IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 2 Ver. VII (Mar-Apr. 2014), PP 05-13
www.iosrjournals.org

Generalised CR-submanifolds of an (ε,δ)-trans-Sasakian manifold with certain connection

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Abstract:*In this paper, generalised CR -submanifolds of an (ε,δ) -trans-Sasakian manifolds with semi-symmetric non-moetric connection are studied. Moreover, integrability conditions of the distributions on generalised CR -submanifolds of an (ε,δ) -trans-Sasakian manifolds with semi-symmetric non-moetric connection and geometry of leaves with semi-symmetric non-metric connection have been discussed.*
2000 Mathematical Subject Classification :*53C21, 53C25, 53C05.*
Keywords:*Distributions, (ε,δ) -trans-Sasakian manifolds, Generalised CR submanifolds, Semi-symmetric non-metric connection.*

I. Introduction

Ion Mihai [1] introduced a new class of submanifolds called Generalised CR-submanifolds of Kaehler manifolds and also studied generalised CR-submanifolds of Sasakian manifolds [2]. In 1985, Oubina [3] introduced a new class of almost contact Riemannian manifolds knows as trans-Sasakian manifolds. After M. H. Shahid studied CR-submanifolds of trans-Sasakian manifold [4] and generic submanifolds of trans-Sasakian manifold [5]. In 2001, A. Kumar and U.C. De [6] studied generalised CR- submanifolds of a trans-Saakian manifolds. In 1993, A. Bejancu and K. L. Duggal [7] introduced the concept of (ε)-Sasakian manifolds. Then U. C. De and A. Sarkar [8] introduced (ε)-Kenmotsu manifolds. The existence of a new structure on indefinite metrices has been discussed. Moreover, Bhattacharyya [9] studied the contact CR-submanifolds of indefinite trans-Sasakian manifolds. Recently, Nagaraja et. al. [10] introduced the concept of (ε,δ) -trans-Sasakian manifolds which generalised the notion of (ε)-Sasakian as well as (δ)-Kenmotsu manifolds. In 2010, Cihan Özgür [11] studied the submanifolds of Riemannian manifold with semi-symmetric non-metric connection. Moreover, Özgür and others also studied the different structures with semi-symmetric non-metric connection in ([12], [13]) . On other hand, some properties of semi-invariant submanifolds, hypersurfaces and submanifolds with semi-symmetric non-metric connection were studied in ([14], [15], [16]). Thus motivated sufficiently from the above studies, in this paper we study generalised CR- submanifolds of an (ε,δ)-trans-Sasakian manifolds with semi-symmetric non-moetric connection.

We know that a connection ∇ with a Riemannian metric g on a manifold M is called metric such that $\nabla g = 0$, otherwise it is non-metric. Further it is said to be a semi-symmetric linear connection [17]. A linear connection ∇ is said to be a semi-symmetric connection it its torsion tensor is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. A study of semi-symmetric connection on Riemannian manifold was enriched by K. Yano [18]. In 1992, Agashe and Chaffle [19] introduced the notion of semi-symmetric non-metric connection. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

Semi-Symmetric metric connection plays an important role in the study of Riemannaian manifolds, there are various physical problems involving the semi-symmetric metric connection. For example if a man is moving on the surface of the earth always facing one definite point, say Mekka or Jaruselam or the North pole, then this displacement is semi-symmetric and metric [20].

In this paper, we study Generalised CR-submanifolds of an (ε,δ) trans-Sasakian manifolds with a semi-symmetric non-metric connection. This paper is organized as follows. In section 2, we give a brief introduction of generalised CR-submanifolds of an (ε,δ) trans-Sasakian manifold and give an example. In section 3, we discuss some Basic Lemmas . In section 4, integrability of some distributions discuss. In section 5, Geometry of leaves of Generalised CR -submanifolds of an (ε,δ) -trans-Sasakian manifold with semi-symmetric non-metric connection have been discussed .

II. (ε,δ)-trans-Sasakian manifolds

Let $\overline{M}$ be an almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatiable Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta o\phi = 0,\ \eta(\xi) = 1, \qquad (2.1)$$

$$g(\phi X,\phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \qquad (2.2)$$

$$g(\xi,\xi) = \varepsilon \qquad (2.3)$$

$$g(X,\phi Y) = -g(\phi X,Y),\ \varepsilon g(X,\xi) = \eta(X) \qquad (2.4)$$

for all vector fields X , Y on $T\overline{M}$, where $\varepsilon = g(\xi,\xi) = \pm 1$. An (ε)- almost contact metric manifold is called an (ε,δ) -trans Sasakian manifold [10] if

$$(\overline{\overline{\nabla}}_X \phi)(Y) = \alpha(g(X,Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X,Y)\xi - \delta\eta(Y)\phi X) \qquad (2.5)$$

for some smooth functions α and β on $\overline{M}$ and $\varepsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0$, $\alpha = 1$, an (ε,δ) -trans-Sasakian manifolds reduces to (ε)-Sasakian and for $\alpha = 0$, $\beta = 1$ it reduces to a (δ)-Kenmotsu manifolds. From (2.5) it follows that

$$\overline{\overline{\nabla}}_X \xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X. \qquad (2.6)$$

for any vector field X tangent to $\overline{M}$.

Example of (ε,δ)-trans Sasakian manifolds

Consider the three dimensional manifold $M = \{(x,y,z) \in R^3 \mid z \neq 0\}$, where (x,y,z) are the cartesian coordinates in R^3 and let the vector fields are

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x}, \quad e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y}, \quad e_3 = \frac{-(\varepsilon+\delta)}{2}\frac{\partial}{\partial z},$$

where e_1,e_2,e_3 are linearly independent at each point of M . Let g be the Riemannain metric defined by

$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = \varepsilon$, $g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$,

where $\varepsilon = \pm 1$.

Let η be the 1-form defined by $\eta(X) = \varepsilon g(X,\xi)$ for any vector field X on M , let ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2,\ \phi(e_2) = -e_1,\ \ \phi(e_3) = 0$.

Then by using the linearty of ϕ and g , we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$.

Further $g(\phi X,\phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y)$ for any vector fields X and Y on M . Hence for $e_3 = \xi$, the structure defines an (ε)-almost contact structure in R^3.

Let ∇ be the Levi-Civita connection with respect to the metric g , then we have

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y])$$

which is know as Koszul's formula.

We, also have

$$\overline{\overline{\nabla}}_{e1} e_3 = -\frac{(\varepsilon+\delta)}{z} e_1, \quad \overline{\overline{\nabla}}_{e2} e_3 = -\frac{(\varepsilon+\delta)}{z} e_2, \quad \overline{\overline{\nabla}}_{e1} e_2 = 0,$$

using the above relation, for any vector X on M, we have

$\overline{\overline{\nabla}}_X \xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ,ξ,η,g) structure defines the (ε,δ)-trans-Sasakian structure in R^3.

III. Semi-symmetric non-metric connection

We remark that owing to the existence of the 1-form η, we can define a semi-symmetric non-metric connection $\overline{\nabla}$ in almost contact metric manifold by

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \eta(Y)X, \qquad (3.1)$$

where $\overline{\overline{\nabla}}$ is the Riemannian connection with respect to g on n-dimensional Riemannian manifold and η is a 1-form associated with the vector field ξ on M defined by

$$\eta(X) = g(X,\xi). \qquad (3.2)$$

[19] BY (3.1) the torsion tensor T of the connection $\overline{\nabla}$ is given by

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]. \qquad (3.3)$$

Also, we have

$$T(X,Y) = \eta(Y)X - \eta(X)Y. \qquad (3.4)$$

A linear connection $\overline{\nabla}$, satisfying (3.4) is called a semi-symmetric connection. $\overline{\nabla}$ is called a metric connection if

$$\overline{\nabla} g = 0$$

otherwise, if $\overline{\nabla} g \neq 0$, then $\overline{\nabla}$ is said to be non-metric connection. Furthermore, from (3.1), it is easy to see that

$$\overline{\nabla}_X g(Y,Z) = (\overline{\nabla}_X g)(Y,Z) + g(\overline{\nabla}_X Y,Z) + g(Y,\overline{\nabla}_X Z)$$

$$= (\overline{\nabla}_X g)(Y,Z) + \overline{\nabla}_X g(Y,Z) + \eta(Y)g(X,Z) + \eta(Z)g(X,Y)$$

which implies

$$(\overline{\nabla}_X g)(Y,Z) = \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \qquad (3.5)$$

for all vector fields X, Y, Z on M. Therefore in view of (3.4) and (3.5) $\overline{\nabla}$ is a semi-symmetric non-metric connection.

for all $X,Y \in TM$. Now from (3.1), (2.5) and (2.6), we have

$$(\overline{\nabla}_X \phi)Y = \alpha\{(g(X,Y)\xi - \varepsilon\eta(Y)X\} + \beta(g(\phi X,Y))\xi \qquad (3.6)$$

$$+ (1-\beta\delta)\eta(Y)\phi X.$$

From (3.6) it follows that

$$\overline{\nabla}_X \xi = X - \varepsilon\alpha\phi X - \beta\delta\phi^2 X \qquad (3.7)$$

for any vector field X tangent to $\overline{M}$.

Now, let M be a submanifold isomertically immeresed in an (ε,δ)-trans-Sasakian manifold $\overline{M}$ such that the structure vector field ξ of $\overline{M}$ is tangent to submanifolds M. We denote by { is the 1- dimensional distribution spanned by ξ on M and by $\{\xi\}^{\perp}$ the complementary orthogonal distribution to ξ in TM.

For any $X \in$ TM , we have $g(\phi X,\xi) = 0$. Then we have

$$\phi X = BX + CX, \qquad (3.8)$$

where $BX \in \{\xi\}^{\perp}$ and $CX \in T^{\perp}M$. Thus $X \to BX$ is an endomorphism pf the tangent bundle TM and $X \to CX$ is a normal bundel valued 1-form on M.

Definition. A submanifold of M of an almost contact metric manifolds $\overline{M}$ with an (ε,δ)-trans-Sasakian metric sructure (ϕ,ξ,η,g) is said to be a generalised CR-submanifold if

$$D_x^{\perp} = T_xM \cap \phi T_x^{\perp}M; \qquad for\ x \in M$$

defines a differentiable sub-bundle of T_xM. Thus for $X \in D^{\perp}$ one has $BX = 0$.

We denote by D the complementary orthogonal sub-bundle to $D^{\perp} \oplus \{\xi\}$ in TM.

For any $X \in D$, $BX \neq 0$. Also we have $BD = D.$

Thus for a generalised CR-submanifold M, we have the orthogonal decomposition

$$TM = D \oplus D^{\perp} \oplus \{\xi\}. \qquad (3.9)$$

IV. Basic Lemmas

Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$. We denote by g both Riemannian metrices on $\overline{M}$ and M.

For each $X \in TM$, we can write

$$X = PX + QX + \eta(X)\xi, \qquad (4.1)$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.

For any $N \in T_x^{\perp}M$, we can write

$$\phi X = tN + fN, \qquad (4.2)$$

where tN is the tangential part of ϕN and fN is the normal part of ϕN.

By using (2.2) we have

$$g(\phi X, CY) = g(\phi X, BY + CY) = g(\phi X, \phi Y) = g(X,Y) = 0,$$

for $X \in D_x^{\perp}$ and $Y \in D_x$. Therefore

$$g(\phi D_x^{\perp}, CD_x) = 0. \qquad (4.3)$$

We denote by ν the orthogonal complementray vector bundle to $\phi D^{\perp} \oplus CD$ in $T^{\perp}M$.

Thus, we have

$$T^{\perp}M = \phi D^{\perp} \oplus CD \oplus \nu \qquad (4.4)$$

Lemma 4.1. The morphism t and f satisfy

$$t(\phi D^{\perp}) = D^{\perp} \qquad (4.5)$$

$$t(CD) \subset D \qquad (4.6)$$

Proof. For $X \in D^{\perp}$ and $Y \in D$.

$$g(t\phi, Y) = g(t\phi X + f\phi Y, Y) = g(\phi^2 X, Y) = -g(X,Y) = 0$$

$$g(t\phi X, \xi) = g(\phi^2 X, \xi) = -g(\phi X, \phi\xi) = 0.$$

Therefore, $t(\phi D^{\perp}) \subset D^{\perp)}$.

For $X \in D^{\perp}$, we have

$$-X = \phi^2 X = t\phi X + f\phi X, \text{ which implies} -X = t\phi X.$$

Consequently, $D^{\perp} \subset t(\phi D^{\perp})$. Hence the equation (4.5) follows. The equation (4.6) is trivial.

Let M be a submanifold of a Riemannian manifold $\overline{M}$ with Riemannian metric g. Then Gauss and Weingarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad (X,Y \in TM), \qquad (4.7)$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N + \eta(N)X \qquad (N \in T^{\perp}M), \qquad (4.8)$$

where $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ respectively the semi-symmetric non-metric, induced connection and induced normal connections in $\overline{M}$, M and the normal bundle $T^{\perp}M$ of M respectively and h is the second fundamental form related to A by

$$g(h(X,Y),N) = g(A_N X,Y) \qquad (4.9)$$

for $X,Y,\in TM$ and $N \in T^{\perp}M.$

We denote

$$u(X,Y) = \nabla_X BPY - A_{CPY}X - A_{\phi QY}X. \qquad (4.10)$$

Lemma 4.2. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then we have

$$P(u(X,Y)) - BP\nabla_X Y - Pth(X,Y) = -\varepsilon\alpha\eta(Y)PX \qquad (4.11)$$

$$-(1-\beta\delta)\eta(Y)PBX - 2\eta(CPY)PX,$$

$$Q(u(X,Y)) - Qth(X,Y) = -\varepsilon\alpha\eta(Y)QX - (1-\beta\delta)\eta(Y)QBX \qquad (4.12)$$

$$-2\eta(CPY)QX,$$

$$\eta(u(X,Y)) = \alpha g(\phi X,\phi Y) + \beta g(\phi BX,\phi Y) - 2\eta(CPY)\eta(X)\xi, \qquad (4.13)$$

$$h(X,BPY) + \nabla_X^{\perp} CPY + \nabla_X^{\perp}\phi QY - CP\nabla_X Y - \phi Q\nabla_X Y - fh(X,Y) \qquad (4.14)$$

$$= (1-\beta\delta)\eta(Y)CX,$$

for $X,Y \in TM.$

Proof. For $X,Y \in TM$ by using (3.8), (4.1), (4.2), (4.7), (4.8) in (3.6), we have

$$\nabla_X PBY + h(X,BPY) - A_{CPY}X + \nabla_X^{\perp} CPY + \eta(CPY)X - A_{\phi QY}X + \nabla_X^{\perp}\phi QY$$

$$-BP\nabla_X Y - CP\nabla_X Y - \phi Q\nabla_X Y - Pth(X,Y) - Qth(X,Y) - fh(X,Y)$$

$$= \alpha\{(g(X,Y)\xi - \varepsilon\eta(Y)X\} + \beta(g(\phi X,Y))\xi + (1-\beta\delta)\eta(Y)\phi X.$$

Then (4.11), (4.12), (4.13) and (4.14) are obtaining by taking the components of each vector bundles D, $D^{\perp}$, $\{\xi\}$ and $T^{\perp}(M)$ respectively.

Lemma 4.3. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then we have

$$P(t\nabla_X^{\perp} N + A_{fN}X - \nabla_X tN) = BPA_N X - \eta(fN)PX, \qquad (4.15)$$

$$Q(t\nabla_X^{\perp} N + A_{fN}X - \nabla_X tN) = -\eta(fN)QX, \qquad (4.16)$$

$$\eta(A_{fN}X - \nabla_X tN) = -\beta g(CX,N) + \eta(fN)\eta(X)\xi, \qquad (4.17)$$

$$h(X,tN) + \phi QA_N X + \nabla_X^{\perp} fN + CPA_N X = f\nabla_X^{\perp} N \qquad (4.18)$$

for $X \in TM$ and $N \in T^{\perp}M$.

Proof. For $X \in TM$ and $N \in T^{\perp}M$ by using the equations (3.8), (4.1), (4.2), (4.7) and (4.8) in (3.6), we get

$$P\nabla_X tN + Q\nabla_X tN + \eta(\nabla_X tN) + h(X, tN) - PA_{fN}X - \eta(fN)PX - QA_{fN}X$$

$$-\eta(fN)QX - \eta(A_{fN}X) + \nabla_X^{\perp} fN + \eta(fN)\eta(X)\xi + BPA_N X + CPA_N X$$

$$+\phi QA_N X - Pt\nabla_X^{\perp} N - Q\nabla_X^{\perp} N - f\nabla_X^{\perp} N = \beta g(CX, N)$$

Then (4.15), (4.16), (4.17) and (4.18) are obtaining by taking the components of each vector bundles D, $D^{\perp}$, $\{\xi\}$ and $T^{\perp}(M)$ respectively.

Lemma 4.4. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then we have

$$\nabla_X \xi = PX + \beta\delta X - \varepsilon\alpha BX, \text{ for } X \in D \qquad (4.19)$$

$$h(X,\xi) = QX - \varepsilon\alpha CX \text{ and } (1-\beta\delta)\eta(X) = 0, \quad \text{for } X \in D \qquad (4.20)$$

$$\nabla_Y \xi = PY + \beta\delta Y, \qquad \text{for } Y \in D^{\perp} \qquad (4.21)$$

$$h(Y,\xi) = QY - \varepsilon\alpha\phi Y; \quad \eta(Y)(1-\beta\delta) = 0, \text{for } Y \in D^{\perp} \qquad (4.22)$$

$$\nabla_{\xi}\xi = P\xi \qquad (4.23)$$

$$h(\xi,\xi) = Q\xi; \quad \beta\delta = 1. \qquad (4.24)$$

Proof. The proof of above lemma from (3.7) by using (3.8), (4.1) and (4.7).

Lemma 4.5 Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then we have

$$A_{\phi X}Y = A_{\phi Y}X, \qquad \text{for } X, Y \in D^{\perp}. \qquad (4.25)$$

Proof. By using (2.2), (2.3), (4.7) and (4.9), we get

$$g(A_{\phi X}Y, Z) = g(h(Y,Z), \phi X) = g(\overline{\nabla}_X Y, \phi X) = -g(\phi\overline{\nabla}_Z Y, X)$$

$$= -g(\overline{\nabla}_{\phi Z} Y, X) = g(\phi Y, \overline{\nabla}_Z X) = g(h(Z,X), \phi Y) = g(h(X,Z), \phi Y)$$

$$= g(A_{\phi} YX, Z),$$

for $X, Y \in D^{\perp}$ and $Z \in TM$. Hence the Lemma follows.

Lemma 4.6. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then we have

$$\nabla_{\xi} V \in D^{\perp}, \qquad \text{for } V \in D^{\perp} \text{ and} \qquad (4.26)$$

$$\nabla_{\xi} W \in D, \qquad \text{for } W \in D. \qquad (4.27)$$

Proof. Let us take $X = \xi$ and $V = \phi N$ in (4.15), where $N \in \phi D$. Taking account that $tN = \phi N$, $fN = 0$ we get

$$P\nabla_{\xi} V = Pt\nabla_{\xi}^{\perp} N - BPA_N \xi. \qquad (4.28)$$

The first relation of (4.20) gives

$$g(PA_N\xi, W) = g(A_N\xi, W) = g(h(W,\xi,N)) = -\varepsilon\alpha g(CW, N) + g(QW, N) = 0$$

for $W \in D$. Hence, (4.28) becomes

$$P\nabla_{\xi}V = Pt\nabla^{\perp}_{\xi}N. \qquad (4.29)$$

On the other hand (4.18) implies

$$h(\xi,V) = f\nabla^{\perp}_{\xi}N - \phi QA_N\xi. \qquad (4.30)$$

$$\text{For} \quad V\in D^{\perp},\ h(\xi,V) = h(V,\xi) = -\varepsilon\alpha\phi V \in \phi D^{\perp},\ \text{by}\,(3.22)$$

Now for $X\in D^{\perp}$ by using the lemma (4.5) and of (4.9), we have

$$g(h(\xi,V),\phi X) = g(h(V,\xi),\phi X) = g(A_{\phi X}V,\xi) = g(A_{\phi V}X,\xi)$$

$$= g(h(X,\xi),\phi V) = g(h(X,\xi),-N) = -g(A_N\xi,X) = -g(\phi A_N\xi,\phi X)$$

$$= -g(\phi PA_N\xi,\phi X) - g(\phi QA_N\xi,\phi X) = -g(\phi QA_N\xi,\phi X)$$

since $CD^{\perp}\in\phi D^{\perp}$.

Therefore, $h(\xi,V) = -\phi QA_N\xi$, which together with (4.30) implies $f\nabla^{\perp}_{\xi}N = 0$.

Hence $\nabla^{\perp}_{\xi}N\in\phi D^{\perp}$, since f is an automorphism of $CD\oplus\nu$. Thus, $t\nabla^{\perp}_{\xi}N\in D^{\perp}$ and from (4.29) it follows that

$$P\nabla_{\xi}V = 0, \qquad \text{for all}\, V\in D^{\perp} \qquad (4.31)$$

Next from (4.17), we have

$$\eta(\nabla_{\xi}V) = 0 \qquad (4.32)$$

for all $V = \phi D\in D^{\perp}$, where $N\in\phi D^{\perp}$. Hence (4.26) follows from (4.31) and (4.32).
Finally using the (4.1), (4.23) and (4.26), we have

$$g(\nabla_{\xi}W,X) = g(\nabla_{\xi}W,PX)$$

for $X\in TM$ and $W\in D$. Thus we have $\nabla_{\xi}W\in D$, for $W\in D$ and this completes the proof.

Corollary 4.1. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then we have

$$[Y,\xi]\in D^{\perp}, \quad \text{for}\, Y\in D^{\perp} \qquad (4.33)$$

$$[X,\xi]\in D, \qquad \text{for}\ X\in D \qquad (4.34)$$

The above corollary follows immediate consiqueces of the Lemma (4.4) and Lemma (4.6).

V. Integrability of Distributions

Theorem 5.1. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D^{\perp}$ is always involutive if and only if

$$g([X,Y],\xi) - 2\beta\delta g(X,Y) = 0. \qquad (5.1)$$

Proof. For $X,Y\in D^{\perp}$ by using (4.21), we get

$$g([X,Y],\xi) = g(\nabla_X Y,\xi) - g(\nabla_Y X,\xi)$$

$$g([X,Y],\xi) = g(X,\nabla_Y\xi) - g(Y,\nabla_X\xi) = 2\beta\delta g(X,Y). \qquad (5.2)$$

On the other hand, from (4.10), we have

$$BP\nabla_X Y = -PA_{\phi Y}X - Pth(X,Y), \qquad (5.3)$$

for $X,Y\in D^{\perp}$. Then using lemma (4.5), we get from equation (5.3)

$$BP[X,Y] = 0, \text{for}\ X,Y\in D^{\perp}. \qquad (5.4)$$

Theorem 5.2. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution D is never involutive.

Proof. For $X,Y \in D$ by using (4.19), we have

$$g([X,Y],\xi) = g(\nabla_X Y,\xi) - g(\nabla_Y X,\xi)$$

$$= 2\varepsilon\delta g(Y,BX) + 2\beta\delta g(X,Y) + g(X,PY) - g(Y,PX). \qquad (5.5)$$

Taking $X \neq 0$ and $Y = BX$ in (5.5), it follows that D is not involutive.

Next we have the following theorem.

Theorem 5.3. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D \oplus \{\xi\}$ is involutive if and only if

$$h(BX,Y) - h(X,BY) + \nabla_Y^{\perp} CX - \nabla_X^{\perp} CY \in CD \oplus \nu \qquad (5.6)$$

Proof. Applying ϕ to equation (4.14) and taking component in $D^{\perp}$, we have

$$Q\nabla_X Y = -Qt(h(X,BY) + \nabla_X^{\perp} CPY - fh(X,Y))$$

for $X,Y \in D$.

Thus

$$Q[X,Y] = Qt(h(X,BY) - h(X,BY) + \nabla_Y^{\perp} CX - \nabla_X^{\perp} CY \qquad (5.7)$$

for $X,Y \in D$. Hence, the theorem follows from (5.7) and (4.34).

VI. Geometry of Leaves

Theorem 6.1. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the leaves of distribution $D^{\perp}$ are totally geodesic in M if and only if

$$h(X,BZ) + \nabla_X^{\perp} CZ + \eta(CZ)X \in CD \oplus \nu \qquad (6.1)$$

for $X \in D^{\perp}$ and $Z \in D \oplus \{\xi\}$.

Proof. For $X,Y \in D^{\perp}$ and $Z \in D \oplus \{\xi\}$ by using (2.2), (2.3), (4.7) and (4.8), we get

$$g(\overline{\nabla}_X Y,Z) = -g(Y,\overline{\nabla}_X Z) = -g(\overline{\nabla}_X Z,Y) = -g(\phi\overline{\nabla}_X Z,\phi X)$$

$$= g((\overline{\nabla}_X \phi)Z,\phi Y) - g(\overline{\nabla}_X \phi Z,\phi Y) = -g(\overline{\nabla}_X BZ + \overline{\nabla}_X CZ,\phi Y)$$

$$= -g(\nabla_X BZ + h(X,BZ) - A_{CZ}X + \eta(CZ)X + \nabla_X^{\perp} CZ,\phi Y)$$

$$= -g(h(X,BZ) + \nabla_X^{\perp} CZ + \eta(CZ)X,\phi Y). \qquad (6.2)$$

Hence the theorem follows from the (6.2).

Theorem 6.2. Let M be a generalised CR-submanifold of an (ε,δ)-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D^{\perp} \oplus \{\xi\}$ is involutive and its leaves are totally geodesic in M if and only if

$$h(X,BY) + \nabla_X^{\perp} CY + \eta(CY)X \in CD \oplus \nu \qquad (6.3)$$

for $X,Y \in D^{\perp} \oplus \{\xi\}$.

Proof. For $X,Y \in D^{\perp} \oplus \{\xi\}$ and $Z \in D^{\perp}$ by using (2.2),(2.3),(3.8), (4.7) and (4.8), we get

$$g(\overline{\nabla}_X Y,Z) = g(\nabla_X Y,Z) = g(\phi\nabla_X Y\phi Z) = g(\nabla_X \phi Y,\phi Z)$$

$$= g(\nabla_X BY + h(X,BY) - A_{CY}X + \eta(CY)X + \nabla_X^{\perp} CY,\phi Z). \qquad (6.4)$$

Hence, the theorem follows from the equation (6.4).

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