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CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric non-metric connection

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Abstract: *This paper deals with the study of CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric non-metric connection. We study parallel distribution relating to $\xi-$vertical CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with quarter symmetric non-metric connection. Further, we obtain the parallel distributions on CR-submanifolds.*

Keywords: *CR-submanifolds, nearly trans-hyperbolic Sasakian manifold, quarter symmetric non-metric connection, parallel distribution.*

I. Introduction

In 1978, Aurel Bejancu introduced the notion of *CR*-submanifold of Kaehler manifold [1, 2]. On the other hand *CR*-submanifold have been studied by kobayashi [3]. J. A. Oubina introduced a new class of almost contact metric manifold known as trans-Sasakian manifold [4]. Gherghe studied on harmonicity on nearly trans-Sasaki geometry of *CR*-submanifold of manifold [5]. *CR*-submanifold of a trans-Sasakian manifold have been studied by Shahid [6]. Later Al-Solamy studied the *CR*-submanifold of a nearly trans-Sasakian manifold [7]. In 1976, Upadhyay and Dube have studied almost contact hyperbolic structure [8]. Bhatt and Dube studied on *CR*-submanifold of trans-hyperbolic Sasakian manifold [9]. Gill and Dube have also worked on *CR*-submanifold of trans-hyperbolic Sasakian manifold [10]. Kumar and Dube studied *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold [11]. In this paper we study *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold endowed with a quarter symmetric non-metric connection. Let ∇ be a linear connection in an $n-$dimensional differentiable manifold M. The torsion tensor T and curvature tensor R of ∇ are given respectively by

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]$$

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g=0,$ otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [12] S. Golab introduced the idea of a quarter symmetric connection. A linear connection is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(X,Y)=\eta(Y)\phi X-\eta(X)\phi Y.$$

Some properties of quarter symmetric non-metric connection was studied by several authors in ([13], [14], [15], [16]).

This paper is organized as follows: In section 2, we give a brief introduction of nearly trans-hyperbolic Sasakian manifold. In section 3, we have proved some basic lemmas on nearly trans-hyperbolic Sasakian manifold with a quarter symmetric non-metric connection. In section 4, we have discussed parallel distributions.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure $(\phi,\xi,\eta,g),$ where a tensor ϕ of type (1,1), a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following

(2.1) $$\phi^2 X=X-\eta(X)\xi,\quad g(X,\xi)=\eta(X)$$

(2.2) $$\eta(\xi)=-1,\qquad \phi(\xi)=0,\qquad \eta o\phi=0$$

(2.3) $$g(\phi X,\phi Y)=-g(X,Y)-\eta(X)\eta(Y)$$

for any X,Y tangent to $\bar{M}$ [17]. In this case

(2.4) $$g(\phi X, Y) = -g(X, \phi Y).$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans-hyperbolic Sasakian [10] if and only if

(2.5) $$(\overline{\overline{\nabla}}_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X)$$

for all X, Y tangent to $\overline{M}$ and α, β are functions on $\overline{M}$. On a trans-hyperbolic Sasakian manifold $\overline{M}$, we have

(2.6) $$\overline{\overline{\nabla}}_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi),$$

where g is the Riemannian metric and $\overline{\overline{\nabla}}$ is the Riemannian connection.

Let M be an $m-$dimensional isometrically immersed submanifold of nearly trans-hyperbolic Sasakian manifold $\overline{M}$. We denote by g the Riemannian metric tensor field on M as well as $\overline{M}$.

Definition 2.1. An $m-$dimensional Riemannian submanifold M of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ is called a *CR*-submanifold if ξ is tangent to M and there exists differentiable distribution $D : x \in M \to D_X \subset T_x(M)$ such that

(i) the distribution D_x is invariant under ϕ, that is $\phi D_x \subset D_x$ for each $x \in M$;

(ii) the complementary orthogonal distribution $D^{\perp} : x \to D^{\perp}{}_x \subset T_x(M)$ of the distribution D on M is anti-invariant under ϕ that is, $\phi D^{\perp}{}_x(M) \subset T_x{}^{\perp}(M)$ for all $x \in M$, where $T_x(M)$ and $T^{\perp}{}_x(M)$ are tangent space and normal space of M at $x \in M$ respectively.

If $\dim D^{\perp}{}_x = 0$ (resp. $\dim D_x = 0$), then *CR*-submanifold is called an invariant (resp. anti-invariant). The distribution D (resp. $D^{\perp}$) is called horizontal (resp. vertical) distribution. The pair (D, $D^{\perp}$) is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D^{\perp}$) for $x \in M$.

For any vector field X tangent to M, we write

(2.8) $$X = PX + QX,$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.

For any vector field N normal to M, we put

(2.9) $$\phi N = BN + CN,$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

Now, we remark that owing to the existence of the 1-form η, we can define a quarter symmetric non-metric connection $\overline{\nabla}$ in almost contact metric manifold by

(2.10) $$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \eta(Y)\phi X$$

such that $$(\overline{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y)$$

for any $X, Y \in TM$, where $\overline{\overline{\nabla}}$ is the induced connection with respect to g on M, η is a 1-form and ξ is a vector field.

Using (2.5) and (2.10), we get

(2.11) $$(\overline{\nabla}_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) - \eta(Y)X + \eta(Y)\eta(X)\xi.$$

Similarly, we have

$$(\overline{\nabla}_Y \phi)X = \alpha(g(Y,X)\xi - \eta(X)\phi Y) + \beta(g(\phi Y,X)\xi - \eta(X)\phi Y) - \eta(X)Y + \eta(X)\eta(Y)\xi.$$

On adding above equations, we obtain

(2.12)
$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) - \eta(X)Y - \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

This is the condition for an almost contact structure (ϕ,ξ,η,g) with a quarter symmetric non-metric connection to be nearly trans-hyperbolic Sasakian manifold.
From (2.10) and (2.6), we get

(2.13)
$$\overline{\nabla}_X \xi = -(\alpha+1)(\phi X) + \beta(X - \eta(X)\xi).$$

We denote by g the metric tensor of $\overline{M}$ as well as that induced on M. Let $\overline{\nabla}$ be the quarter symmetric non-metric connection on $\overline{M}$ and ∇ be the induced connection on M with respect to the unit normal N.

Theorem 2.2. The connection induced on the *CR*-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric non-metric connection is also a quarter symmetric non-metric connection.

Proof. Let ∇ be the induced connection with respect to the unit normal N on a *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric non-metric connection $\overline{\nabla}$. Then

(2.14)
$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y),$$

where m is a tensor field of type (0, 2) on *CR*-submanifold M. If ∇^* be the induced connection on *CR*-submanifolds from Riemannian connection $\overline{\overline{\nabla}}$, then

(2.15)
$$\overline{\overline{\nabla}}_X Y = \nabla^*_X Y + h(X,Y),$$

where h is a second fundamental tensor.
Now, from (2.14) and (2.15) we have

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) + \eta(Y)\phi X.$$

Equating the tangential and normal components from both the sides in the above equation, we get

$$h(X,Y) = m(X,Y)$$

and

$$\nabla_X Y = \nabla^*_X Y + \eta(Y)\phi X.$$

Thus ∇ is also a quarter symmetric non-metric connection.
Now, the Gauss formula for a *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric non-metric connection is

(2.16)
$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and the Weingarten formula for M is given by

(2.17)
$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for $X,Y \in TM$, $N \in T^{\perp}M$, where h and A are called the second fundamental tensors of M and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

(2.18)
$$g(h(X,Y),N) = g(A_N X, Y).$$

III. Some Basic Lemmas

Lemma 3.1 Let M be a *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter-symmetric non-metric connection. Then

(3.1)
$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QX} Y) - P(A_{\phi QY} X) = \phi P\nabla_X Y + \phi P\nabla_Y X + 2\alpha g(X,Y)P\xi - \alpha\eta(X)\phi PY - \alpha\eta(Y)\phi PX - \beta\eta(Y)\phi PX - \beta\eta(X)\phi PY - \eta(X)PY - \eta(Y)PX + 2\eta(X)\eta(Y)P\xi,$$

(3.2)
$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QX} Y) - Q(A_{\phi QY} X) = 2Bh(X,Y) + 2\alpha g(X,Y)Q\xi - \eta(X)QY - \eta(Y)QX + 2\eta(X)\eta(Y)Q\xi,$$

(3.3)
$$h(X,\phi PY)+h(Y,\phi PX)+\nabla^{\perp}_{X}\phi QY+\nabla^{\perp}_{Y}\phi QX$$
$$=\phi Q\nabla_{Y}X+\phi Q\nabla_{X}Y+2Ch(X,Y)-(\alpha+\beta)\eta(Y)\phi QX$$
$$-(\alpha+\beta)\eta(X)\phi QY,$$

for all $X,Y\in TM.$

Proof. By direct covariant differentiation, we have

$$\overline{\nabla}_{X}\phi Y=(\overline{\nabla}_{X}\phi)Y+\phi\nabla_{X}Y+\phi h(X,Y),$$
$$\overline{\nabla}_{X}\phi Y=\overline{\nabla}_{X}\phi PY+\overline{\nabla}_{X}\phi QY.$$

By virtue of (2.8), (2.11), (2.16) and (2.17), we get

$$(\overline{\nabla}_{X}\phi)Y+\phi\nabla_{X}Y+\phi h(X,Y)=P\nabla_{X}\phi PY+Q\nabla_{X}\phi PY+h(X,\phi PY)$$
$$+\nabla^{\perp}_{X}\phi QY-PA_{\phi QY}X-QA_{\phi QY}X.$$

Similarly,

$$(\overline{\nabla}_{Y}\phi)X+\phi\nabla_{Y}X+\phi h(Y,X)=P\nabla_{Y}\phi PX+Q\nabla_{Y}\phi PX+h(Y,\phi PX)$$
$$+\nabla^{\perp}_{Y}\phi QX-PA_{\phi QX}Y-QA_{\phi QX}Y.$$

Adding, we obtain

(3.4)
$$((\overline{\nabla}_{X}\phi)Y+(\overline{\nabla}_{Y}\phi)X)+\phi P\nabla_{X}Y+\phi Q\nabla_{X}Y+\phi P\nabla_{Y}X+\phi Q\nabla_{Y}X$$
$$+2Bh(X,Y)+2Ch(X,Y)=\alpha(2g(X,Y)P\xi+\alpha(2g(X,Y)Q\xi$$
$$-\alpha\eta(Y)\phi PX-\alpha\eta(Y)\phi QX-\alpha\eta(X)\phi PY-\alpha\eta(X)\phi QY$$
$$-\beta\eta(X)\phi PY-\beta\eta(X)\phi QY-\beta\eta(Y)\phi QX-\beta\eta(Y)\phi PX$$
$$-\eta(X)PY-\eta(X)QY-\eta(Y)PX-\eta(Y)QX+2\eta(X)\eta(Y)P\xi$$
$$+2\eta(X)\eta(Y)Q\xi+\phi P\nabla_{X}Y+\phi Q\nabla_{X}Y+\phi P\nabla_{Y}X+\phi Q\nabla_{Y}X$$
$$+2Bh(X,Y)+2Ch(X,Y)$$

for any $X,Y\in TM.$

Now, equating horizontal, vertical and normal components in (3.4) we get the desired result.

Lemma 3.2. Let M be a *CR*-Submanifod of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric non-metric connection. Then

(3.5)
$$2(\overline{\nabla}_{X}\phi)Y=\nabla_{X}\phi Y-\nabla_{Y}\phi X+h(X,\phi Y))-h(Y,\phi X)-\phi[X,Y]$$
$$+\alpha(2g(X,Y)\xi-\eta(Y)\phi X+\eta(X)\phi Y)-\beta(\eta(X)\phi Y$$
$$+\eta(Y)\phi X)-(\eta(Y)X+\eta(X)Y-2\eta(X)\eta(Y)\xi),$$

(3.6)
$$2(\overline{\nabla}_{Y}\phi)X=\alpha(2g(X,Y)\xi-\eta(Y)\phi X-\eta(X)\phi Y)-\beta(\eta(X)\phi Y$$
$$+\eta(Y)\phi X)-(\eta(X)Y+\eta(Y)X-2\eta(X)\eta(Y)\xi)-(\nabla_{X}\phi Y)$$
$$-h(X,\phi Y)+\nabla_{Y}\phi X+h(Y,\phi X)+\phi[X,Y]$$

for any $X,Y\in D.$

Proof. From Gauss formula (2.16), we have

(3.7)
$$\overline{\nabla}_{X}\phi Y-\overline{\nabla}_{Y}\phi X=\nabla_{X}\phi Y+h(X,\phi Y)-\nabla_{Y}\phi X-h(Y,\phi X).$$

Also, we have

(3.8)
$$\overline{\nabla}_{X}\phi Y-\overline{\nabla}_{Y}\phi X=(\overline{\nabla}_{X}\phi)Y-(\overline{\nabla}_{Y}\phi)X+\phi[X,Y].$$

From (3.7) and (3.8), we get

$$(\overline{\nabla}_{X}\phi)Y-(\overline{\nabla}_{Y}\phi)X=\nabla_{X}\phi Y+h(X,\phi Y)-\nabla_{Y}\phi X$$
$$-h(Y,\phi X)-\phi[X,Y].$$

Also for nearly trans-hyperbolic Sasakian manifold with quarter symmetric non-metric connection, we have

(3.10)
$$(\overline{\nabla}_X\phi)Y+(\overline{\nabla}_Y\phi)X=\alpha(2g(X,Y)\xi-\eta(Y)\phi X-\eta(X)\phi Y) \\ -\beta(\eta(X)\phi Y+\eta(Y)\phi X)-(\eta(Y)X \\ +\eta(X)Y-2\eta(X)\eta(Y)\xi.$$

Adding (3.9) and (3.10), we obtain

$$2(\overline{\nabla}_X\phi)Y=\nabla_X\phi Y-\nabla_Y\phi X+h(X,\phi Y))-h(Y,\phi X)-\phi[X,Y] \\ +\alpha(2g(X,Y)\xi-\eta(Y)\phi X+\eta(X)\phi Y)-\beta(\eta(X)\phi Y \\ +\eta(Y)\phi X)-(\eta(Y)X+\eta(X)Y-2\eta(X)\eta(Y)\xi).$$

Subtracting (3.9) from (3.10), we get

$$2(\overline{\nabla}_Y\phi)X=\alpha(2g(X,Y)\xi-\eta(Y)\phi X-\eta(X)\phi Y)-\beta(\eta(X)\phi Y \\ +\eta(Y)\phi X)-(\eta(X)Y+\eta(Y)X-2\eta(X)\eta(Y)\xi) \\ -(\nabla_X\phi Y)-h(X,\phi Y)+\nabla_Y\phi X+h(Y,\phi X)+\phi[X,Y].$$

Hence Lemma is proved.

Lemma 3.3. Let M be a *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric non-metric connection, then

$$2(\overline{\nabla}_Y\phi)Z=A_{\phi Y}Z-A_{\phi Z}Y+\nabla^{\perp}_Y\phi Z-\nabla^{\perp}_Z\phi Y-\phi[Y,Z]+\alpha(2g(Y,Z)\xi \\ -\eta(Y)\phi Z-\eta(Z)\phi Y)-\beta(\eta(Y)\phi Z+\eta(Z)\phi Y)-(\eta(Y)Z \\ +\eta(Z)Y-2\eta(Y)\eta(Z)\xi)$$

and

$$2(\overline{\nabla}_Z\phi)Y=-A_{\phi Y}Z+A_{\phi Z}Y-\nabla^{\perp}_Y\phi Z+\nabla^{\perp}_Z\phi Y+\phi[Y,Z]+\alpha(2g(Y,Z)\xi \\ -\eta(Y)\phi Z-\eta(Z)\phi Y)-\beta(\eta(Y)\phi Z+\eta(Z)\phi Y)-(\eta(Y)Z \\ +\eta(Z)Y-2\eta(Y)\eta(Z)\xi)$$

for any $Y,Z\in D^{\perp}$.

Proof. From Weingarten formula (2.17), we have

(3.11)
$$\overline{\nabla}_Z\phi Y-\overline{\nabla}_Y\phi Z=A_{\phi Y}Z-A_{\phi Z}Y+\nabla^{\perp}_Y\phi Z-\nabla^{\perp}_Z\phi Y.$$

Also, we have

(3.12)
$$\overline{\nabla}_Z\phi Y-\overline{\nabla}_Y\phi Z=(\overline{\nabla}_Y\phi)Z-(\overline{\nabla}_Z\phi)Y+\phi[Y,Z].$$

From (3.11) and (3.12), we get

(3.13)
$$(\overline{\nabla}_Y\phi)Z+(\overline{\nabla}_Z\phi)Y=\alpha(2g(Y,Z)\xi-\eta(Y)\phi Z-\eta(Z)\phi Y) \\ -\beta(\eta(Y)\phi Z+\eta(Z)\phi Y)-(\eta(Y)Z \\ +\eta(Z)Y-2\eta(Y)\eta(Z)\xi).$$

On adding (3.13) and (3.14), we obtain

$$2(\overline{\nabla}_Y\phi)Z=A_{\phi Y}Z-A_{\phi Z}Y+\nabla^{\perp}_Y\phi Z-\nabla^{\perp}_Z\phi Y-\phi[Y,Z] \\ +\alpha(2g(Y,Z)\xi-\eta(Y)\phi Z-\eta(Z)\phi Y)-\beta(\eta(Y)\phi Z \\ +\eta(Z)\phi Y)-(\eta(Y)Z+\eta(Z)Y-2\eta(Y)\eta(Z)\xi).$$

Subtracting (3.13) and (3.14), we find

$$2(\overline{\nabla}_Z\phi)Y=-A_{\phi Y}Z+A_{\phi Z}Y-\nabla^{\perp}_Y\phi Z+\nabla^{\perp}_Z\phi Y+\phi[Y,Z] \\ +\alpha(2g(Y,Z)\xi-\eta(Y)\phi Z-\eta(Z)\phi Y)-\beta(\eta(Y)\phi Z \\ +\eta(Z)\phi Y)-(\eta(Y)Z+\eta(Z)Y-2\eta(Y)\eta(Z)\xi).$$

This proves our assertions.

Lemma 3.4. Let M be a *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric non-metric connection, then

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] + \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) - (\eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi),$$

$$2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] + \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X)$$

for any $X \in D$ and $Y \in D^{\perp}$.

Proof. By using Gauss and Weingarten equation for $X \in D$ and $Y \in D^{\perp}$ respectively, we get

(3.15) $$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X).$$

Also, we have

(3.16) $$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[X,Y].$$

From (3.15) and (3.16), we obtain

(3.17) $$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y].$$

Also for nearly trans-hyperbolic Sasakian manifold with a quarter symmetric non-metric connection, we have

(3.18) $$(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) - (\eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi).$$

Adding (3.17) and (3.18), we find

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] + \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) - (\eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi).$$

Subtracting (3.17) from (3.18), we get

$$2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] + \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X).$$

Hence Lemma is proved.

IV. Parallel Distributions

Definition 4.1. The horizontal (resp. vertical) distribution D (resp. $D^{\perp}$) is said to be parallel with respect to the quarter symmetric non-metric connection on M if $\nabla_X Y \in D$ (resp. $\nabla_Z W \in D^{\perp}$) for any vector field $X, Y \in D$ (resp. $W, Z \in D^{\perp}$).

Proposition 4.1. Let M be a ξ-vertical *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric non-metric connection. If the horizontal distribution D is parallel, then

(4.1) $$h(X,\phi Y) = h(Y,\phi X)$$

for all $X, Y \in D.$

Proof. For horizontal distribution $D,$ we have

(4.2) $$\nabla_X\phi Y \in D,\ \nabla_Y\phi X \in D \text{ for any } X, Y \in D.$$

Using the fact that $QX = QY = 0$ for $X, Y \in D,$ (3.2) gives

(4.3) $$Bh(X,Y) = -\alpha g(X,Y)Q\xi, \quad \text{for any } X, Y \in D.$$

Also, since

(4.4) $$\phi h(X,Y) = Bh(X,Y) + Ch(X,Y),$$

Therefore,

(4.5) $$\phi h(X,Y) = -\alpha g(X,Y)Q\xi + Ch(X,Y) \quad \text{for any } X, Y \in D.$$

From (3.3), we have

(4.6) $$h(X,\phi Y) + h(Y,\phi X) = 2Ch(X,Y) = 2\phi h(X,Y) + 2\alpha g(X,Y)Q\xi$$

for any $X, Y \in D$. Putting $X = \phi X \in D$ in (4.6), we get

(4.7) $$h(\phi X, \phi Y) + h(Y, \phi^2 X) = 2\phi h(\phi X, Y) + 2\alpha g(\phi X, Y)Q\xi$$

or

(4.8) $$h(\phi X, \phi Y) - h(Y, X) = 2\phi h(\phi X, Y) + 2\alpha g(\phi X, Y)Q\xi.$$

Similarly, putting $Y = \phi Y \in D$ in (4.6), we find

(4.9) $$h(\phi Y, \phi X) - h(X, Y) = 2\phi h(X, \phi Y) + 2\alpha g(X, \phi Y)Q\xi.$$

Hence from (4.8) and (4.9), we have

(4.10) $$\phi h(X, \phi Y) - \phi h(Y, \phi X) = \alpha g(\phi X, Y)Q\xi - \alpha g(X, \phi Y)Q\xi.$$

Operating ϕ on both sides of (4.10) and using $\phi\xi = 0$, we get

(4.11) $$h(X, \phi Y) = h(Y, \phi X)$$

for all $X, Y \in D.$

Now, for the distribution $D^\perp$ we prove the following proposition.

Proposition 4.2. Let M be a ξ-vertical *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric non-metric connection. If the distribution $D^\perp$ is parallel with respect to the connection on M, then

(4.12) $$A_{\phi Y}Z + A_{\phi Z}Y \in D^\perp \quad \text{for any } Y, Z \in D^\perp.$$

Proof. Using Gauss and Weingarten formula, we obtain

(4.13) $$\begin{aligned} &-A_{\phi Z}Y + \nabla^\perp_Y \phi Z - A_{\phi Y}Z + \nabla^\perp_Z \phi Y = \phi\nabla_Y Z + \phi\nabla_Z Y + 2\phi h(Y, Z) \\ &+ \alpha(2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \beta(\eta(Z)\phi Y + \eta(Y)\phi Z) \\ &-(\eta(Y)Z + \eta(Z)Y - 2\eta(Y)\eta(Z)\xi) \end{aligned}$$

for any $Y, Z \in D^\perp$. Taking inner product with $X \in D$ in (3.13), we get

(4.14) $$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X).$$

If the distribution $D^\perp$ is parallel, then $\nabla_Y Z \in D^\perp$ and $\nabla_Z Y \in D^\perp$ for any $Y, Z \in D^\perp$.
So from (4.14), we get

(4.15) $$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = 0 \quad \text{or} \quad g(A_{\phi Y}Z + A_{\phi Z}Y, X) = 0$$

which is equivalent to

(4.16) $$A_{\phi Y}Z + A_{\phi Z}Y \in D^\perp$$

for any $Y, Z \in D^\perp.$

This completes the proof.

Definition 4.3. A *CR*-submanifold with a quarter-symmetric non-metric connection is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^\perp.$

The following Lemma is an easy consequence of (2.18).

Lemma 4.4. Let M be a *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter-symmetric non-metric connection. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.

Definition 4.5. A normal vector field $N \neq 0$ is called D-parallel normal section if $\nabla^\perp_X N = 0$ for all $X \in D.$

Now, we have the following proposition.

Proposition 4.6. Let M be a mixed totally geodesic ξ-vertical *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric non-metric connection. Then the normal section $N \in \phi D^\perp$ is D-parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D.$

Proof. Let $N \in \phi D^\perp$. Then from (3.2), we have

(4.17) $$Q(\nabla_Y \phi X) = 0 \text{ for any } X \in D, Y \in D^{\perp}.$$

In particular, we have $$Q(\nabla_Y X) = 0.$$

By using it in (3.3), we get

(4.18) $$\nabla_X^{\perp} \phi QY = \phi Q \nabla_X Y \quad \text{or} \quad \nabla_X^{\perp} N = -\phi Q \nabla_X \phi N.$$

Thus, if the normal section $N \neq 0$ with quarter symmetric non-metric connection is D-parallel, then by definition and (4.18), we get

(4.19) $$\phi Q(\nabla_X \phi N) = 0$$

which is equivalent to $\nabla_X \phi N \in D$ for all $X \in D$.

The converse part easily follows from (4.18).

This completes the proof of the proposition.

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