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Doubt fuzzy ideals of BF-algebra

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Abstract : *The aim of this paper is to introduce the notion of Doubt fuzzy ideals of BF -algebra and to investigate some of their basic properties.*
Keywords:*BF -algebra, Subalgebra, Doubt fuzzy BF -Subalgebra, Doubt fuzzy ideals of BF -algebra*

I. Introduction

In 1966, Imai and Iseki[1] introduced two classes of abstract algebras viz. BCK-algebras and BCI-algebras. The class of BCK-algebras is a proper subclass of the class of BCI-algebras. J. Neggers and H. S. Kim [2] introduced the notion of B-algebra which is a generalisation of BCK-algebras. Walendziak [3] introduced the notion of BF-algebras, which is a generalization B-algebras andsubsequently fuzzy BF-subalgebra were introduced by Saeid and Rezvani [4, 5] in 2009. Y. B. Jun [6] introduced the notion of Doubt fuzzy ideals in BCK/BCI-algebras. R. Biswas [7] introduced the concept of anti fuzzy subgroup. Modifying their idea, in this paper we apply the idea of BF-algebras to introduce the notion of Doubt fuzzy ideal of BF-algebras and establish some of their basic properties.

II. Preliminaries

In this section, we recallsome basic concepts which would beused in the sequel.

Definition 2.1. A BF-algebra is a non-empty set X with a constant 0 and a binary operation * satisfying the following axioms:

(i) $x * x = 0$

(ii) $x * 0 = x$

(iii) $0 * (x * y) = y * x$ for all $x, y \in X$

For brevity we also call X a BF-algebra.A binary relation '≤' on X can be defined by $x \leq y$ if and only if $x*y=0$.

Example 2.2. Let R be the set of real numbers and X = (R, *, 0) be the algebra with the operation * defined by

$$x*y=\begin{cases} x & if \quad y=0 \\ y & if \quad x=0 \\ 0 & otherwise \end{cases}$$

then X is a BF-algebra.

Definition 2.3. A non-empty subset S of a BF-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 2.4. A nonempty subset I of a BF-algebra X is said to be an ideal of X if

(i) $0 \in I$

(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$

Definition 2.5. A fuzzy subset μ of X is called a fuzzy subalgebra of a BF-algebra X if

$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.6. A fuzzy set μ of a BF-algebra X is called a fuzzy ideal of X if it satisfies the following conditions.

(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

Definition 2.7. A fuzzy set μ of a BF-algebra X is called a doubt fuzzy subalgebra of X if

$\mu(x * y) \leq \max\{\mu(x), \mu(y)\} \forall x, y \in X.$

III. Doubt fuzzy ideal

Definition 3.1. A fuzzy set μ of BF-algebra X is called a doubt fuzzy (DF) ideal of X if

(i) $\mu(0) \leq \mu(x)$
(ii) $\mu(x) \leq \max\{\mu(x * y), \mu(y)\}$

Example 3.2. Let X = {0, 1, 2} with the following Cayley table.

*	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

Then (X, *, 0) is a BF-algebra.Define a fuzzy set $\mu : X \rightarrow [0\ 1]$ by $\mu(0) = 0.1$, $\mu(1) = \mu(2) = 0.4$ then μ is a doubt fuzzy ideal of X.

Example 3.3. In above algebra if we take $\mu(0) = 0.2$, $\mu(1) = 0.3$, $\mu(2) = 0.6$ then μ is a doubt fuzzy subalgebra of X.

Definition 3.4. Let μ be a fuzzy set of a BF-algebra Xfor $t \in [0\ 1]$, then the sets

$\mu_t = \{ x \in X \mid \mu(x) \geq t\}$,

$\mu^t = \{ x \in X \mid \mu(x) \leq t\}$

could be empty sets.The set $\mu_t = \varphi$ (respt. $\mu^t \neq \varphi$)is called the t (respt t-doubt) confidenceset of μ.

Theorem 3.5. μ is a fuzzy subalgebra of BF-algebra X iff $\mu_t$isempty or subalgebra of X for all $t \in [0\ 1]$.

Proof.Suppose μ is fuzzy subalgebra of X

Therefore $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ (1)

To prove μ_t is a subalgebra of X

Let $x,y \in \mu_t \Rightarrow \mu(x), \mu(y) \geq t$

Now (1) $\Rightarrow \mu(x*y) \geq \min\{t,t\}=t$

$\Rightarrow x*y \in \mu_t$

Conversely

Let μ_t is a subalgebra of X.

To prove μ is a fuzzy subalgebra of X. Let $x, y \in X$ such that $\mu(x) = t$ and $\mu(y) = s$ where $t \leq s$

Then $x, y \in \mu_t$and so $(x*y) \in \mu_t$ [μ_t is a subalgebra of X]

$\Rightarrow \mu(x*y) \geq t = \min\{\mu(x), \mu(y)\}$

Hence μ is a fuzzy subalgebra of X.

Theorem 3.6. μ is a fuzzy ideal of BF-algebra X iff $\mu_t$is idealof X$t \in [0\ 1]$.

Proof.Assume μ is a fuzzy ideal of X. Here $\mu_t = \{ x \in X \mid \mu(x) \geq t\}$

Clearly $0 \in \mu_t$ since $\mu(0) \geq t$

Let $x*y, y \in \mu_t$i.e$\mu(x*y) \geq t$, $\mu(y) \geq t$

$\mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq \min\{t, t\}$

$\Rightarrow x \in \mu_t$

Therefore $x*y, y \in \mu_t \Rightarrow x \in \mu_t$

$\Rightarrow \mu_t$is an ideal of BF-algebra X.

Conversely

Let μ_t is an ideal, to prove μ is fuzzy ideal. Let $x, y \in X$ such that $\mu(x*y) = t$ and $\mu(y) = s$ where $t \leq s$

Then $x*y, y \in \mu_t$ and hence$x \in \mu_t$,[since μ_t is ideal]

whichimplies$\mu(x) \geq t = \min\{t, s\} = \min\{\mu(x*y), \mu(y)\}$. Therefore μ is a fuzzy ideal of X

Proposition 3.7.Let μ be a Doubt fuzzy (DF)ideal of a BF-algebra X. Then the following hold.

(a) If $x \leq y$ then $\mu(x) \leq \mu(y)$, i.e.μ preserves order.

(b) If $\mu(x*y) = 0$ then $\mu(x) \leq \mu(y)$

(c) If $x*y \leq z$ then $\mu(x) \leq \max\{\mu(y), \mu(z)\}$,for all x, y, $z \in X$

Proof.

(a) Let $x \leq y$,then $x * y = 0$

Now,$\mu(x) \leq \max\{\mu(x*y), \mu(y)\}$[Since$\mu$ is DF ideal]

$= \max\{\mu(0), \mu(y)\} = \mu(y)$ [since $\mu(0) \leq \mu(y)$ for DF ideal]

i.e.$\mu(x) \leq \mu(y)$ i.e.μ preserves order.

(b)If $\mu(x*y) = 0$,then we have

$\mu(x) \leq \max\{\mu(x * y), \mu(y)\}$[Since μ is DF ideal]

$= \max\{\mu(0), \mu(y)\} = \mu(y)$, [Since $\mu(0) \leq \mu(y)$ for DF ideal]

i.e.$\mu(x) \leq \mu(y)$.

(c) Here,

$x * y \leq z$, therefore $(x * y) * z = 0$

Now $\mu(x) \leq \max\{\mu(x*y), \mu(y)\}$(2)

In particular,

$\mu(x * y) \leq \max\{\mu((x * y) * z), \mu(z)\}$

$= \max\{\mu(0), \mu(z)\} = \mu(z)$ [Since $\mu(0) \leq \mu(z)$ for DF ideal]

$\therefore$ $\mu(x * y) \leq \mu(z)$

$\therefore$ $\max\{\mu(x * y), \mu(y)\} \leq \max\{\mu(z), \mu(y)\}$ (3)

(2) and (3) $\Rightarrow \mu(x) \leq \max\{\mu(y), \mu(z)\}$.

Theorem 3.8.If μ is a Doubt fuzzy (DF)ideal of a BF-algebraX. Then the set$X_\mu=\{x \in X | \mu(x) = \mu(0)\}$ is an ideal of X.

Proof. Clearly,$0 \in X_\mu$
Let $x*y$, $y \in X_\mu$
$\Rightarrow \mu(x * y) = \mu(y) = \mu(0)$

$\mu(x) \leq \max\{\mu(x * y), \mu(y)\}$[Since μ is DF ideal]

$= \max\{\mu(0), \mu(0)\} = \mu(0)$

$\therefore$ $\mu(x) \leq \mu(0)$ also $\mu(0) \leq \mu(x)$[Since μ is DF ideal]

$\therefore$ $\mu(x) = \mu(0)$

$\therefore \quad x \in X_\mu$
$\therefore \quad x * y, y \in X_\mu \Rightarrow x \in X_\mu$
$\Longrightarrow X_\mu$ is an ideal.

Theorem 3.9.A fuzzy subset μ of BF-algebra X is a fuzzy idealof Xiff its complement μ^c is DF ideal of X.

Proof.Let μ be a fuzzy ideal of X,To prove μ^c is DF ideal. Let x, y∈ X.

(i) $\mu^c(0) = 1-\mu(0) \leq 1-\mu(x) = \mu^c(x)$
i.e.,$\mu^c(0) \leq \mu^c(x)$[since $\mu(0) \geq \mu(x) \forall x \in X$]

(ii) $\mu^c(x) = 1-\mu(x) \leq 1-\min\{\mu(x*y), \mu(y)\}$
[since$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$]
$= 1 - \min\{1 - \mu^c(x * y), 1 - \mu^c(y)\}$

$= \max\{\mu^c(x * y), \mu^c(y)\}$

$\Rightarrow \mu^c$ is DF ideal.

Conversely,
Let μ^c is DF ideal of X. To prove μ is fuzzy ideal of X

(i) $\mu^c(0) \leq \mu^c(x)$

(ii) $\mu^c(x) \leq \max\{\mu^c(x*y), \mu^c(y)\}$

Now (i) $\Rightarrow 1-\mu(0) \leq 1-\mu(x)$

$\Rightarrow \mu(0) \geq \mu(x)$

(ii)$\Rightarrow 1-\mu(x) \leq \max\{1-\mu(x*y), 1-\mu(y)\}$

$\Rightarrow \quad 1 - \mu(x) \leq 1 - \min\{\mu(x * y), \mu(y)\}$

$\Rightarrow \quad -\mu(x) \leq -\min\{\mu(x * y), \mu(y)\}$

$\Rightarrow \quad \mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

$\Rightarrow$ μ is fuzzy ideal.

Theorem 3.10.Let μ be a fuzzy subset of a BF-algebra X.If μ is a DF ideal of X,then the lower level cut μ^t is an ideal of X for all t∈ [0 1], t >μ(0).
Proof.
Let μ be a DF ideal of X. Therefore we have

(i) $\mu(0) \leq \mu(x)$

(ii)$\mu(x) \leq \max\{\mu(x*y), \mu(y)\}$

To prove μ^t is an ideal of X.We know that $\mu^t = \{x \in X | \mu(x) \leq t\}$
Let x, y∈ μ^t
since$\mu(0) \leq \mu(x) \leq t \Rightarrow 0 \in \mu^t \forall t \in [0\ 1]$
Again let x*y, y∈ μ^t
$\therefore \mu(x * y) \leq t, \mu(y) \leq t$
Now,$\mu(x) \leq \max\{\mu(x*y), \mu(y)\}$
$\leq \max\{t, t\} = t$
hence,$\mu(x) \leq t \Rightarrow x \in \mu^t$
$x * y, y \in \mu^t \Rightarrow x \in \mu^t$
$\therefore \mu^t$ is an ideal.

Theorem 3.11.Let μ be a DF ideal of BF-algebra X.Then twolower level cuts μ^{t_1}, $\mu^{t_2}$where ($t_1 < t_2$) of μ are equaliff there is no x∈ X such that $t_1 < \mu(x) < t_2$.
Proof.Recall that $\mu^t=\{x\in X|\mu(x)\leq t\}$
Let $\mu^{t_1}=\mu^{t_2}$ where ($t_1 < t_2$) and there exists x∈ X such that $t_1 < \mu(x) < t_2$.
then$\mu^{t_1}\subset\mu^{t_2}$,then $x\in\mu^{t_2}$ but$x\notin\mu^{t_1}$which contradicts the fact that $\mu^{t_1}=\mu^{t_2}$. Hence there is no x∈ X such that $t_1 < \mu(x) < t_2$.
Conversely, suppose that there is no x∈ X such that $t_1 < \mu(x) < t_2$. Therefore $\mu^{t_1}\subset\mu^{t_2}$(since $t_1 < t_2$).Again if $x \in\mu^{t_2}$then $\mu(x) \leq t_2$ andby hypothesis we get $\mu(x)\leq t_1 \Rightarrow \mu^{t_2}\subset\mu^{t_1}$.Hence $\mu^{t_1}=\mu^{t_2}$.

Theorem 3.12.Let $\mu_1$and $\mu_2$be two DF ideal of BF-algebraX. Then $\mu_1\cup\mu_2$ is also a DF ideal of X.
Proof.Let x, y∈ X. Now
$(\mu_1\cup \mu_2)(0) = \max\{\mu_1(0), \mu_2(0)\}\leq\max\{\mu_1(x), \mu_2(x)\} =(\mu_1\cup \mu_2)(x)$
$\therefore\ (\mu_1\cup \mu_2)(0) \leq (\mu_1\cup \mu_2)(x)$

Again$(\mu_1\cup\mu_2)(x) =\max\{\mu_1(x), \mu_2(x)\}$
$\leq\max\{\max\{\mu_1(x * y), \mu_1(y)\}, \max\{\mu_2(x * y), \mu_2(y)\}\}$
$= \max\{\max\{\mu_1(x * y), \mu_2(x * y)\}, \max\{\mu_1(y), \mu_2(y)\}\}$
$= \max\{(\mu_1\cup \mu_2)(x * y), (\mu_1\cup \mu_2)(y)\}$

Therefore $\mu_1\cup\mu_2$ is a DF ideal of X.
The above theorem can be generalised as
Theorem 3.13.Let$\{\mu_i|i = 1, 2, 3, ...\}$be a family of DF idealof BF-algebra X,then $\cup^n_{i=1}\mu_i$ is also a DF ideal of X. where $\cup\mu_i=\max\{\mu_i(x) : i= 1, 2, . . .\}$.

(iii) Product of DF ideals of BF-algebra

Definition 4.1. Let μ_1 and μ_2 be two DF idealsof a BF-algebra X. Then their cartesian product is defined by $(\mu_1\times \mu_2)(x, y) = \max\{\mu_1(x), \mu_2(y)\}$ where $(\mu_1\times \mu_2)$: $X \times X \to [0\ 1]\forall x, y \in X$.
Theorem 4.2 Let X be a BF-algebra,then the cartesian product$X\times X$.= {(x,y) | x,y∈ X } is also a BF-algebra under the binary operation * defined in X×X by (x,y)*(p,q)=(x*p,y*q) for all (x,y),(p,q) ∈ X×X .
Proof. Clearly (0,0) ∈ X×X
(i) (x,y)*(x,y)=(x*x,y*y)=(0,0)
(ii) (x,y)*(0,0)=(x*0,y*0)=(x,y)
(iii) (0,0)*{(x,y)*(p,q)}=(0,0)*(x*p,y*q)
={0*(x*p),0*(y*q)}
=(p*x,q*y)=(p,q)*(x,y)
Which shows that (X×X,(0,0),*) is a BF-algebra.
Theorem 4.3.Let $\mu_1$and $\mu_2$be two DF ideal of BF-algebraX. Then $\mu_1\times\mu_2$ is also a DF ideal ofX×X.
Proof.For any (x, y)∈ X×X,
we have $(\mu_1\times\mu_2)(0,0) =\max\{\mu_1(0), \mu_2(0)\}$
$\leq \max\{\mu_1(x), \mu_2(y)\}$
$=(\mu_1\times\mu_2) (x, y)$
Therefore $(\mu_1\times\mu_2)(0,0) \leq (\mu_1\times\mu_2) (x, y)$ (4)
Againlet $(x_1, x_2), (y_1, y_2)\in X\times X$
then$(\mu_1\times\mu_2)(x_1, x_2) =\max\{\mu_1(x_1), \mu_2(x_2)\}$
$\leq\max\{\max\{\mu_1(x_1* y_1), \mu_1(y_1)\}, \max\{\mu_2(x_2* y_2), \mu_2(y_2)\}\}$
$= \max\{\max\{\mu_1(x_1* y_1), \mu_2(x_2* y_2)\}, \max\{\mu_1(y_1), \mu_2(y_2)\}\}$
$= \max\{(\mu_1\times \mu_2)(x_1* y_1, x_2* y_2), (\mu_1\times \mu_2)(y_1, y_2)\}$
$= \max\{(\mu_1\times \mu_2)((x_1, x_2)*(y_1,y_2)), (\mu_1\times \mu_2)(y_1, y_2)\}$ (5)
(4) and(5) shows that $\mu_1\times\mu_2$ is also a DF ideal of X×X.

(iv) Investigation of DF ideals under homomorphism

In this section homomorphism of BF-algebra is defined and some results are studied.
Definition 5.1. Let X and $X^/$be two BF-algebras. A mapping $f : X\to X^/$ is said to be homomorphismif f (x *y) = f (x) * f (y) for all x, y ∈ X.
Theorem 5.2 Let X and $X^/$ be two BF-algebras and $f : X \to X^/$ be a homomorphism Then $f(0)= 0^/$
Proof. Let x ∈ X therefore $f(x) \in X^/$
Now $f(0)= f (x * x) = f (x) * f (x) = 0*0=0^/$
Theorem 5.3.Letf :$X\to X^/$be an epimorphism of BF-algebras if ν be a DF ideal of $X^/$,then the pre image of

νunder f is also a DF ideal of X.

Proof. Recall that $f^{-1}(\nu)$ is defined as $f^{-1}(\nu)(x) = \nu(f(x))$. Let μ be the pre image of ν under f then $\nu(f(x)) = \mu(x) \forall x \in X$. Since ν is DF ideal therefore $\nu(0^{/}) \leq \nu(f(x)) = \mu(x)$.On the other hand$\nu(0^{/}) = \nu(f(0)) = \mu(0)$.
$\Rightarrow \mu(0) \leq \mu(x) \forall x \in X$
Again
$\mu(x) = \nu(f(x)) \leq \max\{\nu(f(x) * y^{/}), \nu(y^{/})\}$for any $y^{/} \in X^{/}$
Let $y \in X$ such that $f(y) = y^{/}$,then $\mu(x) \leq \max\{\nu(f(x) * y^{/}), \nu(y^{/})\}$
$= \max\{\nu((f(x) * f(y)), \nu(f(y))\}$
$= \max\{\nu(f(x * y), \nu(f(y))\}$
$= \max\{\mu(x * y), \mu(y)\}$
$\therefore \mu(x) \leq \max\{\mu(x * y), \mu(y)\}$which is true forallx, $y \in X$.Hence μ is a Doubt fuzzy ideal of X.

Theorem 5.4.Let $f : X \rightarrow X^{/}$be an epimorphism where X and $X^{/}$ are two BF-algebras if ν be a fuzzy subset of $X^{/}$,suchthat $f^{-1}(\nu)$ is DF ideal of X,then ν is also a DF ideal of $X^{/}$.

Proof .Let u, $v \in X^{/}$ therefore there exists x, $y \in X$ such that f (x) = u, f (y) = v
Let μ be the pre image of ν under f, then $\nu(f(x)) = \mu(x)$ [since $f^{-1}(\nu)(x) = \nu(f(x))$]
sinceμ is DF ideal of X
$\therefore \mu(0) \leq \mu(x)$
$\Rightarrow \quad \nu(f(0)) \leq \nu(f(x))$
$\Rightarrow \quad \nu(0^{/}) \leq \nu(u) \forall u \in X$
Again
$\mu(x) \leq \max\{\mu(x * y), \mu(y)\} \forall x, y \in X$
$\Rightarrow \nu(f(x)) \leq \max\{\nu(f(x * y)), \nu f(y)\}$
$\Rightarrow \nu(u) \leq \max\{\nu(f(x) * f(y)), \nu f(y)\}$
$= \max\{\nu(u * v), \nu(v)\}$
$\nu(u) \leq \max\{\nu(u * v), \nu(v)\}$for all u, $v \in X^{/}$
Hence ν is a DF ideal of $X^{/}$.

(v) Conclusion

In this paper we studiedabout ideals of BF-algebras in context of fuzzy set and we introduced Doubt fuzzy ideals of BF-algebras. We discussed some characterizations of BF- algebras in terms of Doubt fuzzy ideals.In future, the following studies may be carried out (1) Rough fuzzy ideals of BF-algebras (2)($\in, \in \vee q$)-Doubt fuzzy ideals of BF-algebras.

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