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The Co-action of Lie group on Hom Space

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Abstract: *let G be a Lie group, and Π^* is a dual representation of G, In this paper we will present and study the concepts of co-action of Lie group on Hom space, we recall the definition of tensor product of two representations of lie groups and construct the definition of co-action of Lie group, and also study the properties of this action. we use the co-action of Lie group on Hom space $(Hom(W_4, W_3^*))^*$combining with another Hom space having the same structure with different vector space,$(Hom(W_2, W_1^*))^*$.so we have new co-action which called double co-action of Lie group G, denoted by Co-AAC_Lie group which acting on $(Hom(Hom(W_4, W_3^*), Hom(W_2, W_1^*)))^*$.*
Keywords: *Action Group, CO-AC-Lie group, CO-AAC-Lie group, Tensor space, Hom space.*

I. Introduction:

Throughout this paper**,** in 2004 Hall B. C. [1] wrote a book of Lie group for manifold theory and the relationship between Lie groups and Lie algebras. The reason of studying the representation is that a representation can be thought of as an action of group on some vector space. Such actions (representations) arise naturally in many branches of both mathematics and physics [2], [3], and it is important to understand them.

In [1], the Schur's lemma introduced the concept of action of Lie algebra on the space of linear maps from W_2 into W_1, which denoted by $\mathrm{Hom}\,(W_2, W_1)$, also introduce the concept of action on tensor product of two representation of Lie algebra. Schur's lemma state: Suppose that $\pi_1$and π_2 are representation of lie algebra acting on finite –dimensional space W_1 and W_2, respectively. Define an action of g on $\mathrm{Hom}\,(W_2, W_1)$, such that π: g⟶ gl(Hom (W_2, W_1)),

$\pi(x) = \pi_1 f - f\pi_2$, for all x∈g and f ∈ $\mathrm{Hom}\,(W_2, W_1)$. and $\mathrm{Hom}\,(W_2, W_1) \cong W_2^* \otimes W_1$, as equivalence of representation.

In this paper we will present and study the concept of co-action on $(\mathrm{Hom}(W_2, W_1^*))^*$and the equivalent relation with the tensor product space. since $\mathrm{Hom}(W_2, W_1^*)$ is a vector space of all linear functional from W_2 into W_1^*, so $(\mathrm{Hom}(\mathrm{Hom}(W_4, W_3^*), \mathrm{Hom}(W_2, W_1^*)))^*$ is dual vector space of all linear functional from $(\mathrm{Hom}(W_2, W_1^*))^*$ into $(\mathrm{Hom}(W_4, W_3^*))^*$, then the representation of G acting on this dual vector space is co-action of G on this Hom space. Also we give an equivalent relation between CO-AC_Lie group and CO-AAC_Lie group on Hom and AC_Lie group with AAC_Lie group on Tensor products, and explain the actions structure by using diagram.

II. Basic Concept:

In this section, we give the main definitions and some examples of group action, group representation, and tensor product, for more details, see[9,10].

(2.1) Definition, [8]:

Let 𝔸 be a non empty set and let G be a group with neutral element e ∈G, a left action of G on 𝔸 is a map φ : G×𝔸 ⟶ 𝔸 such that satisfies the following $\varphi(e, x) = x$ and $\varphi(g, \varphi(k, x)) = \varphi(gk, x)$, for all $x \in$ 𝔸 and g, $k \in$ G. By definition of an action, for $g \in$ G with inverse g^{-1}, we get:
$\varphi(g^{-1}, \varphi(g, x)) = \varphi(e, x) = x$, for any $x \in$ 𝔸. Thus for any $g \in$ G the map $x \longmapsto \varphi(e, x)$ is a bijection from 𝔸 ⟶ 𝔸, so we may also view φ as a mapping of G to the group of bijections of 𝔸.

(2.1) Definition, [4]:

A Lie group G is a finite dimensional smooth manifold G together with a group structure on G, such that the multiplication G×G⟶G and the attaching of an inverse $g \longrightarrow g^{-1}$: G ⟶ G are smooth maps**.**

(2.2) Definition, [7]:

A matrix Lie group is any subgroup G of GL(n,ℂ)with the following property. If A_m is any sequence of matrices in G and A_m converges to some matrix A then A ∈ G, or A is not invertible.

(2.3) Definition, [6]:

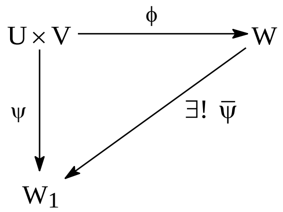
A finite-dimensional real (complex) representation of G is a Lie group homomorphism $\Pi: G \longrightarrow GL(n, \mathbb{R})$, $(n \geq 1)$. Generally, a Lie group homomorphism $\Pi : G \longrightarrow GL(V)$, where V is a finite dimensional real (complex) vector space with dim $V \geq 1$.

(2.4) Definition, [6]:

Let G and H be two Lie groups. A map f from G to H is called a Lie group homomorphism if f is a group homomorphism and $\overset{\infty}{C}$-map on H.

(2.5) Definition, [1]:

If U and V are finite dimensional real or complex vector spaces, then a tensor product of U and V is a vector space W, together with a bilinear map $\phi : U\times V \longrightarrow W(U \otimes V)$ with the following property: If ψ is any bilinear map of $U\times V$ into a vector space W_1, then there exists a unique linear map $\overline{\psi}$ of W into W_1, such that the following diagram commutes:



(2.6) Definition, [1]:

Suppose G is a Lie group and Π is representation of G acting on a finite dimensional vector space V. Then the dual representation Π^* to Π is the representation of G acting on V^* given by $\Pi^*(g) = [\Pi(g^{-1})]^{tr}$ The dual representation is also called contragredient representation.

(2.7) Example:

Let $\Pi: S^1 \longrightarrow Sl(2, \mathbb{C})$, where $S^1 = \{(\cos\theta, \sin\theta), 0 \leq \theta \leq 2\pi\}$, $\overset{1}{S} = e^{i\theta} = \cos\theta + i\sin\theta$

$$Sl(2, \mathbb{C}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, 0 \leq \theta \leq 2\pi \right\}$$

Such that:

$$\Pi(e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, 0\leq\theta\leq 2\pi$$

$$\Pi(g)^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$[\Pi(g)^{-1}]^{tr} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \Pi^*(g)$$

III. The Co-Action Of Lie Group On Hom And Tensor Product

In this section we study the co-action of G on the Hom space and on the tensor product.

(3. 1) Proposition:

Let $\Pi_i^*: G \longrightarrow GL(W_i^*)$, for i=1,2 be a dual representation of Π_i ,then the co-action of G on $(Hom(W_2, W_1^*))^*$ is given by a dual representation of G, such that: $\Pi^*(a) = \Pi_2^*(a)^{-1} {}_o F^* {}_o \Pi_1(a)$, for all $a \in G$.

Proof:

since $\Pi^*(a) = (\Pi(a))^* = (\Pi_1^*(a) {}_o F {}_o \Pi_2(a)^{-1})^*$

$= \Pi_2^*(a)^{-1} {}_o F^* {}_o \Pi_1^{**}(a)$

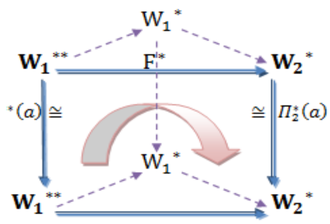
$= \Pi_2^*(a)^{-1} {}_o F^* {}_o \Pi_1(a)$

Where $F^*: W_1^{**} \longrightarrow W_2^*$, and $\Pi^*(ab) = (\Pi(ab))^*$

$= (\Pi(b) {}_o \Pi(a))^*$

$= \Pi^*(a) {}_o \Pi^*(b)$.

Thus, the co-action of G is a group homomorphism ■



(3.2) Corollary:

Let $\Pi_i: G \longrightarrow GL(n,k) \cong GL(W_i)$, i= 1,2 be a matrix representation ,then the co-action of G on GL(n,k) is a matrix representation defined by:
$\Pi(a) = (\Pi_1(a)^{-1})^{tr}{}_{o}F_{o}\Pi_2(a)^{-1}$, with duality $\Pi^*: G \longrightarrow GL(Hom\ (W_2, {W_1}^*))^*$
, such that $\Pi^*(a) = (\Pi_2(a))^{tr}{}_{o}F^*{}_{o}\Pi_1(a)$, for all a∈G.

Proof:

Since $\Pi^*(a) = (\Pi(a)^{-1})^{tr}$

$= ((\Pi_1^*(a)_o F_o \Pi_2(a)^{-1})^{-1})^{tr}$

$= (\Pi_2(a))^{tr}{}_{o}F^*{}_{o}\Pi_1(a)$

And $\Pi^*(ab) = ((\Pi(ab))^{-1})^{tr}$

$= (\Pi(a)^{-1}{}_{o}\Pi(b)^{-1})^{tr}$

$= (\Pi(a)^{-1})^{tr}{}_{o}(\Pi(b)^{-1})^{tr}$

$= \Pi^*(a)\ {}_{o}\Pi^*(b)$ ■

(3.4) proposition:

Let G be a lie group, ${W_1}^*$ and W_2 are two finite vector spaces, the following assertion are equivalent:

(1) $(Hom\ (W_2, {W_1}^*))^*$
(2) $Hom\ (W_1^{**}, W_2^*)$
(3) $Hom\ (W_1, W_2^*)$
(4) $Hom\ (W_1, Hom(W_2, k))$
(5) $Hom\ (Hom(Hom(W_1, k), Hom(W_2, k))$
(6) $(Hom\ (W_2, {W_1}^*))^{\overset{**\ldots\ldots*}{\leftarrow n \rightarrow}} = \begin{cases} Hom\ (W_1, W_2^*) & \text{if n is odd} \\ Hom\ (W_2, {W_1}^*) & \text{if n is even} \end{cases}$

Proof:

(1) To show $(Hom\ (W_2, {W_1}^*))^* \cong Hom\ (W_1^{**}, W_2^*)$, let $F \in Hom\ (W_2, {W_1}^*)$, $F: W_2 \longrightarrow {W_1}^*$, $F^* \in (Hom\ (W_2, {W_1}^*))^*$, thus

$F^*: W_1^{**} \longrightarrow W_2^*$, $F^* \in Hom\ (W_1^{**}, W_2^*)$, and there exist intertwining map
$\psi: (Hom\ (W_2, {W_1}^*))^* \longrightarrow Hom\ (W_1^{**}, W_2^*)$, $(\Pi^*(a))(v) = \Pi^*(a)\psi(v)$
For all $v \in W_2^*$ and ψ is invertible map.

(2) To show $(Hom\ (W_2, {W_1}^*))^* \cong Hom\ (W_1, Hom(W_2, k))$

Since W_2^* can be written as $Hom(W_2, k)$, by proof of (1), thus:
$Hom\ (W_1, W_2^*) = Hom\ (W_1, Hom(W_2, k))$, by the same method we have the other parts ■

(3.5) Example:

Let $\Pi_1 : G \rightarrow SU(2)$,such that $\Pi_1(g) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

And $\Pi_2: \mathbb{R} \rightarrow SO(2)$, such that $\Pi_2(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be matrix representation of G, then the co-action is given by : $\Pi^*(a) = (\Pi_2(a))^{tr}{}_{o}F^*{}_{o}\Pi_1(a)$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} {}_{o}F^*{}_{o} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\theta}\cos\theta & e^{i\theta}\sin\theta & 0 & 0 \\ -e^{i\theta}\sin\theta & e^{i\theta}\cos\theta & 0 & 0 \\ 0 & 0 & e^{-i\theta}\cos\theta & e^{-i\theta}\sin\theta \\ 0 & 0 & -e^{-i\theta}\sin\theta & e^{-i\theta}\cos\theta \end{pmatrix}.$$

(3. 6) proposition:

If Π is an action of G on $(W_2^* \otimes W_1^*)$, such that: $(a) = \Pi_2^*(a)^{-1} \otimes \Pi_1^*(a)$,for all a∈G. then the co-action of G on $(W_2^* \otimes W_1^*)^*$ is a dual representation of G, Π^*, such that $\Pi^*: G \longrightarrow GL(W_2^* \otimes W_1^*)^*$ defined by:
$\Pi^*(a) = \Pi_2^{**}(a)^{-1} \otimes \Pi_1^{**}(a)$, for all a∈G.

Proof:

Since $(a) = \Pi_2^*(a)^{-1} \otimes \Pi_1^*(a)$, then $\Pi^*(a) = (\Pi(a))^*$

$= (\Pi_2^*(a)^{-1} \otimes \Pi_1^*(a))^*$

$= \Pi_2^{**}(a)^{-1} \otimes \Pi_1^{**}(a)$, for all a∈G.

And $\Pi^*(ab) = \Pi_2^{**}(ab)^{-1} \otimes \Pi_1^{**}(ab)$

$= \Pi_2^{**}(b)^{-1}(\Pi_2^{**}(a)^{-1} \otimes \Pi_1^{**}(a))\Pi_1^{**}(b)$…..(1)

$\Pi^*(a)\ \Pi^*(b) = \Pi^*(b)_o(\Pi_2^{**}(a)^{-1} \otimes \Pi_1^{**}(a))$

$= \Pi_2^{**}(b)^{-1}(\Pi_2^{**}(a)^{-1} \otimes \Pi_1^{**}(a))\Pi_1^{**}(b)$ ……. (2)

Thus (1) & (2) are equal then the co-action is a dual representation of G on $W_2^* \otimes W_1^*)^*$ ■

(3.7)Corollary:

If the AC-lie group of G on GL(nm,k)≅GL($W_2^* \otimes W_1^*$),is a matrix representation, then the AC-lie group of G on $(W_2^* \otimes W_1^*)^*$ is a matrix representation defined by:

$\Pi^*(a) = \Pi_2(a)^{-1} \otimes \Pi_1(a)$, for all a∊G.

Proof:

$\because \Pi_1^{**}(a) = \Pi_1(a)$

Then $\Pi^*(a) = \Pi_2^{**}(a)^{-1} \otimes \Pi_1^{**}(a)$

$= \Pi_2(a)^{-1} \otimes \Pi_1(a)$, for all a∊G.

And $\Pi^*(ab) = ((\Pi(ab))^{-1})^{tr}$

$= (\Pi(b)^{-1}\Pi(a)^{-1})^{tr}$

$= (\Pi(a)^{-1} {}_o \Pi(b)^{-1})^{tr}$

$= (\Pi(a)^{-1})^{tr} {}_o (\Pi(b)^{-1})^{tr}$

$= \Pi^*(a) {}_o \Pi^*(b)$ ■

(3.8) Proposition:

Let W_1 and W_2 are two finite vector spaces and $W_1{}^*$, $W_1{}^*$ is the dual space of W_i,i-=1,2 then the following assertions are equivalent:

(1) $(W_2^* \otimes W_1^*)^*$

(2) $W_2^{**} \otimes W_1^{**}$

(3) $W_2 \otimes W_1$

(4) $(W_2^* \otimes W_1^*)^{\overleftarrow{n}\overrightarrow{} **.....*} = \begin{cases} W_2^* \otimes W_1^*, & \text{if n is even integer number} \\ W_2^{**} \otimes W_1^{**}, & \text{if n is odd integer number.} \end{cases}$

Proof:

(1) $(W_2^* \otimes W_1^*)^* \cong W_2^{**} \otimes W_1^{**}$, we must show that

ψ: $W_2^{**} \times W_1^{**} \longrightarrow W_2^{**} \otimes W_1^{**}$ is a bilinear map, by:

$(w_2^{**}, w_1^{**}) = w_2^{**}(v)w_1^{**}$, for all v∊W_2, $w_2^{**} \in W_2^{**}$, $w_1^{**} \in W_1^{**}$.

$(\alpha w_2^{**} + \beta w_2^{**\prime}, w_1^{**}) = (\alpha w_2^{**} + \beta w_2^{**\prime})(v)\, w_1^{**} = (\alpha w_2^{**}(v)\, w_1^{**} + \beta w_2^{**\prime}(v)\, w_1^{**}) = \alpha\, \psi(w_2^{**}, w_1^{**}) + \beta\, \psi(w_2^{**\prime}, w_1^{**})$.

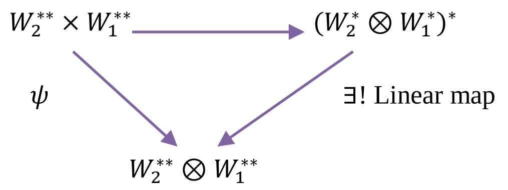
Other for all $w_1^{**}, w_1^{**\prime} \in W_1^{**}$, $w_2^{**} \in W_2^{**}$.

$\psi(w_2^{**}, \alpha w_1^{**} + \beta w_1^{**\prime}) = w_2^*(v)(\alpha w_1^{**} + \beta w_1^{**\prime})$

$= w_2^{**}(v)(\alpha w_1^{**}) + w_2^{**}(v)(\beta w_1^{**\prime})$

$= (w_2^{**}(v) w_1^{**}) + \beta(w_2^{**}(v)\, w_1^{**\prime})$

$= \alpha\, (w_2^{**}, w_1^{**}) + \beta\, \psi(w_2^{**}, w_1^{**\prime})$, for all α , β∊k.



So ψ : $W_2^{**} \times W_1^{**} \longrightarrow (W_2^* \otimes W_1^*)^*$ is a bilinear map, thus we use the tensor product and universal property of this tensor product , we get a unique linear map Ω. So by universal property of tensor product there exist a unique linear map Ω: $(W_2^* \otimes W_1^*)^* \longrightarrow W_2^{**} \times W_1^{**}$, defined by:

$\Omega(w_2^* \otimes w_1^*) = w_2^{**}(v) w_1^{**}$, and this make the above diagram commutative.

Define ξ: $W_2^{**} \otimes W_1^{**} \longrightarrow (W_2^* \otimes W_1^*)^*$, defined by:

$\xi(w_2^{**}(v) w_1^{**}) = (w_2^* \otimes w_1^*)^*$, for all v∊$W_2$, $w_2^{**} \in W_2^{**}$, $w_1^{**} \in W_1^{**}$.

Since $\xi^{-1} = \Omega$ is a linear map and Ω is an intertwining map and invertible,

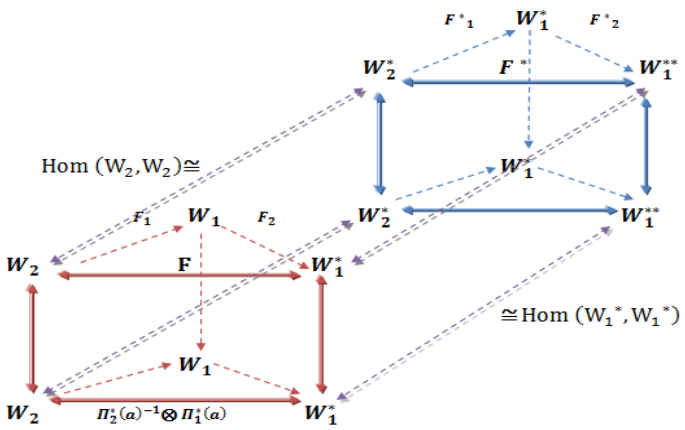
where: $\xi(w_2^* \otimes w_1^*) = w_2^{**}(v) w_1^{**}$

The representation of $(W_2^* \otimes W_1^*)^*$is:

$\Pi^*(a) = (\Pi_2^*(a)^{-1} \otimes \Pi_1^*(a))^*$

$= \Pi_2^{**}(a)^{-1} \otimes \Pi_1^{**}(a)$,

$(\Pi^*(a))(v) = \Pi^*(a)\Omega(v)$, for all v∊W_2.



(4)If n=2

$$(W_2^* \otimes W_1^*)^{**} = (W_2^{**} \otimes W_1^{**})^*$$
$$= W_2^{***} \otimes W_1^{***} = W_2^* \otimes W_1^*,\ (W_2^{***} \cong W_2^*,\ W_1^{***} \cong W_1^*).$$

In general, if n is odd $(W_2^* \otimes W_1^*)^{\overset{**\ldots*}{\leftarrow n \rightarrow}} = W_2^* \otimes W_1^*$.
All parts are equivalence representation in the same way of (1) ■

(3.9)Example:

let Π_1:G →SO(3), $\Pi_1(g)=\begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$,and Π_2:G→$SUT_1(\mathbb{R})$,
$\Pi_2(g)=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$,n∈ z.be matrix representation of lie group G then the co-action is given by:
$\Pi^*(g) = \Pi_2(g)^{-1} \otimes \Pi_1(g)$,

$$=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}^{-1} \otimes \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 & -n\cos 2\theta & n\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 & -n\sin 2\theta & -n\cos 2\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & -n \\ 0 & 0 & 0 & \cos 2\theta & -\sin 2\theta & 0 \\ 0 & 0 & 0 & \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

IV. The Co-Action Of Lie Group On Hom Space And Tensor Product:

(4.1) Proposition:

Let Π_i ,i=1,2,3,4,be a matrix representation of W_i, i=1,2,3,4 and the AAC_Lie group of commutative G on $\mathrm{Hom}(\mathrm{Hom}(W_4, W_3^*), \mathrm{Hom}(W_2, W_1^*))$, given by: $\Pi(a) = \left(\left(\left(\Pi_4(a)^{-1}F_3\ \Pi_3^*(a)\right)F_2\Pi_2(a)^{-1}\right)F_1\Pi_1^*(a)\right)$, for all a∈G. Then the co-action of AAC_Lie group is a dual representation$\Pi^*(a)$, given by:
$\Pi^*(a) = ((\Pi_1(a)F_1{}^*\Pi_2^*(a)^{-1})F_2{}^*\Pi_3(a))F_3{}^*\Pi_4{}^*(a)^{-1}$, for all a∈G.
Proof:

Let $\Pi^*(a)$:G⟶ $GL(\mathrm{Hom}(\mathrm{Hom}(W_4, W_3^*), \mathrm{Hom}(W_2, W_1^*)))^*$, thus:

$$\Pi^*(a) = \left(\left(\left(\Pi_4(a)^{-1}F_3\ \Pi_3^*(a)\right)F_2\Pi_2(a)^{-1}\right)F_1\Pi_1^*(a)\right)^*$$
$$=((\Pi_1^{**}(a)F_1{}^*\Pi_2^*(a)^{-1})F_2{}^*\Pi_3^{**}(a))F_3{}^*\Pi_4{}^*(a)^{-1}$$
$$=((\Pi_1(a)F_1{}^*\Pi_2^*(a)^{-1})F_2{}^*\Pi_3(a))F_3{}^*\Pi_4{}^*(a)^{-1},$$

Where
$F_1{}^*: W_2{}^* \longrightarrow W_1$, $F_2^*: W_3 \longrightarrow W_2^*$, $F_3^*: W_4^* \longrightarrow W_3$,
To show that co_action is homo. Of Lie group:

$\Pi^*(ab) = (\Pi(ab))^*$
$=(\Pi(a)\Pi(b))^*$, since Π is a representation of G
$=\Pi^*(a)\Pi^*(b)$, since G is commutative Lie gp.

Hence the co-action is a dual representation of G.

(4.2) Proposition:

Let W_i,i=1,2,3,4 be a finite dimensional vector space having duality W_i^*,i=1,2,3,4 then we have the following assertions are equivalent :

1) $(\mathrm{Hom}(\mathrm{Hom}(W_4, W_3^*), \mathrm{Hom}(W_2, W_1^*)))^*$
2) $\mathrm{Hom}(\mathrm{Hom}(W_2, W_1^*)^*, \mathrm{Hom}(W_4, W_3^*)^*)$
3) $\mathrm{Hom}(\mathrm{Hom}({W_1}^{**}, W_2^*), \mathrm{Hom}({W_3}^{**}, W_4^*))$
4) $\mathrm{Hom}(\mathrm{Hom}(W_1, W_2^*), \mathrm{Hom}(W_3, W_4^*))$
5) $\mathrm{Hom}(\mathrm{Hom}(W_1, \mathrm{Hom}(W_2, k)), \mathrm{Hom}(W_3, \mathrm{Hom}(W_4, k)))$
6) $\mathrm{Hom}(\mathrm{Hom}(W_1, W_2^*), \mathrm{Hom}(W_3, \mathrm{Hom}(W_4, k)))$
7) $(\mathrm{Hom}(\mathrm{Hom}(W_4, W_3^*)\mathrm{Hom}(W_2, W_1^*)))^{\overset{**\ldots\ldots*}{\leftarrow n \rightarrow}} =$
$$\begin{cases} \mathrm{Hom}(\mathrm{Hom}(W_4, W_3^*), \mathrm{Hom}(W_2, W_1^*)) \text{ if n even} \\ \mathrm{Hom}(\mathrm{Hom}(W_1, W_2^*), \mathrm{Hom}(W_3, W_4^*)) \text{ if n odd .} \end{cases}$$

Proof:

1) To prove (1) ≅ (2), let

$$\Psi\colon \left(\mathrm{Hom}(\mathrm{Hom}(W_4, W_3^*), \mathrm{Hom}(W_2, W_1^*))\right)^* \longrightarrow \mathrm{Hom}(\mathrm{Hom}(W_2, W_1^*)^*, \mathrm{Hom}(W_4, W_3^*)^*)$$

$\Psi\,(\Pi^*(a))(v) = \Pi^*(a)\Psi(v)$, for all $v \in \mathrm{Hom}(\mathrm{Hom}(W_2, W_1^*)^*, \mathrm{Hom}(W_4, W_3^*)^*)$

Ψ is an isomorphism map.

2) To prove (4) ≅(5)

Since $W_2^* \cong \mathrm{Hom}(W_2, k)$ and $W_4^* \cong \mathrm{Hom}(W_4, k)$, hence $\mathrm{Hom}(W_1, W_2^*) \cong \mathrm{Hom}(W_1, \mathrm{Hom}(W_2, k))$ ……….(1)

And $\mathrm{Hom}(W_3, W_4^*) \cong \mathrm{Hom}(W_3, \mathrm{Hom}(W_4, k))$ …………(2)

Then from (1)&(2)

$\mathrm{Hom}(\mathrm{Hom}(W_1, W_2^*), \mathrm{Hom}(W_3, W_4^*)) \cong \mathrm{Hom}(\mathrm{Hom}(W_1, \mathrm{Hom}(W_2, k)), \mathrm{Hom}(W_3, \mathrm{Hom}(W_4, k)))$.

** The other parts proving by the same method.

(4.3) Proposition:

Let $\Pi\colon G \longrightarrow GL((W_4 \otimes W_3) \otimes (W_2^* \otimes W_1^*))$ be a representation of comm. G acting on $((W_4 \otimes W_3) \otimes (W_2^* \otimes W_1^*))$, then the co-action is a dual representation of G, acting on $(({W_4}^* \otimes {W_3}^*) \otimes (W_2 \otimes W_1))$, such that : $\Pi^*(a) = ((({\Pi_4}^*(a)^{-1} \otimes {\Pi_3}^*(a)) \otimes \Pi_2(a)^{-1}) \otimes \Pi_1(a))$, a∈G.

Proof:

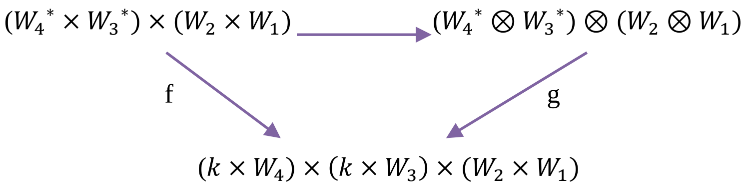
Since (a)= $(((\Pi_4(a)^{-1} \otimes \Pi_3(a)) \otimes \Pi_2^*(a)^{-1}) \otimes \Pi_1^*(a))$,

Then $\Pi^*(a) = (((\Pi_4(a)^{-1} \otimes \Pi_3(a)) \otimes \Pi_2^*(a)^{-1}) \otimes \Pi_1^*(a))^*$

$= ((({\Pi_4}^*(a)^{-1} \otimes {\Pi_3}^*(a)) \otimes \Pi_2^{**}(a)^{-1}) \otimes \Pi_1^{**}(a))$

$= ((({\Pi_4}^*(a)^{-1} \otimes {\Pi_3}^*(a)) \otimes \Pi_2(a)^{-1}) \otimes \Pi_1(a))$, for all a∈G.

Which is a representation of G acting on $(({W_4}^* \otimes {W_3}^*) \otimes (W_2 \otimes W_1))$.



Dim $((W_4 \otimes W_3) \otimes (W_2^* \otimes W_1^*)) = n_1.n_2.n_3.n_4$

$= \dim (({W_4}^* \otimes {W_3}^*) \otimes (W_2 \otimes W_1))$.

Here dim $WW_4 = n_1$, dim $W_3 = n_2$, dim $W_2 = n_3$, dim $W_1 = n_4$,

$\Pi^*(ab) = ((({\Pi_4}^*(ab)^{-1} \otimes {\Pi_3}^*(ab)) \otimes \Pi_2(ab)^{-1}) \otimes \Pi_1(ab)$

$= ({\Pi_4}^*(b)^{-1}{\Pi_4}^*(a)^{-1}) \otimes ({\Pi_3}^*(a){\Pi_3}^*(b)) \otimes (\Pi_2(b)^{-1}\Pi_2(a)^{-1}) \otimes (\Pi_1(a)\Pi_1(b))$

$= ((({\Pi_4}^*(b)^{-1} \otimes {\Pi_3}^*(b)) \otimes \Pi_2(b)^{-1}) \otimes \Pi_1(b)\ )\ (\,((({\Pi_4}^*(a)^{-1} \otimes {\Pi_3}^*(a)) \otimes \Pi_2(a)^{-1}) \otimes \Pi_1(a)\ )$,since G is commutative lie group,for all a,b ∈G.

(4.4) Corollary:

Let G be a matrix representation acting on $GL((W_4 \otimes W_3) \otimes (W_2^* \otimes W_1^*))$, then the co-AAC_lie group of G on $(({W_4}^* \otimes {W_3}^*) \otimes (W_2 \otimes W_1))$is:

$\Pi^*(a) = (((\Pi_4(a))^{tr} \otimes (\Pi_3(a)^{-1})^{tr} \otimes \Pi_2(a)^{-1}) \otimes \Pi_1(a))$, a∈G.

Proof:
$\Pi^*(a) = (\Pi(a)^{-1})^{tr}$
$=((((\Pi_4(a)^{-1} \otimes \Pi_3(a)) \otimes \Pi_2^*(a)^{-1}) \otimes \Pi_1^*(a))^{-1})^{tr}$
$=(((\Pi_4(a))^{tr} \otimes (\Pi_3(a)^{-1})^{tr}) \otimes \Pi_2(a)^{-1}) \otimes \Pi_1(a))$, for all a∈G.
And $\Pi^*(ab)=(\Pi(ab)^{-1})^{tr}$
$=(\Pi(b)^{-1}\Pi(a)^{-1})^{tr}$
$= (\Pi(a)^{-1})^{tr}(\Pi(b)^{-1})^{tr}$
$=\Pi^*(a)\Pi^*(b)$
Hence the co-action is a matrix dual representation.

V. Conclusion:

The main aim in this study is to look for an interesting action with new properties from a lemma of Schure, which states that the action of the tensor product of Lie algebras representations has interesting property.

Putting in mind that one of the two representations is usual and the other is dual to obtain results by relating the tensor product of dual representations with usual representations.

Our main work here is to study the concept of co-action on $(Hom(W_2, W_1^*))^*$and the equivalent relation with the tensor product space combining these co-actions (dual representations) and to give a dual representation which acting on $(Hom(Hom(W_4, W_3^*), Hom(W_2, W_1^*)))^*$ and then generalize them.

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