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A Further Refinement of Van Der Corput's Inequality

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Abstract: *In this paper, we obtain a further refinement of Van der Corput's inequality using an analytical technique.*
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I. Introduction

Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ be the harmonic number and $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} (n+1) a_n < \infty$.
The Van der Corput's inequality states that

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} (n+1) a_n, \qquad (1.1)$$

where $\gamma = 0.57721566...$ denotes the Euler-Mascheroni's constant. The constant $e^{1+\gamma}$ is the best possible.
In 2003, Hu [5], gave the following version of (1.1):

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n - \frac{\ln n}{4} \right) a_n \qquad (1.2)$$

This inequality is a refinement of (1.1).
In 2005, YANG [1] obtained a better result than Hu's inequality (1.2) as

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n - \frac{\ln n}{3} \right) a_n. \qquad (1.3)$$

This inequality is also a refinement of (1.1). He further extended the original Van der Corput's inequality (1.1) as follows.
Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} \left(n + \frac{1}{2} + \beta \right) a_n < \infty$ and $T_n(\beta) = \sum_{k=1}^{n} \frac{1}{k+\beta}$ .Then

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{\frac{1}{k+\beta}} \right)^{\frac{1}{T_n(\beta)}} < e^{1+\gamma_1(\beta)} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} + \beta \right) a_n, \qquad (1.4)$$

where $\gamma_1(\beta) = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \frac{1}{k+\beta} - \ln(n+\beta) \right\}$ for $\beta \in (-1, \infty)$ and $T_n(0) = S_n = \sum_{k=1}^{n} \frac{1}{k}$.
Setting $\beta = 0$ in (1.4), it becomes

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{\frac{1}{S_n}} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} \right) a_n. \qquad (1.5)$$

Clearly, inequality (1.5) improves inequality (1.1)
In 2006, Cao et al [3] established another version of (1.1) as follows.

Let $a_n \geq 0$ for $n \in N$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/\sqrt{k(k+\lambda)}}\right)^{\frac{1}{U_n(\lambda)}} < e^{1+(1+\lambda/3)\gamma(\lambda)} \sum_{n=1}^{\infty}(n+1)^{\frac{\lambda}{3}}\left\{1-\frac{\ln(n+1)}{4\left(n+1+\frac{\lambda}{2}\right)}\right\} a_n, \tag{1.6}$$

where $\lambda \in [0, \infty)$, $\quad U_n(\lambda) = \sum_{k=1}^{n} \frac{1}{\sqrt{k(k+\lambda)}}$ and $\gamma(\lambda) = \lim_{n\to\infty}\left[U_n(\lambda) - 2\ln\frac{\sqrt{n}+\sqrt{n+\lambda}}{1+\sqrt{n+\lambda}}\right]$.

Consequently, they established a sharper inequality that further refines (1.1), (1.2), (1.3) and (1.5), which is given by

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{s_n}} < e^{1+\gamma} \sum_{n=1}^{\infty}\left(n - \frac{n\ln n}{3n-1/4}\right) a_n. \tag{1.7}$$

The aim of this paper is to further refine inequality (1.7) to obtain sharper inequality than that of (1.7). Our main results are the following.

II. Main Results

Theorem 2.1: Let $a_n \geq 0$ and $S_n = \sum_{k=1}^{n} \frac{1}{k}$ such that for $n \in$ □ and

$0 < \sum_{n=1}^{\infty}\left(n - \frac{n\ln n}{2n+\ln n+11/6}\right) a_n < \infty$. Then

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{s_n}} < e^{1+\gamma} \sum_{n=1}^{\infty}\left(\frac{(6n+1)(12n+11)}{(6n+1)(12n+11)+6(6n+1)\gamma-9}\right)\left(n - \frac{n\ln n}{2n+\ln n+11/6}\right) a_n, \tag{2.1}$$

where $\gamma = 0.57721566...$ is the Euler-Mascheroni's constant and $e^{1+\gamma}$ is the best possible.

Remark 2.2. Let

$$U_n = e^{1+\gamma}\left(\frac{(6n+1)(12n+11)}{(6n+1)(12n+11)+6(6n+1)\gamma-9}\right)\left(n - \frac{n\ln n}{2n+\ln n+11/6}\right) \quad \text{and} \quad V_n = e^{1+\gamma}\left(n - \frac{n\ln n}{3n-1/4}\right).$$

For $n \in$ □ , the numerical computations of U_n and V_n gives the following table of values:

n	U_n	V_n
1	4.4230	4.8415
2	8.0200	8.5157
3	11.9770	12.7008
4	16.1260	17.0810
5	20.3990	21.5659

Clearly $U_n < V_n$, for $n \geq 1$.

Also, we consider $2n + \ln n + 11/6$ and $3n - 1/4$.

For $n \geq 4$: $2n + \ln n + \frac{11}{6} < 3n - \frac{1}{4} \quad \Rightarrow 1 - \frac{\ln n}{2n+\ln n+\frac{11}{6}} < 1 - \frac{\ln n}{3n-\frac{1}{4}}$.

Clearly, inequality (2. 1) is an improvement and the refinement of the (1.7).

Theorem 2.3: Let $a_n \geq 0$ and $S_n = \sum_{k=1}^{n} \frac{1}{k}$ such that for $n \in$ □ and

$0 < \sum_{n=1}^{\infty}\left(n - \frac{n\ln n}{2n+\ln n+11/6}\right) a_n < \infty$. Then

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{S_n}} < e^{1+\gamma}\sum_{n=1}^{\infty}\left(n-\frac{n\ln n}{2n+\ln n+11/6}\right)a_n,$$
(2.2)

where $\gamma = 0.57721566...$ is the Euler-Mascheroni's constant and $e^{1+\gamma}$ is the best possible.

Remark 2.4: Let $T_n = e^{1+\gamma}\left(n-\frac{n\ln n}{2n+\ln n+11/6}\right)$ and $V_n = e^{1+\gamma}\left(n-\frac{n\ln n}{3n-1/4}\right)$.

For $n \geq 4$: Numerical computations of T_n and V_n give the following table of values

n	T_n	V_n
1	4.8415	4.8415
2	8.6545	8.5157
3	12.7379	12.7008
4	16.9730	17.0810
5	21.3091	21.5659
6	25.7177	26.1164
7	30.1810	30.7120
8	34.6870	35.3405
9	39.2273	39.9941
10	43.7958	44.6674

Clearly, inequality (2. 2) is a refinement of (1.7) since $T_n < V_n$.
In order to prove our main results, we consider the following lemmas.

Lemma 2.5 [3]. For $n \in \mathbb{N}$, $\frac{1}{2n+\frac{1}{1-\gamma}-2} < S_n - \ln n - \gamma < \frac{1}{2n+\frac{1}{3}}$,

where the constant $\frac{1}{1-\gamma}-2$ and $\frac{1}{3}$ are the best possible.

Lemma 2.6 [3]. *If* $x > 0$, *then* $\left(1+\frac{1}{x}\right)^x < e\left(1-\frac{1}{2x+\frac{11}{6}}\right)$

Lemma 2.7. *For* $n \in N$, *let* $A_n = \left[\frac{(n+1)S_n+1}{nS_n}\right]^{nS_n}$, *then*

$$A_n < e^{1+\gamma}\left(\frac{(6n+1)(12n+11)}{(6n+1)(12n+11)+6(6n+1)\gamma-9}\right)\left(n-\frac{n\ln n}{2n+\ln n+11/6}\right).$$

Proof of Lemma 2.7:

We have that $A_n = \left[\frac{(n+1)S_n+1}{nS_n}\right]^{nS_n} = \left(\left[\frac{(n+1)S_n+1}{nS_n}\right]^{\frac{nS_n}{S_n+1}}\right)^{S_n+1}$.

(2.3)

Suppose we set $B_n = \left[\frac{(n+1)S_n+1}{nS_n}\right]^{\frac{nS_n}{S_n+1}} = \left[\frac{n+1}{n}+\frac{1}{nS_n}\right]^{\frac{nS_n}{S_n+1}} = \left[1+\frac{S_n+1}{nS_n}\right]^{\frac{nS_n}{S_n+1}}$

Then applying Lemma 2.6 in (1.10), we get

$$B_n < e\left(1-\frac{S_n+1}{\left(2n+\frac{11}{6}\right)S_n+\frac{11}{6}}\right) < e\left(1-\frac{S_n}{\left(2n+\frac{11}{6}\right)S_n}\right) = e\left(1-\frac{1}{2n+\frac{11}{6}}\right).$$

using Lemma 2.5, in view of (2.3), we obtain

$$A_n < \left[e\left(1-\frac{1}{2n+\frac{11}{6}}\right)\right]^{S_n+1} = e^{S_n+1}\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{S_n+1}. \qquad (2.4)$$

and $S_n - \ln n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad \Rightarrow \quad S_n < \frac{1}{2n+\frac{1}{3}} + \ln n + \gamma.$

Hence, Equation (2.4) yields

$$A_n < e^{\frac{1}{2n+\frac{1}{3}}+\ln n+\gamma+1}\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{\frac{1}{2n+\frac{1}{3}}+\ln n+\gamma+1}. \qquad (2.5)$$

To proceed from here, we consider the following inequalities:

$$\text{If } m>1, \text{ then } \left(1-\frac{1}{m}\right)^{-m} > e. \qquad (2.6)$$

$$\text{If } m>-1, \text{ then } e^{-m} < \frac{1}{1+m}. \qquad (2.7)$$

Observe from (2.6) that $\left(1-\frac{1}{m}\right)^{m} < e^{-1}.$

$$\text{Thus,} \quad \left(1-\frac{1}{2n+\frac{11}{6}}\right)^{\ln n} = \left\{\left(1-\frac{1}{\left(2n+\frac{11}{6}\right)}\right)^{\left(2n+\frac{11}{6}\right)}\right\}^{\frac{\ln n}{2n+\frac{11}{6}}} < e^{-\frac{\ln n}{\left(2n+\frac{11}{6}\right)}} \qquad (2.8)$$

Using (2.7) in (2.8), we have

$$\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{\ln n} < e^{-\frac{\ln n}{2n+\frac{11}{6}}} < \frac{1}{1+\frac{\ln n}{2n+11/6}} = 1-\frac{\ln n}{2n+\ln n+\frac{11}{6}}$$

$$\text{Therefore,} \left(1-\frac{1}{2n+\frac{11}{6}}\right)^{\ln n} e^{\ln n} < \left(n-\frac{n\ln n}{2n+\ln n+\frac{11}{6}}\right). \qquad (2.9)$$

Similarly,

$$\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{1+\gamma+\frac{1}{2n+1/3}} e^{\frac{1}{2n+1/3}} < e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}}. \qquad (2.10)$$

And, from (2.7), Setting $m = \frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}$, inequality (2.10) becomes

$$\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{1+\gamma+\frac{1}{2n+1/3}} e^{\frac{1}{2n+1/3}} < \left(\frac{(6n+1)(12n+11)}{(6n+1)(12n+11)+6(6n+1)\gamma-9}\right) \qquad (2.11)$$

Combining inequalities (2.5), (2.9) and (2.11), we obtain

$$A_n < e^{\frac{1}{2n+\frac{1}{3}}+\ln n+\gamma+1}\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{\frac{1}{2n+\frac{1}{3}}+\ln n+\gamma+1}$$

$$= e^{1+\gamma}\left\{\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{\frac{1}{2n+\frac{1}{3}}+\gamma+1} e^{\frac{1}{2n+\frac{1}{3}}}\right\}\left\{\left(1-\frac{1}{2n+\frac{11}{6}}\right)^{\ln n} e^{\ln n}\right\}.$$

Thus, $A_n < e^{1+\gamma}\left(\frac{(6n+1)(12n+11)}{(6n+1)(12n+11)+6(6n+1)\gamma-9}\right)\left(n-\frac{n\ln n}{2n+\ln n+\frac{11}{6}}\right).$ $\qquad (2.12)$

Hence Lemma 2.3 is proved.
Inequality (2.12) can further be written as:

$$A_n < e^{1+\gamma}\left(\frac{(6n+1)(12n+11)}{(6n+1)(12n+11)}\right)\left(n-\frac{n\ln n}{2n+\ln n+\frac{11}{6}}\right) = e^{1+\gamma}\left(n-\frac{n\ln n}{2n+\ln n+\frac{11}{6}}\right).$$

I.

III. Proof Of Theorems 2.1 And 2.3

II.

We now give proofs of our main results.

Proof of Theorem 2.1:

Let $c_k = \left[\frac{(k+1)S_k+1}{kS_{k-1}}\right]^{kS_k}$, where S_o is assumed zero i.e. $S_o = 0$. Then

$$\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}} = \frac{1}{(n+1)S_{n+1}}.$$

$$\therefore \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{S_n}} = \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{S_n}}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{\frac{1}{S_n}}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}}$$

$$= \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}(a_k c_k)^{\frac{1}{k}}\right)^{\frac{1}{S_n}}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}} \qquad (3.1)$$

By using arithmetic mean – geometric mean inequality and interchanging the order of the inequality (3.1), we have

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{S_n}} = \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}(a_k c_k)^{\frac{1}{k}}\right)^{\frac{1}{S_n}}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}} \le \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}}\frac{1}{S_n}\sum_{k=1}^{n}\frac{a_k c_k}{k}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{S_n}} \le \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}}\frac{1}{S_n}\sum_{k=1}^{n}\frac{a_k c_k}{k} = \sum_{k=1}^{\infty}\frac{a_k c_k}{k}\sum_{n=k}^{\infty}\frac{1}{(n+1)S_{n+1}S_n}.$$

Letting $\frac{1}{S_k} = \sum_{n=k}^{\infty}\frac{1}{(n+1)S_{n+1}S_n}.$

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{S_n}} \le \sum_{k=1}^{\infty}\frac{a_k c_k}{kS_k} = \sum_{k=1}^{\infty}\left(\frac{(k+1)S_k+1}{kS_k}\right)^{kS_k} a_k. \qquad (3.2)$$

Applying inequality (2.12) in (3.2), we get

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{s_n}} \le \sum_{k=1}^{\infty}\left(\frac{(k+1)S_k+1}{kS_k}\right)^{kS_k}a_k$$

$$< \sum_{k=1}^{\infty}e^{1+\gamma}\left(\frac{(6k+1)(12k+11)}{(6k+1)(12k+11)+6(6k+1)\gamma-9}\right)\left(k-\frac{k\ln k}{2k+\ln k+\frac{11}{6}}\right)a_k. \quad (3.3)$$

Replacing k by n in the right hand side of (3.3), this becomes

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{s_n}} < \sum_{n=1}^{\infty}e^{1+\gamma}\left(\frac{(6n+1)(12n+11)}{(6n+1)(12n+11)+6(6n+1)\gamma-9}\right)\left(n-\frac{n\ln n}{2n+\ln n+\frac{11}{6}}\right)a_n \quad (3.4)$$

This completes the proof of theorem 2.1.

Proof of Theorem 2.3:

Substituting (2.13) into (3.2), we obtain

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{s_n}} < e^{1+\gamma}\sum_{n=1}^{\infty}\left(n-\frac{n\ln n}{2n+\ln n+\frac{11}{6}}\right)a_n. \quad (3.4)$$

This proves Theorem 2.3.

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