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Numerical Solution of Nonlinear Diffusion Equation with Convection Term by Homotopy Perturbation Method

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Abstract: *In this paper, an application of homotopy perturbation method (HPM) is applied to finding the approximate solution of nonlinear diffusion equation with convection term, We obtained the numerically solution and compared with the exact solution.The results reveal that the homotopy perturbation method is very effective, simple and very close to the exact solution.*
Keywords: *Diffusion equation with convection term, homotopy perturbation method.*

I. INTRODUCTION

In recent years, the application of the homotopy perturbation method (HPM) [1, 2] in nonlinear problems has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem which is easy to solve. The homotopy perturbation method [3], proposed first by He in 1998 and was further developed and improved by He [2, 4, 5]. The method yields a very rapid convergence of the solution series in the most cases. Usually, one iteration leads to high accuracy of the solution. Although goal of He's homotopy Perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. A review of recently developed nonlinear analysis methods can be found in [6]. Recently, the applications of homotopy perturbation theory among scientists were appeared [7-11], which has become a powerful mathematical tool, when it is successfully coupled with the perturbation theory [2, 5, 12].

I.1 Mathematical Model

we consider the nonlinear diffusion equation with convection term of the form:

$$u_t = (A(u)u_x)_x + B(u)u_x + C(u)$$

Where $u=u(x, t)$ is the unknown function *and* $A(u)$, $B(u)$, and $C(u)$ are arbitrary smooth functions on u. The indices t and x denotes differentiating with respect to the variables. The study of solution of this problem has over the years attracted the interest of many researches. Chemiha and Serov [13] considered the lie and non-lie symmetries of non linear diffusion with convection term. In this work, we will apply homotopy perturbation method to approximate solution of the generalized non linear diffusion equation with convection term of the form:

$$u_t = au_{xx} + buu_x + cu(u-k)(u+k) \quad (1)$$

with taking $A(u) = a, B(u) = bu, and\ C(u) = cu(u-k)(u+k)$

with the initial condition

$$u(x,0) = f(x)\ , \quad \alpha \le x \le \beta$$

II. Materials And Methods

II.1 Basic Idea of Homotopy Perturbation Method (HPM)

To illustrate HPM consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \quad (2)$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \quad (3)$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω.The operator A can be generally divided into two parts F and N, where F is linear, Whereas N is nonlinear. Therefore, equation (2) can be rewritten as follows:

$$F(u) + N(u) - f(r) = 0 \quad (4)$$

He [14] constructed a homotopy $\nu: \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[F(v) - F(v_0)] + p[A(v) - f(r)] = 0, \quad (5)$$

or

$$H(v,p) = F(v) - F(v_0) + pF(v_0) + p[N(v) - f(r)] = 0, \quad (6)$$

Where r∈ Ω, $p \in [0, 1]$ that is called homotopy parameter, and v_0 is an initial approximation of equation (2). Hence, it is obvious that:

$$H(v,0) = F(v) - F(v_0) = 0, \qquad H(v,1) = A(v) - f(r) = 0, \tag{7}$$

and the changing process of p from 0 to 1, is just that of $H(v, P)$ from $F(v) - F(v_0)$ to $A(v) - f(r)$.

In topology, this is called deformation, $F(v) - F(v_0)$ and $A(v) - f(r)$ are called homotopic. Applying the perturbation technique [15], due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of equation(5) equation (6) can be expressed as a series in *p*, as follows:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots, \tag{8}$$

when $p \to 1$, equation(5) or equation (6) corresponds to equation(4) and becomes the approximate solution of equation (4), i.e.,

$$u = \lim_{p\to 1}(v) = v_0 + v_1 + v_2 + v_3 + \cdots \tag{9}$$

The series (9) is convergent for most cases, and the rate of convergence depends on A(v), [3].

II.2 Solution of the nonlinear diffusion equation with convection term by HPM

For solving equation (1), by homotopy perturbation method, we construct the following homotopy:

$$H(v,p) = (1-p)\left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial v}{\partial t} - a\frac{\partial^2 v}{\partial x^2} - bv\frac{\partial v}{\partial x} - cv[(v-k)(v+k)]\right] = 0 \tag{10}$$

hence, we get

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} - p\frac{\partial v}{\partial t} + p\frac{\partial u_0}{\partial t} + p\frac{\partial v}{\partial t} - p\left(a\frac{\partial^2 v}{\partial x^2}\right) - p\left(bv\frac{\partial v}{\partial x}\right) - p[cv(v-k)(v+k)] = 0 \tag{11}$$

Substituting equation (8) in equation (11), we get:

$$\begin{aligned}
&\frac{\partial}{\partial t}(v_0 + pv_1 + p^2v_2 + \cdots) - \frac{\partial u_0}{\partial t} - p\frac{\partial}{\partial t}(v_0 + pv_1 + p^2v_2 + \cdots) \\
&+p\frac{\partial u_0}{\partial t} + p\frac{\partial}{\partial t}(v_0 + pv_1 + p^2v_2 + \cdots) - pa\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + \cdots) \\
&-pb\left[(v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots)\left(\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + \cdots)\right)\right] \\
&-pc\begin{bmatrix}(v_0 + pv_1 + p^2v_2 + \cdots)(v_0 + pv_1 + p^2v_2 + \cdots - k) \\ (v_0 + pv_1 + p^2v_2 + \cdots + k)\end{bmatrix} = 0
\end{aligned} \tag{12}$$

Collecting the terms with the same power of p, we get:

$$p^0: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \qquad , v_o(x,t) = u_o(x,t) \tag{13}$$

$$p^1: \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} - a\frac{\partial^2 v_0}{\partial x^2} - bv_0\frac{\partial v_0}{\partial x} - c({v_0}^3 - v_0k^2) = 0 \qquad , v_1(x,0) = 0 \tag{14}$$

$$p^2: \frac{\partial v_2}{\partial t} - a\frac{\partial^2 v_1}{\partial x^2} - b\left(v_0\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_0}{\partial x}\right) - c(3{v_0}^2v_1 - v_1k^2) = 0 \qquad , v_2(x,0) = 0 \tag{15}$$

$$p^3: \frac{\partial v_3}{\partial t} - a\frac{\partial^2 v_2}{\partial x^2} - b\left(v_0\frac{\partial v_2}{\partial x} + v_1\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_0}{\partial x}\right) - c(3{v_0}^2v_2 + 3v_0{v_1}^2 - v_2k^2) = 0\,, \quad v_3(x,t) = 0 \tag{16}$$

⋮

So we can calculate the equations (13, 14, 15,16,…) by applying the inverse operator on the interval (0, t)

$$L_t^{-1} = \int_0^t (.)dt \;, \tag{17}$$

From equation (13), we get

$$v_0 = u_0 \tag{18}$$

From equation (14), we get

$$v_1(x,t) = \int_0^t \left(a\frac{\partial^2 v_0}{\partial x^2} + bv_0\frac{\partial v_0}{\partial x} + c({v_0}^3 - k^2v_0) - \frac{\partial u_0}{\partial t}\right)dt$$

$$= \int_0^t a\frac{\partial^2 v_0}{\partial x^2}dt + \int_0^t bv_0\frac{\partial v_0}{\partial x}dt + \int_0^t c({v_0}^3 - k^2 v_0)dt$$

$$= a\frac{\partial^2 v_0}{\partial x^2}t + bv_0\frac{\partial v_0}{\partial x}t + c({v_0}^3 - k^2 v_0)t$$

$$= \left(a\frac{\partial^2 v_0}{\partial x^2} + bv_0\frac{\partial v_0}{\partial x} + c({v_0}^3 - k^2 v_0)\right)t \tag{19}$$

By the same way we can continue, Then the approximate solution of equation (1) is:

$$u(x,t) = \lim_{p\to 1} v = \lim_{p\to 1}(v_0 + v_1p^1 + v_2p^2 + v_3p^3 + \cdots + v_jp^j + \cdots)$$

$$= v_0 + v_1 + v_2 + v_3 + \cdots + v_j + \cdots \tag{20}$$

III. Nunerical Applications

We will apply homotopy perturbation method (HPM) to solve the nonlinear diffusion equation with convection term, and present numerical results to verify the effectiveness of this method, we take the following example:

III.1. Numerical Example and Results

In this section, we present examples of nonlinear diffusion equation with convection term and results will be compared with the exact solutions.

III.2. Example

Consider the following nonlinear diffusion equation with convection term. [16]

$$u_t = au_{xx} + buu_x + \frac{b^2}{9a}u(u-k)(u+k) \tag{21}$$

with the initial condition

$$u(x,o) = \frac{k(-1 + c_1 e^{\frac{bkx}{3a}})}{1 + c_1 e^{\frac{bkx}{3a}} + c_2 e^{\frac{bkx}{6a}}} \tag{22}$$

and boundary conditions

$$u(0,t) = \frac{k(-1+c_1)}{1 + c_1 + c_2 e^{\frac{b^2k^2t}{12a}}} \quad , \quad (1,t) = \frac{k\left(-1 + c_1 e^{\frac{bk}{3a}}\right)}{1 + c_1 e^{\frac{bk}{3a}} + c_2 e^{\frac{b^2k^2t}{12a}+\frac{bk}{6a}}} \tag{23}$$

Where a ≠ 0, b and k are arbitrary constants. In this example, $A(u) = a$, $B(u) = bu$, and

$$C(u) = \frac{b^2}{9a}u(u-k)(u+k)$$

The exact solutions of this equation have been derived by Andrei D. Polyanin and Valentin F. Zaitsev. [16]

$$u(x,t) = \frac{k[-1 + c_1 e^{\frac{bkx}{3a}}]}{1 + c_1 e^{\frac{bkx}{3a}} + c_2 e^{\frac{b^2k^2t}{12a}+\frac{bkx}{6a}}} \tag{24}$$

where c_1 and c_2 are arbitrary constants.

Apply the HPM, and equating the terms with the identical powers of p as:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \qquad , v_o(x,0) = u_o(x,t) \tag{25}$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} + a\frac{\partial^2 v_0}{\partial x^2} + bv_o\frac{\partial v_0}{\partial x} + \frac{b^2}{9a}({v_o}^3 - v_ok^2) \qquad , v_1(x,0) = 0 \tag{26}$$

$$p^2: \frac{\partial v_2}{\partial t} = a\frac{\partial^2 v_1}{\partial x^2} + b\left(v_o\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_0}{\partial x}\right) + \frac{b^2}{9a}(3{v_o}^2 v_1 - v_1k^2) \qquad , v_2(x,0) = 0 \tag{27}$$

then solving equation (26),and equation (27),we obtained V_1 and V_2 as follows:

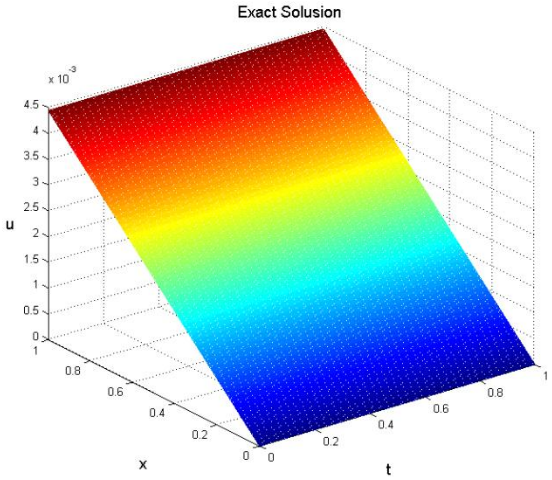
$$v_1(x,t) = -\frac{b^2 e^{\frac{bkx}{6a}} k^3 t\left(-1 + c_1 e^{\frac{bkx}{3a}}\right)c_2}{12a(1 + c_1 e^{\frac{bkx}{3a}} + c_2 e^{\frac{bkx}{6a}})^2} \tag{28}$$

$$v_2(x,t) = -\frac{b^4 e^{\frac{bkx}{6a}} k^5 t^2 \left(-1 + c_1 e^{\frac{bkx}{3a}}\right) c_2 (1 + c_1 e^{\frac{bkx}{3a}} - c_2 e^{\frac{bkx}{6a}})}{288 a^2 (1 + c_1 e^{\frac{bkx}{3a}} + c_2 e^{\frac{bkx}{6a}})^3} \qquad (29)$$

Then the approximate solution of second-order is $u_2(x,t) = \lim_{p\to 1}(v_0 + pv_1 + p^2 v_2) = v_0 + v_1 + v_2$.
The results are given in the following table and figures:

Table 1: The numerical results for the approximate solution obtained by **HPM** and comparison with the **Exact** solution for $c_1 = c_2 = 1$, k = a = b = 0.2

Space X	Time t	Exact Solution	HPM Solution	Error $\lvert u_{Exact}(x,t) - u_2(x,t) \rvert$
0	0	0	0	0
	0.2	0	0	0
	0.4	0	0	0
	0.6	0	0	0
	0.8	0	0	0
	1	0	0	0
0.2	0	0.000888882304590150	0.000888882304590149	2.1684E-19
	0.2	0.000888842798417329	0.000888842798417290	3.9248E-17
	0.4	0.000888803290488899	0.000888803290488587	3.1236E-16
	0.6	0.000888763780805095	0.000888763780804042	1.0538E-15
	0.8	0.000888724269366151	0.000888724269363653	2.4978E-15
	1	0.000888684756172300	0.000888684756167422	4.8786E-15
0.4	0	0.001777725104909530	0.001777725104909530	2.1684E-19
	0.2	0.001777646097830930	0.001777646097830850	7.8063E-17
	0.4	0.001777567087241040	0.001777567087240410	6.2428E-16
	0.6	0.001777488073140310	0.001777488073138210	2.1070E-15
	0.8	0.001777409055529230	0.001777409055524240	4.9945E-15
	1	0.001777330034408260	0.001777330034398500	9.7554E-15
0.6	0	0.002666488904294910	0.002666488904294910	4.3368E-19
	0.2	0.002666370406842970	0.002666370406842850	1.1753E-16
	0.4	0.002666251904123900	0.002666251904122960	9.3675E-16
	0.6	0.002666133396138390	0.002666133396135230	3.1602E-15
	0.8	0.002666014882887140	0.002666014882879650	7.4897E-15
	1	0.002665896364370860	0.002665896364356240	1.4629E-14
0.8	0	0.003555134221294890	0.003555134221294890	4.3368E-19
	0.2	0.003554976249264190	0.003554976249264030	1.5656E-16
	0.4	0.003554818270210270	0.003554818270209020	1.2477E-15
	0.6	0.003554660284134070	0.003554660284129860	4.2115E-15
	0.8	0.003554502291036530	0.003554502291026550	9.9829E-15
	1	0.003554344290918590	0.003554344290899090	1.9499E-14
1	0	0.004443621597267900	0.004443621597267900	0
	0.2	0.004443424171710220	0.004443424171710030	1.9429E-16
	0.4	0.004443226737372940	0.004443226737371380	1.5595E-15
	0.6	0.004443029294257220	0.004443029294251960	5.2623E-15
	0.8	0.004442831842364240	0.004442831842351770	1.2473E-14
	1	0.004442634381695170	0.004442634381670800	2.4362E-14



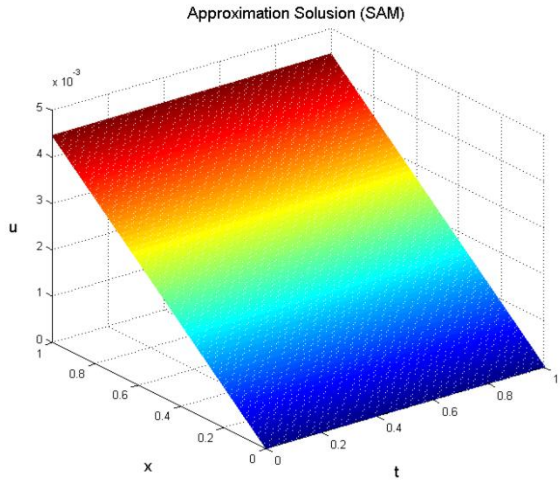


Fig. 1: Exact solution
When $c_1 = c_2 = 1$, k = a = b = 0.2

Fig. 2: Approximate solution
When $c_1 = c_2 = 1$, k = a = b = 0.2

IV. CONCLUSION

In this paper we have applied homotopy perturbation method for approximate solution of nonlinear diffusion equation with convection term and compared with exact solution . We can conclude from the numerical results **Fig. (1)** and **Fig. (2)** shows the comparison between the exact solution and the numerical solution obtained by homotopy perturbation method (HPM), For $c_1 = 1$, $c_2 = 1$, k=0.2, a=0.2,and b=0.2 .It can be seen that the solution obtained by the present method is nearly identical with that given by exact solution. The absolute error $|u_{exact}(x,t) - u_2(x,t)|$ of the example be observed in **Table (1)**.Therefore, this method can be seen as efficient method for solving nonlinear diffusion equation with convection term. In our work, we use the Mat lab program to calculate the series obtained from this method.

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