IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 1 Ver. I. (Jan. 2014), PP 01-03
www.iosrjournals.org

On Generalized Projective ϕ-Recurrent Sasakian Manifold

S. Shashikala, Venkatesha*
Department of Mathematics, Kuvempu University, Shankaraghatta- 577451, Shimoga, Karnataka, INDIA.

Abstract: *The object of the present paper is to study generalized projective ϕ-recurrent Sasakian manifolds. Here we find a relation between the associated 1-forms A and B. We also proved that the characteristic vector field ξ and vector field ρ associated to the 1-forms A and B are co-directional. Finally we proved that generalized projective ϕ-recurrent Sasakian manifold is of constant curvature.*
Key Words: *Generalized projective ϕ-recurrent, Sasakian manifold, Sectional curvature.*
AMS Subject Classification (2000): *53C05, 53C20, 53C25, 53D15.*

I. Introduction

In 1977, T.Takahashi [13] introduced the notion of locally ϕ-symmetric Sasakian manifold and obtain few of its interesting properties. The authors like [7] and [15] have extended this notion to 3-dimensional Kenmotsu and trans-Sasakian manifolds respectively. Also ϕ-recurrent Sasakian and Kenmotsu manifolds was studied by authors [6]. In this paper, we study generalized projective ϕ-recurrent Sasakian manifold.

The paper is organized as follows. In preliminaries, we give a brief account of Sasakian manifolds. In section 3, we find a relation between the associated 1-forms A and B. We also proved that the characteristic vector field ξ and vector field ρ associated to the 1-forms A and B are co-directional. Finally we proved that a generalized projective ϕ-recurrent Sasakian manifold is of constant curvature.

II. Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Sasakian manifold with the structure (ϕ,ξ,η,g). Then the following relations hold [1]:

(2.1) $$\phi^2(X) = -X + \eta(X)\xi, \qquad \phi\xi = 0,$$
(2.2) $$\eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0,$$
(2.3) $$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
(2.4) $$R(\xi,X)Y = (\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X,$$
(2.5) $$(a)\ \nabla_X\xi = -\phi X, \qquad (b)\ (\nabla_X\eta)(Y) = g(X,\phi Y),$$
(2.6) $$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$
(2.7) $$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi,$$
(2.8) $$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y),$$
(2.9) $$S(X,\xi) = 2n\eta(X),$$
(2.10) $$S(\phi X,\phi Y) = S(X,Y) - 2n\eta(X)\eta(Y),$$

for all vector fields X,Y,Z where ∇ denotes the operator of covariant differentiation with respect to g, ϕ is a $(1,1)$ tensor field, S is the Ricci tensor of type $(0,2)$ and R is the Riemannian curvature tensor of the manifold.

Definition 2.1. A Sasakian manifold is said to be locally ϕ-symmetric if

(2.11) $$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

for all vector fields X,Y,Z,W orthogonal to ξ.

Definition 2.2. A Sasakian manifold is said to be locally projective ϕ-symmetric if

(2.12) $$\phi^2((\nabla_W P)(X,Y)Z) = 0,$$

for all vector fields X,Y,Z,W orthogonal to ξ.

Definition 2.3. A Sasakian manifold is said to be projective ϕ-recurrent manifold if there exists a non-zero 1-form A such that

(2.13) $$\phi^2((\nabla_W P)(X,Y)Z) = A(W)P(X,Y)Z,$$

for arbitrary vector fields X,Y,Z,W, *where P is a projective curvature tensor given by*

(2.14) $$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y].$$

If the 1-form A vanishes, then the manifold reduces to locally projective ϕ-symmetric manifold.

III. Generalized Projective ϕ-Recurrent Sasakian Manifold

Definition 3.1. A Sasakian manifold M^{2n+1} is called generalized projective ϕ-recurrent if its curvature tensor R satisfies the condition

(3.1) $$\phi^2((\nabla_W P)(X,Y)Z) = A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y],$$

where A and B are 1-forms, β is non-zero and these are defined by

$$A(W) = g(W,\rho_1), B(W) = g(W,\rho_2),$$

and where ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively.

Let us consider generalized projective ϕ-recurrent Sasakian manifold. Then by virtue of (2.1) and (3.1) we have

(3.2) $$-((\nabla_w P)(X,Y)Z) + \eta((\nabla_w P)(X,Y)Z)\xi = A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y].$$

From which it follows that

(3.3) $$-g(((\nabla_W P)(X,Y)Z,U) + \eta((\nabla_W P)(X,Y)Z)\eta(U) = A(W)g(P(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Let $\{e_i\}$, $i = 1,2,\dots,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (3.3) and taking summation over $i, 1 \le i \le 2n+1$, we get

(3.4) $$-(\nabla_w S)(X,U) + \frac{\nabla_w r}{2n} g(X,U) - \frac{(\nabla_w S)(X,U)}{2n} + (\nabla_w S)(X,\xi)\eta(U) - \frac{\nabla_w r}{2n} \eta(X)\eta(U) + \frac{(\nabla_w S)(X,U)}{2n}\eta(U) = A(W)[\frac{2n+1}{2n}S(X,U) - \frac{r}{2n} g(X,U)] + 2nB(W)g(X,U).$$

Replacing U by ξ in (3.4) and using (2.2)(b) and (2.9), we get

(3.5) $$A(W)[(2n+1) - \frac{r}{2n}]\eta(X) + 2nB(W)\eta(X) = 0.$$

Putting $X = \xi$ in (3.5), we obtain

(3.6) $$B(W) = [\frac{r}{4n^2} - \frac{2n+1}{2n}]A(W).$$

This leads to the following result:

Theorem 3.1. *In a generalized projective* ϕ *−recurrent Sasakian manifold* M^{2n+1} *, the 1-forms A and B are related as in* (3.6).

From (3.2) we have,

(3.7) $$(\nabla_W P)(X,Y)Z = \eta((\nabla_W P)(X,Y)Z)\xi - A(W)P(X,Y)Z - B(W)[g(Y,Z)X - g(X,Z)Y],$$

this implies,

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z + \frac{1}{2n}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y] - \frac{1}{2n}[(\nabla_W S)(Y,Z)\eta(X) - (\nabla_W S)(X,Z)\eta(Y)]\xi + \frac{1}{2n} A(W)[S(Y,Z)X - S(X,Z)Y]$$
(3.8) $$+ B(W)[g(Y,Z)X - g(X,Z)Y].$$

From (3.8) and the Bianchi identity we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = \frac{1}{2n}A(W)[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] + B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \frac{1}{2n} A(X)[S(W,Z)\eta(Y) - S(Y,Z)\eta(W)] + B(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]$$
(3.9) $$+ \frac{1}{2n}A(Y)[S(X,Z)\eta(W) - S(W,Z)\eta(X)] + B(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)].$$

By virtue of (2.8) we obtain from (3.9) that

$$A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + A(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] + A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)] = \frac{1}{2n}A(W)[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] + B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \frac{1}{2n}A(X)[S(W,Z)\eta(Y) - S(Y,Z)\eta(W)] + B(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]$$

(3.10) $$+ \frac{1}{2n}A(Y)[S(X,Z)\eta(W) - S(W,Z)\eta(X)] + B(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)].$$

Putting $Y = Z = e_i$ in (3.10) and taking summation over i, $1 \le i \le 2n+1$, we get

$$(a)\ A(W)\eta(X) = A(X)\eta(W)$$
(3.11) $$(b)\ B(W)\eta(X) = B(X)\eta(W),$$

for all vector fields X, W.

Replacing X by ξ in (3.11), we get

$$(a) A(W) = \eta(W)\eta(\rho_1)$$
(3.12) $$(b) B(W) = \eta(W)\eta(\rho_2),$$
for any vector field W, where $A(\xi) = g(\xi, \rho_1) = \eta(\rho_1)$ and $B(\xi) = g(\xi, \rho_2) = \eta(\rho_2)$, ρ_1 and ρ_2 being the vector fields associated to the 1-forms A and B.

From (3.11) and (3.12), we state the following theorem:

Theorem 3.2. *In a generalized projective ϕ-recurrent Sasakian manifold (M^{2n+1}, g), $n \geq 1$, the characteristic vector field ξ and the vector fields ρ_1 and ρ_2 associated to the 1-forms A and B respectively are codirectional and the 1-forms A and B are given by* (3.12).

From (2.14) it follows that

(3.13) $$(\nabla_W P)(X,Y)\xi = (\nabla_W R)(X,Y)\xi - \frac{1}{2n}[(\nabla_W S)(Y,\xi)X - (\nabla_W S)(X,\xi)Y].$$

Using (2.5), (2.6) and (2.9) in the above equation, we have

(3.14) $$(\nabla_W P)(X,Y)\xi = [g(W,\phi Y)X - g(W,\phi X)Y] + R(X,Y)\phi W.$$

By virtue of (2.8) and (2.9) it follows from (3.14) that,

(3.15) $$\eta(\nabla_W P)(X,Y)\xi = 0.$$

Also in a Sasakian manifolds, the following result holds:

$$R(X,Y)\phi W = g(\phi X,W)Y - g(Y,W)\phi X$$
(3.16) $$- g(\phi Y,W)X + g(X,W)\phi Y + \phi R(X,Y)W.$$

Using (3.14) and (3.16) it follows that

(3.17) $$(\nabla_W P)(X,Y)\xi = g(X,W)\phi Y - g(Y,W)\phi X + \phi R(X,Y)W.$$

In view of (3.14) and (3.16), we obtain from (3.1) that

$$g(X,W)\phi Y - g(Y,W)\phi X + \phi R(X,Y)W$$
(3.18) $$= -A(W)R(X,Y)\xi - B(W)[g(Y,Z)X - g(X,Z)Y].$$

Using (2.6) and (3.12) in (3.18) we have

$$g(X,W)\phi Y - g(Y,W)\phi X + \phi R(X,Y)W$$
(3.19) $$= -\eta(W)\eta(\rho)[\eta(Y)X - \eta(X)Y] - B(W)[\eta(Y)X - \eta(X)Y].$$

Thus if X and Y are orthogonal to ξ, (3.19) reduces to

(3.20) $$\phi R(X,Y)W = g(Y,W)\phi X - g(X,W)\phi Y.$$

Operating ϕ on both sides of (3.20) and using (2.1), we get

(3.21) $$R(X,Y)W = g(Y,W)X - g(X,W)Y,$$

for all X, Y, W.

Hence we can state the following:

Theorem 3.3. *A generalized projective ϕ-recurrent Sasakian manifold (M^{2n+1}, g), $n \geq 1$, is a space of constant curvature, provided that X and Y are orthogonal to ξ.*

References

[1] Blair, D.E., Contact manifolds in Riemannian geometry, Lecture Notes in Math. Springer 509(1976).

[2] Bhagwat Prasad, A pseudo projective curvature tensor on a Riemannian manifolds Bull.Cal.Math.Soc., 94:3(2002), 163-166.

[3] Asli Basari, Cengizhan Murathan, Generalized ϕ-recurrent kenmotsu Manifolds, Sdu fen debiyat fakultesi, 3(1)(2008), 91-97.

[4] M.C.Chaki, On Pseudo Symmetric manifolds, Analele Stiintifice Ale Univeritatti. 33(1987), 53-58.

[5] U.C.De, N.Guha, On generalized recurrent manifolds, J.National Academy of Math. India, 9(1991), 85-92.

[6] U.C.De, A.A.Shaikh, S.Biswas, On $\phi-$recurrent Sasakian Manifolds, Novi Sad J.Math., 33(2003), 13-48.

[7] U.C.De and G. Pathak, On 3-dimensional kenmotsu manifolds, Indian J.Pure Appl.Math., 35: 2(2004), 159-165.

[8] K.Kenmotsu, A Class of almost contact Riemannian manifolds,Tohoku Math.J., 24(1972), 93-103.

[9] D.A.Patil, D.G.Prakasha and C.S.Bagewadi, On Generalized $\phi-$recurrent Sasakian Manifolds, Bulletin of Mathematical Analysis and Applications. 3(2009), 1821-1291.

[10] D.G.Prakasha and A.Yildiz, Generalized ϕ-recurrent Lorentzian $\alpha-$ Sasakian Manifold, Commun. Fac.Sci.Uni.Ank.Series A1.59(2010), 1303-5991.

[11] Absos Ali Shaikh and Helaluddin Ahmad, On Generalized $\phi-$recurrent Sasakian Manifolds, Applied Mathematics, 2(2011), 1317-1322.

[12] T.Takahashi, Sasakian $\phi-$symmetric spaces, Tohoku Math.J. 29, (1977), 91-113.

[13] L. Tamassy and T.Q.Binh, OnWeakly Symmetic and weakly projective Symmetic Riemannian Manifolds, Colloquia Mathematica Societatis. 50(1989), 663-670.

[14] Venkatesha, C.S.Bagewadi, On pseudo projective $\phi-$recurrent kenmotsu Manifolds, Soochow J.Math.,323(2006), 433-439.

[15] Venkatesha, C.S.Bagewadi, On 3-dimensional trans-Sasakian manifolds, AMSE, 42(5) (2005),63-73.