model parameters and their interpretations for the malaria model equation 1.0

Parameter	Description
P	Birth
μ_h	Natural death rate for humans
δ_h	Force of infection for humans from susceptible state to exposed state.
i_h	Incident rate for humans
Q	Treatment of infected humans
η_h	Disease- induced death rate for humans
ε	Rate of loss of immunity for humans
V	Recruitment rate of mosquitoes
δ_v	Force of infection of mosquitoes from susceptible state to exposed state.
i_v	Incident rate for mosquitoes
φ_v	Natural death rate for mosquitoes
Λ_{vh}	Probability of transmission of infection from an infectious mosquito to a susceptible human provided there is a contact.
Λ_{hv}	Probability of transmission of infection from an infectious human to a susceptible mosquitoes provided there is a contact

TOTAL POPULATION DENSITY

The total population sizes are

$$N_h = S_h + E_h + I_h + R_h$$
$$N_v = S_v + E_v + I_v$$
$$\frac{dN_h}{dt} = P - \mu_h N_h - \eta_h I_h$$
$$\text{also} \frac{dN_v}{dt} = V - \varphi_v N_v \qquad 2.0$$
$$\text{where} \delta_h = \Lambda_{vh} \ , \ \delta_v = \Lambda_{hv}$$

where Λ_{vh} denotes the rate at which the susceptible human S_h, become infected by infectious female Anopheles mosquitoes I_v and Λ_{hv} represents the rate at which the susceptible mosquitoes S_v are infected by infectious human I_h.

The model is analysed to examine the critical factors which determine the persistence or eradication of malaria. We hence determine if the mode is well posed.

Theorem (2.1): The solution of (2.0) are feasible for all t>0 and is defined in the subset $\Omega \times [0, \infty)$ of R_+^7

Proof:

Let $\Omega_h = (S_h, E_h, I_h, R_h, S_v, E_v, I_v) \epsilon R_+^7$ be any solution of the system (2.0) with non-negative initial conditions.

In absence of the disease (malaria), $I_h = 0$,

$$\frac{dN_h}{dt} \leq P - \mu_h N_h$$

$$\frac{dN_h}{dt} + \mu_h N_h \leq P$$

$$\frac{d}{dt}(N_h e^{\mu_h t}) \leq P e^{\mu_h t} \qquad 2.1$$

$$N_h e^{\mu_h t} \leq \frac{P e^{\mu_h t}}{\mu_h} + c$$

Where c is a constant of integration

$$N_h \leq \frac{P}{\mu_h} + c e^{\mu_h t}$$

at t= 0

$$N_h(0) \leq \frac{P}{\mu_h} + c$$

$$N_h(0) \leq \frac{P}{\mu_h} \leq c$$

With

$$N_h \leq \frac{P}{\mu_h} + (N_h(0) - \frac{P}{\mu_h}) e^{-\mu_h t} \qquad 2.2$$

$$0 \leq N_h \leq \frac{P}{\mu_h} att \to \infty$$

Therefore, as $t \to \infty$ in (2.2), the human population.

$N_h approaches \frac{P}{\mu_h}, where \frac{P}{\mu_h} = k$Called the carrying capacity.

Hence all feasible solution set of the human of the human population of the model (2.1) enter the region.

$$\Omega_h = \left\{(S_h, E_h, I_h, R_h) \epsilon \mathbb{R}_+^4 : S_h > 0, E_h \geq 0, I_h \geq 0, R_h \geq 0 \; and N_h \leq \frac{P}{\mu_h}\right\}$$

In like manner, the feasible solutions set of the mosquito population enters the region

$$\Omega_v = \left\{(S_v, E_v, I_v) \epsilon \mathbb{R}_+^3 : S_v > 0, E_v \geq 0, I_v \geq 0, N_v \geq 0 \; a \leq \frac{V}{\varphi_v}\right\}$$

$$\frac{dS_h}{dx} = P + \varepsilon R_h - \delta_h S_h - \mu_h S_h \geq -\delta_h S_h - \mu_h S_h$$

$$\frac{dS_h}{dx} - (\delta_h S_h + \mu_h S_h) \geq -(\delta_h S_h - \mu_h) S_h$$

$$\frac{dS_h}{S_h} \geq -(\delta_h + \mu_h) dt$$

$$du S_h \geq -(\delta_h + \mu_h) t + C_1$$

$$S_h \geq e^{-(\delta_h + \mu_h) t + C_1} \geq e^{-(\delta_h + \mu_h)} e^{C_1}$$

$$S_h(t) \geq A e^{-(\delta_h + \mu_h) t}$$

$$Att = 0$$

$$S_h(0) \geq A$$

$$S_h(0) e^{-(\delta_h + \mu) t} \geq 0$$

$$\therefore S_h(t) \geq S_h(0) e^{-(\delta_h + \mu) t} \geq 0$$

Also the remaining equations of system (2.0) are all positive for $t > 0$.

Hence, the domain Ω is positively invariant, because no solution paths leave through any boundary. Since paths cannot leave Ω , solutions exist for all positive time. Thus the model is mathematically and epidemiologically well-posed.

Disease free equilibrium: We now solve the model equations to obtain the equilibrium states. At the equilibrium state in the absence of the disease, we have $E_h = I_h = E_v = I_v = 0$.

From (1.0)

$$\mathrm{P} - \mu_h - N_h = 0$$

$$N_h^* = \frac{P}{\mu_h} \qquad - \quad 3.5$$

Also $V - \varphi_v N_v = 0$

$$N_v^* = \frac{V}{\varphi_v} \qquad - \quad 3.6$$

Therefore, the disease free equilibrium point of the malaria mode (3.2) is $E_o = (S_h, E_h, I_h, R_h, S_v, E_v, I_v) = (\frac{P}{\mu_h}, 0, 0, 0, \frac{V}{\psi_v}, 0, 0)$.

This is the state in which there is no infection (in the absence of malaria) in the society.

THE BASIC REPRODUCTION RATIO R_o

This is the expected number of secondary cases, produced, in a completely susceptible population, by a typical infected individual during its entire period of infectiousness, and mathematically as the dominant eigen value of positive linear operator.

In the original parameters $R_o = \sqrt{\frac{V i_h i_v \Lambda_{hv} \Lambda_{vh} \mu}{P\psi (i_h+\mu)(q+\mu+\eta)(i_v+\psi)\psi}}$

LOCAL STABILITY OF THE DISEASE-FREE EQUILIBRIUM

Theorem 3.1: The disease-free equilibrium point is locally asymptotically stable if $R_o < 1$ and is unstable if $R_o > 1$.

Proof: The matrix of the system of equation is

$$\begin{pmatrix} -(\mu_h i_h) & 0 & 0 & 0 & \Lambda_{vh}\varphi_{vh} \\ i_h & -(\mu_h + \eta + q) & 0 & 0 & 0 \\ 0 & q & -(\mu_h - \varepsilon) & 0 & 0 \\ 0 & \frac{\Lambda_{hv}\varphi v \mu_v}{\psi_v p} & 0 & -(\varphi_v i_v) & 0 \\ 0 & 0 & 0 & i_v & -\psi_v \end{pmatrix}$$

The Jacobian matrix is given as $(A - \lambda I)$

$$J = \begin{pmatrix} -(T_1 + \lambda) & 0 & 0 & 0 & T_2 \\ T_3 & T_4 & 0 & 0 & 0 \\ 0 & T_5 & -(T_6 + \lambda) & 0 & 0 \\ 0 & T_0 & 0 & -(T_7 + \lambda) & 0 \\ 0 & 0 & 0 & T_8 & T_9 \end{pmatrix} = 0$$

$$T_1 = \mu_h + i_h, \quad T_2 = \Lambda_{vh}\varphi, T_3 = i_h, T_4 = (\mu_h + \eta + q)$$

$$T_5 = q, T_6 = (\mu_h - \varepsilon), \quad T_7 = (\psi_v + i_v), T_8 = i_V$$

$$T_9 = \psi_v, T_0 = \frac{\Lambda_{hv}\varphi v \mu_v}{\psi_v p}$$

From the third column, which has a diagonal entry, one of the eigenvalues is $-(T_6 + \lambda)$.

The matrices therefore reduces to

$$J = \begin{vmatrix} -(T_1 + \lambda) & 0 & 0 & T_2 \\ T_3 & -(T_4 + \lambda) & 0 & 0 \\ 0 & T_0 & -(T_7 + \lambda) & 0 \\ 0 & 0 & T_8 & -(T_9 + \lambda) \end{vmatrix} = 0$$

which forms the characteristics equation

$$(T_1 + \lambda)(T_4 + \lambda)(T_7 + \lambda)(T_9 + \lambda) - T_2T_3T_8T_0 = 0$$

$$= (T_1T_4 + \lambda T_1 + \lambda T_4 + \lambda^2)(T_7T_9 + \lambda T_7 + \lambda T_9 + \lambda^2) - T_2T_3T_8T_0 = 0$$

$$\lambda^4 + \lambda^3 \gamma_1 + \lambda^2 \gamma_2 + \lambda \gamma_3 + \gamma_0 = 0$$

For

$\gamma_1 = T_1 + T_4 + T_7 + T_9$
$\gamma_2 = T_1T_7 + T_1T_9 + T_4T_9 + T_7T_9 + T_1T_4$
$\gamma_3 = T_1T_4T_7 + T_1T_4T_9 + T_1T_7T_9 + T_4T_7T_9 + T_4T_7$
$\gamma_0 = T_1T_4T_7T_9 - T_2T_3T_8T_0$

To evaluate the signs of the roots of, we use the Routh – Hurwitz criterion and Descartes' Rule of sign.

Theorem 3.2: Routh-Hurwitz Criteria

Given the polynomial

$P(\lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \ldots \gamma_{n-1} \lambda + \gamma_n$

where the coefficients γ_i are real constants, $i = 1, \ldots, n$; define the n Hurwitz matrices is equal to the number of sign changes γ_i of the characteristic polynomial.

$$F_o = 1,\ F_1 = \gamma_1,\quad F_2 = \begin{vmatrix} \gamma_1 & 1 \\ \gamma_3 & \gamma_2 \end{vmatrix},\ F_3 = \begin{vmatrix} \gamma_1 & 1 & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 \\ \gamma_5 & \gamma_4 & \gamma_3 \end{vmatrix},\ F_4 = \begin{vmatrix} \gamma_1 & 1 & 0 & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 & 1 \\ \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 \end{vmatrix}$$

where $\gamma_i = 0\ if i > n$. All of the roots of the polynomial $P(\lambda)$ are negatives or have negative real parts if and only if the determinants of all Hurwitz matrices are positive: $det(F_j) > 0, j = 1, 2, \ldots\ldots, n.$

We show that when $R_o < 1$, all the coefficients , γ_i, of the characteristics equation andF_j are positive.

From the characteristic equation, with n = 4, the Routh – Hurwitz criteria are

$\gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0, \gamma_4 > 0$ and$\det(F_1) = \gamma_1 > 0$,

$$\det(F_2) = \begin{vmatrix} \gamma_1 & 1 \\ 0 & \gamma_2 \end{vmatrix} = \gamma_1\gamma_2 > 0,$$

$$\det(F_3) = \begin{vmatrix} \gamma_1 & 1 & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 \\ 0 & 0 & \gamma_3 \end{vmatrix}$$

$= \gamma_1\gamma_2\gamma_3 - \gamma_3^2 > 0$
i.e. $= \gamma_1\gamma_2 - \gamma_3 > 0$

and

$= \gamma_1\gamma_2\gamma_3 - \gamma_1^2\gamma_4 - \gamma_3^2 > 0$
$= \gamma_1\gamma_2\gamma_3 - \gamma_3^2 - \gamma_1^2\gamma_4 > 0$

Since all the determinants of the Hurwitz are positive, which implies that all the Eigen values of the Jacobian matrix have negative real part. Hence disease –free equilibrium point is asymptotically stable and $R_o > 1$.

The Endemic Equilibrium Point

Endemic equilibrium points are steady state situations where the disease persist in the population (all state variables are positive). The endemic equilibrium of the model is given as $E^* = (S_h, E_h, I_h, R^*, S_v, E^*, I_v) > 0$.

Consider equation * * * *

$$\frac{dS_h}{dt} = P + \varepsilon R_h - \frac{\Lambda_{vh}\varphi I_v S_h}{N_h} - \mu S_h$$

$$\frac{dE_h}{dt} = \frac{\Lambda_{vh}\varphi I_v S_h}{N_h} - \mu_h E_h - i_h E_h = 0$$

$$\frac{dI_h}{dt} = i_h E_h - (\mu_h + \Lambda_h) I_h - qI_h = 0$$

$$\frac{dR_h}{dt} = qI_h - \mu_h R_h - \varepsilon R_h = 0$$

$$\frac{dS_v}{dt} = v - \psi_v S_v - \frac{\Lambda_{hv}\varphi I_h S_v}{N_h}$$

$$\frac{dE_v}{dt} = \frac{\Lambda_{hv}\varphi I_h S_v}{N_h} - \psi_v E_v - i_v E_v = 0$$

$$\frac{dI_v}{dt} = i_v E_v - \psi_v E_v = 0$$

from

$$i_v E_v - \psi_v I_v = 0$$

$$\psi_v I_v = i_v E_v$$

$$I_v^* = \frac{i_v E_v}{\psi_v} \quad \text{- - - - - - - - (i)}$$

Also

$$E_v(\psi_v + i_v) = \frac{\Lambda_{hv}\varphi I_h S_v}{N_h}$$

$$E_v^* = \frac{\Lambda_{hv}\varphi I_h S_v}{N_h(\psi_v + i_v)} \quad \text{- - - - - - - - (ii)}$$

Substitute (ii) and (i)

$$I_v^* = \frac{\Lambda_{hv}\varphi I_h S_v}{\psi_v N_h(\psi_v + i_v)} \quad \text{- - - - - - - - (iii)}$$

from $\frac{dS_v}{dt} = v - \psi_v S_v - \frac{\Lambda_{hv}\varphi I_h S_v}{N_h} = 0$

$$S_v\left(\psi_v + \frac{\Lambda_{hv}\varphi I_v}{N_h}\ \frac{\hat{}\varphi_h \varphi I_v S_h}{\mu_h}\right) = v$$

$$S_v^* = \frac{vN_h}{N_h\psi_v + \Lambda_{hv}\varphi I_v} \quad \text{- - - - - - - - (iv)}$$

Substitute (iv) in (iii)

$$I_v^* = \frac{i_v \Lambda_{hv}\varphi I_h v N_h}{\psi_v N_h(\psi_v + i_v)(N_h\psi_v + \Lambda_{hv}\varphi I_v)} \quad \text{- - - - - - - - (iv)}$$

III. Conclusion

From our analysis, we found that the disease free equilibrium is stable when the threshold parameter is less than unity. However, the disease could be reduced if the contact rate is avoided. This could be achieved by the use of insecticide bed treated net and or indoor spraying..

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