IOSR Journal of Applied Physics (IOSR-JAP)
e-ISSN: 2278-4861.Volume 8, Issue 1 Ver. I (Jan. - Feb. 2016), PP 19-21
www.iosrjournals

Riemannian Laplacian Formulation in Oblate Spheroidal Coordinate System Using the Golden Metric Tensor

N.E.J. Omaghali[1], S.X.K. Howusu[2]
[1]*Department of Physics, University of Jos, Nigeria*
[2]*Theoretical Physics Programme, National Mathematical Centre Abuja, Nigeria*

Abstract: *Studies have shown that most planetary bodies, the sun and earth inclusive, are better described as oblate spheroids rather than spherical. In this paper, using the Golden Metric tensor, we derive the expression for the Riemannian Laplacian in Oblate Spheroidal coordinate systems by the method of tensor analysis. The derived expression for the Riemannian Laplacian in Oblate Spheroidal coordinate systems reduce to pure Newtonian Laplacian in the limit* c^0 *and contains post-Newtonian correction terms of all orders* c^{-2}*. The result obtained here opens the way for applications of the Riemannian Laplacian to all bodies in the Spheroidal coordinate system.*
Keywords: *Golden metric tensor, Riemannian Laplacian, Oblate Spheroidal coordinates Post-Newtonian correction*

I. Introduction

The shape of the planetary bodies can be assumed to be spherical. To this effect, assumptions of spherical shapes for theses bodies, the earth inclusive, have been made in many treatments involving the planetary bodies. However, studies have shown that these planetary bodies are better described as oblate spheroids rather than spherical[1,2,3]. Therefore it does necessitate the need to describe equation of motion in orthogonal oblate Spheroidal coordinate systems for the planetary bodies, which are a more realistic and natural approximation than the spherical treatment[4]. It is worth noting that the treatment of planetary bodies as mentioned here have been done based on Euclidean Geometry. However, since Riemann's presentation of his work on geometry in 1854, there has been considerable interest Riemannian Geometry ever since, with a key tool in exploring the Riemannian geometry being the Metric Tensors. With knowledge of the Metric Tensors in the Oblate Spheroidal coordinates[5], we proceed to derive the Riemannian Laplacian for all particles in Oblate Spheroidal coordinate system.

II. Theoretical analysis

The oblate spheroid is a surface of revolution generated by the rotation of an ellipse about its minor axis such that the spheroid is flattened about the minor axis. The Oblate Spheroidal coordinate system (ξ, η, ϕ) is defined in terms of the Cartesian coordinate system (x, y, z) [6,7] as follows:

$$x = a \cosh \xi \cos \eta \cos \phi \,, \tag{1}$$

$$y = a \cosh \xi \cos \eta \sin \phi \,, \tag{2}$$

$$z = a \sinh \xi \sin \eta \,, \tag{3}$$

where a is the ellipse's focal distance which is one-half the ellipse's foci and:

$$\xi \geq 0, \quad -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi \tag{4}$$

Let us define the set of four labeled quantities, x^{μ} given by[5]:

$$x^{\mu} = \left\{ x^1, x^2, x^3, x^0 \right\} \tag{5}$$

as the spacetime position tensor in Oblate Spheroidal coordinate system, where:

$$\left(x^1, x^2, x^3 \right) = \left(\xi, \eta, \phi \right) \tag{6}$$

is the coordinate of space in the Oblate Spheroidal coordinate system. In Einstein coordinate, we have that:

$$\left(x^0 \right) = \left(ct \right) \tag{7}$$

where t is the coordinate time and c is the speed of light in vacuum.

Therefore, the spacetime position tensor in equation (5) can be written explicitly as:

$$x^{\mu} = \{\xi, \eta, \phi, ct\} \tag{8}$$

Following the formulation above, we can define the Golden metric tensor needed for the formulation of the Riemannian Laplacian. The Golden Metric tensor for all gravitational fields in nature in Einstein Oblate Spheroidal coordinates (ξ, η, ϕ, ct) is given by [5]:

$$g_{11} = a^2\left(\cosh^2\xi - \cos^2\eta\right)\left(1 + \frac{2}{c^2} f\right)^{-1}, \tag{9}$$

$$g_{22} = a^2\left(\cosh^2\xi - \cos^2\eta\right)\left(1 + \frac{2}{c^2} f\right)^{-1} \tag{10}$$

$$g_{33} = a^2 \cosh^2\xi \cos^2\eta \left(1 + \frac{2}{c^2} f\right)^{-1}, \tag{11}$$

$$g_{00} = \left(1 + \frac{2}{c^2} f\right) \tag{12}$$

$$g_{\mu\nu} = 0; \quad \text{otherwise} \tag{13}$$

where the function f given by:

$$f = f\left(\xi, \eta, \phi, x^0\right) \tag{14}$$

is the gravitational scalar potential. The covariant metric tensor given by equations (9) – (13) is a covariant tensor of rank 2. Therefore, by tensor transformations[8], we can obtain the corresponding conjugate or reciprocal tensor $g^{\mu\nu}$ from equations (9) – (13) as:

$$g^{11} = \frac{\left(1 + \frac{2}{c^2} f\right)}{a^2\left(\cosh^2\xi - \cos^2\eta\right)}, \tag{15}$$

$$g^{22} = \frac{\left(1 + \frac{2}{c^2} f\right)}{a^2\left(\cosh^2\xi - \cos^2\eta\right)}, \tag{16}$$

$$g^{33} = \frac{\left(1 + \frac{2}{c^2} f\right)}{a^2 \cosh^2\xi \cos^2\eta} \qquad 17)$$

$$g^{00} = \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{18}$$

$$g^{\mu\nu} = 0; \quad \text{otherwise} \qquad 19)$$

Following the derivations in the previous section, we can proceed to derive the Riemannian Laplacian in the Oblate Spheroidal coordinates. In general, the Riemannian Laplacian ∇^2_R, is defined as[8]:

$$\nabla^2_R = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}}\left(\sqrt{g}\, g^{\mu\nu} \frac{\partial}{\partial x^{\nu}}\right) \tag{20}$$

where:

$$\sqrt{g} = \sqrt{\left|g_{\mu\nu}\right|}\left(g_{00}\right)^{\frac{1}{2}} \tag{21}$$

Now, using the metric tensors from equations (9) - (13), and after some mathematical simplification, we can write equation (21) explicitly as:

$$\sqrt{g} = \frac{a \cosh \xi \cos \eta \left(a^2 \cosh^2 \xi - a^2 \cos^2 \eta \right)}{\left(1 + \frac{2}{c^2} f \right)} \tag{22}$$

From the general Riemannian Laplacian equation (20), we can write explicitly the Riemannian Laplacian in the Oblate Spheroidal coordinate system as:

$$\nabla^2_R = \left\{ \begin{array}{l} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi} \left(\sqrt{g}\, g^{11} \frac{\partial}{\partial \xi} \right) \\ + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta} \left(\sqrt{g}\, g^{22} \frac{\partial}{\partial \eta} \right) \\ + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \phi} \left(\sqrt{g}\, g^{33} \frac{\partial}{\partial \phi} \right) \\ + \frac{1}{\sqrt{g}} \frac{\partial}{\partial (ct)} \left(\sqrt{g}\, g^{00} \frac{\partial}{\partial (ct)} \right) \end{array} \right\} \tag{23}$$

Substituting equation (22) and the contravariant metric tensors of equations (15) - (19) into equation (23), and after some mathematical simplification, we obtain the Riemannian Laplacian in Oblate Spheroidal coordinate system as:

$$\nabla^2_R (\xi, \eta, \phi) = \left\{ \begin{array}{l} \frac{\left(1 + \frac{2}{c^2} f \right)}{\cosh \xi \left(a^2 \cosh^2 \xi - a^2 \cos^2 \eta \right)} \left[\cosh \xi \frac{\partial^2}{\partial \xi^2} + \sinh \xi \frac{\partial}{\partial \xi} \right] \\ + \frac{\left(1 + \frac{2}{c^2} f \right)}{\cos \eta \left(a^2 \cosh^2 \xi - a^2 \cos^2 \eta \right)} \left[\cos \eta \frac{\partial^2}{\partial \eta^2} - \sin \eta \frac{\partial}{\partial \eta} \right] \\ + \frac{\left(1 + \frac{2}{c^2} f \right)}{a^2 \cosh^2 \xi \cos^2 \eta} \frac{\partial^2}{\partial \phi^2} \\ - \frac{1}{c^2} \left(1 + \frac{2}{c^2} f \right)^{-1} \left[\frac{\partial^2}{\partial t^2} - \frac{1}{c^2} \left(1 + \frac{2}{c^2} f \right)^{-1} \frac{\partial f}{\partial t} \frac{\partial}{\partial t} \right] \end{array} \right\} \tag{24}$$

III. Results and Conclusion

The expression given by equation (24) is the Riemannian Laplacian in Oblate Spheroidal polar coordinate system. A consequence of equation (24) is that the Riemannian Laplacian in Oblate Spheroidal coordinate system reduces to the pure Newtonian Laplacian in the limit of c^0. Also, equation (24) contains post Newtonian correction terms of all orders of c^{-2}. Furthermore, a generalization of Newton's dynamical equations is realized.

References

[1] J. P. Vinti, New approach in the theory of satellite orbits, Physical Review Letters, 3(1), 1960, 8.
[2] J. P. Vinti, Invariant properties of the Spheroidal Potential of an Oblate Planet, Journal of research of the national Bureau of Standards, Vol. 70B, No. 1, Jan-March 1966, 1-6.
[3] P. Goldreich, Inclination of Satellite Orbits about an Oblate processing planet, The Astronomical Journal, 70, 1965, 5-9.
[4] W.L. Gates, Derivation of the equations of atmospheric motion in oblate spheroidal coordinates, Journal of the Atmospheric Science, Vol.61, 2004, 2478-2487.
[5] S.X.K., Howusu, Riemannian revolution in physics and mathematics (Jos University Press Ltd., 2013).
[6] P.M. Morse and H. Feshbach, Methods of theoretical physics (McGraw-Hill, New York, 1953).
[7] G. Arfken, Mathematical methods for physicists (Academic Press, Orlando, FL, 1985).
[8] M.R. Spiegel, Theory and problems of vector analysis and introduction to tensor analysis (MC Graw-Hill, New York, 1974).