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Degree of Approximation of Fourier series of function of bounded variation by (Z, δ,β) method

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Abstract : *The object of the present investigation is to study the degree of approximation of Fourier series of function of bounded variation by generalized Harmonic-Cesaro method of summation.*
Keywords: *Fourier series, bounded variation*

I. Introduction

It is well known that (see [6] vol I p.57) if $f \in BV[-\pi,\pi]$ then

$$\lim_{n\to\infty}(s_n(f,x) - \tfrac{1}{2}\{f(x+0)+f(x-0)\}) = 0 \tag{1}$$

Boianic [1] obtained a sharper version of this result by showing that for n ≥ 1

$$\left|s_n(f,x) - \tfrac{1}{2}\{f(x+0)+f(x-0)\}\right| \le \frac{3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(\phi) \tag{2}$$

Whenever $f \in BV[-\pi,\pi]$.
Extending the above result, Mazhar and Boianic [3] proved the following:
Theorem A
If $f \in BV[-\pi,\pi]$ then for $-1<\delta\le 0$

$$\left|A_n^{\delta}(f,x)\right| \le \frac{C(\delta)}{n^{\delta+1}}\sum_{k=1}^{m} k^{\delta} V_0^{\pi/k}(\phi)$$

where $C(\delta)$ is a constant depending upon δ.
In this connection it may be pointed out that Boianic and Mazhar [2] have also earlier examined with $\phi(t) \in BV$ in Norlund (N, p) mean setup, where it is assumed that P_n >0, monotonic non-increasing with $P_n \to \infty$ and they have obtained:

$$\left|W_n(f,x) - \tfrac{1}{2}\{f(x+0)+f(x-0)\}\right| \le \frac{6}{P_n}\sum_{k=1}^{n} P_k V_0^{\pi/p_k}(\phi) \tag{3}$$

Our object is to obtain degree of approximation of Fourier series of function of bounded variation by (Z, δ,β) method.

II. Main Results

We prove the following theorem.
Theorem
(a) For $f \in BV[0,\pi], -1<\delta<0, \beta \in R$

$$A_n^{\delta,\beta}(f,x) = \frac{O(1)}{n^{\delta+1}(\log n)^{\beta}}\sum_{k=1}^{n}\log^{\beta}\left(\frac{ak}{2}\right)k^{\delta}V_0^{\pi/k}(\phi) + O(1)\frac{1}{n}\sum_{k=1}^{n}V_0^{\pi/k}(\phi)$$

(b) For $f \in BV\left[0,\pi\right], \delta=-1, \beta>0$

$$A_n^{\delta,\beta}(f,x) = \frac{O(1)}{(\log n)^{\beta-1}}\sum_{k=1}^{n}\frac{(\log\frac{ak}{2})^{\beta-1}}{k}V_0^{\pi/k}(\phi)$$

We need the following additional notations for the proof of our theorem.

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin t/2}$$

$$K_n(t) = \frac{1}{P_n}\sum_{k=0}^{n} P_{n-k} D_K(t)$$

$$K_n^{\delta,\beta}(t) = \frac{1}{A_n^{\delta,\beta}}\sum_{k=0}^{n} A_{n-k}^{\delta-1,\beta} D_K(t)$$

$$\wedge_{(t)} = \int_t^{\pi} K_n^{\delta,\beta}(u)du$$

$$A_n^{\delta,\beta}(f,x) = K_n^{\delta,\beta}(f,x) - \tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

where $K_n^{\delta,\beta}(f,x)$ denotes the (Z, δ, β) mean of the Fourier series of f at x.

We need the following Lemmas to prove the theorem.

3 Lemma 1

(see [6], Vol I. p.192)

If $\delta > -1$ and F(u) is slowly varying, then $\sum_{v=1}^{n} v^{\delta} F(v) \square \frac{n^{\delta+1}}{\delta+1} F(n)$

Lemma 2

For $p_n > 0$ and monotonic non increasing

i) $K_n(t) = O(n)$

ii) $K_n(t) = \frac{1}{(2\sin t/2)P_n} I\left(e^{i(n+1/2)t}\sum_{k=0}^{\infty} p_k e^{-ikt}\right) + O(1)\frac{P_n}{P_n(2\sin t/2)^2}$

Proof of Lemma 2(i)

$$K_n(t) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} D_k(t)$$

$$=> |k_n(t)| \le \frac{1}{P_n}\sum_{k=0}^{n} |p_{n-k}||D_k(t)|$$

$$\le \frac{1}{P_n}\sum_{k=0}^{n} |p_{n-k}|(k+1/2)$$

$$\le (n+1/2)$$

$$=> K_n(t) = O(n)$$

Proof of Lemma 2(ii)

$$K_n(t)=\frac{1}{P_n}\sum_{k=0}^{n}P_{n-k}D_k(t)$$
$$=\frac{1}{(2\sin t/2)P_n}\sum_{k=0}^{n}P_{n-k}\sin\left(k+\frac{1}{2}\right)t$$
$$=\frac{1}{(2\sin t/2)P_n}I\left(\sum_{k=0}^{n}P_{n-k}e^{i(k+1/2)t}\right)$$
$$=\frac{1}{(2\sin t/2)P_n}I\left(e^{i(n+1/2)t}\sum_{k=0}^{n}p_ke^{-ikt}\right)$$
$$=\frac{1}{(2\sin t/2)P_n}I\left(e^{i(n+1/2)t}\sum_{k=0}^{\infty}P_ke^{-ikt}-\sum_{k=n+1}^{\infty}p_ke^{-ikt}\right) \tag{4}$$

In order to deal with the second summation formula of (3.1), by Abel's transformation we note that

$$\left|\sum_{k=n+1}^{m}P_ke^{-ikt}\right|=\left|\left(\sum_{k=n+1}^{m-1}(p_k-p_{k+1})\sum_{v=n+1}^{k}e^{-ivt}+\sum_{v=n+1}^{m}e^{-ivt}\right)p_m\right|$$
$$\le\sum_{k=n+1}^{m-1}(p_k-p_{k+1})\frac{1}{\left|1-e^{-it}\right|}+\frac{p_m}{\left|I-e^{-it}\right|}$$
$$=\frac{1}{\left|1-e^{-it}\right|}\left(\sum_{k=n+1}^{m-1}(p_k-p_{k+1})+p_m\right)$$

By using the fact that p_n is monotonic non-increasing and $p_n\to 0$ as $n\to\infty$ we have

$$\left|\sum_{k=n+1}^{\infty}P_ke^{-ikt}\right|\le\frac{1}{2\sin t/2)}\sum_{k=n+1}^{\infty}(p_k-p_{k+1})=\frac{O(p_n)}{2\sin t/2} \tag{5}$$

Now using(3.2) in (3.1) we have

$$K_n(t)=\frac{1}{(2\sin t/2)P_n}I\left(e^{i(n+1/2)t}\sum_{k=0}^{\infty}p_ke^{-ikt}\right)+O(1)\frac{P_n}{P_n(2\sin t/2)^2}$$

which proves Lemma 2(ii).

Lemma 3

For $a\ge 2,\beta\in R,-1<\delta<1$, we write

$$R=R(t)=\left[\log^2\left(\frac{a}{2\sin t/2}\right)+\left(\frac{\pi-t}{2}\right)^2\right]^{1/2}$$

$$\phi=\phi(t)=\tan^{-1}\left(\frac{t-\pi}{2\log\left(\frac{a}{2\sin t/2}\right)}\right)$$

Then for $\frac{\pi}{n}\le t\le\pi$

i) $$K_n^{\delta,\beta}(t)=\begin{cases}\dfrac{R^{\beta}\sin\left[\left(n+\frac{1}{2}+\frac{\delta}{2}\right)t-\frac{\pi\delta}{2}+\phi\beta\right]}{n^{\delta}(\log n)^{\beta}(2\sin t/2)^{\delta+1}}+\dfrac{O(1)}{n(2\sin t/2)^{2}}, \delta\neq -1,-2\ldots\ ,\beta\in R\\ \dfrac{R^{\beta}}{n^{-1}(\log n)^{\beta-1}}\left[\sin nt+\frac{\pi}{2}+\phi\beta\right]+\dfrac{O(1)}{n(2\sin t/2)^{2}}, \delta=-1,-2,\ldots\ ,\beta>0\end{cases}$$

ii) $$\wedge(t)=O(1)\begin{cases}\dfrac{(\log\frac{a\pi}{2t})^{\beta}}{n^{\delta+1}(\log n)^{\beta}t^{\delta+1}}+\dfrac{1}{nt}, \quad \delta>-1,\beta\in R\\ \dfrac{(\log\frac{a\pi}{2t})^{\beta}}{(\log n)^{\beta-1}}+\dfrac{1}{nt}, \delta=-1,\ \beta>0\end{cases}$$

Proof of Lemma 3(i)

We note that $K_n(t)$ reduces to $K_n^{\delta,\beta}(t)$ when $p_n=A_n^{\delta-1,\beta}$ *and* $P_n=A_n^{\delta,\beta}$

From Lemma 2(ii), we get

$$K_n(t)=\frac{1}{(2\sin t/2)P_n}I\left(e^{i(n+1/2)t}\sum_{k=0}^{\infty}P_k e^{-ikt}\right)+\frac{O(1)P_n}{P_n(2\sin t/2)^2}$$

Now (see [6] Vol I. p.192) putting $P_n=A_n^{\delta,\beta}$ and $p_n=A_n^{\delta-1,\beta}$ we have

$$K_n^{\delta,\beta}(t)=\frac{1}{(2\sin t/2)A_n^{\delta,\beta}}I\left(e^{i(n+1/2)t}\sum_{k=0}^{\infty}A_k^{\delta-1,\beta}e^{-ikt}\right)+O(1)\frac{A_k^{\delta-1,\beta}}{A_n^{\delta,\beta}(2\sin t/2)^2}$$

$$=\frac{1}{(2\sin t/2)A_n^{\delta,\beta}}I\left(e^{i(n+1/2)t}\frac{R^{\beta}e^{i\phi\beta}}{\left[(2\sin t/2)e^{i\left(\frac{\pi-t}{2}\right)}\right]^{\delta}}\right)+O(1)\frac{A_n^{\delta-1,\beta}}{A_n^{\delta,\beta}(2\sin t/2)^2}$$

$$\sum_{k=0}^{\infty}A_k^{\delta-1,\beta}e^{-ikt}=\frac{1}{\left(1-e^{-it}\right)^{\delta}}\log^{\beta}\left(\frac{a}{\left(1-e^{-it}\right)}\right)=\frac{1}{\{(1-\cos t)+i\sin t\}^{\delta}}\log^{\beta}\left(\frac{a}{1-e^{-it}}\right)$$

$$=\left[\frac{1}{\left[(2\sin t/2)e^{i(\frac{\pi-t}{2})}\right]^{\delta}}\right]R^{\beta}e^{i\phi\beta}$$

Where $\mathrm{Re}^{i\phi}=\log\left(\frac{a}{1-e^{-it}}\right)$

$$R=\left[\log^2\left(\frac{a}{2\sin t/2}\right)+(\frac{\pi-t}{2})^2\right]^{1/2}$$

and

$$\phi = \tan^{-1}\left(\frac{t-\pi}{2\log\left(\frac{a}{2\sin t/2}\right)}\right) \tag{6}$$

$$= \frac{R^{\beta}}{\left(2\sin t/2\right)^{\delta+1} A_n^{\delta,\beta}}\sin\left[\left(n+1/2+\delta/2\right)t-\frac{\pi\delta}{2}+\phi\beta\right]+\frac{O(1)A_n^{\delta-1,\beta}}{A_n^{\delta,\beta}\left(2\sin t/2\right)^2}$$

Which ensures Lemma 3(i) in the case $\delta \neq -1, \beta \in R$, since

$A_n^{\delta,\beta} \sim \frac{n^{\delta}}{\Gamma(\delta+1)}(\log n)^{\beta}$ (see [6] (Vol I. p. 192)

We omit the proof of the case $\delta = -1, \beta > 0$ as its proof is similar to that of the case $\delta \neq -1$ described above.

Proof of Lemma 3(ii)

Case – 1 : For $\delta \neq -1, -2\ldots$ and using Lemma 3(i)

$$\wedge(t) = \int_t^{\pi} K_n^{\delta,\beta}(u)du$$

$$\frac{1}{A_n^{\delta,\beta}}\int_t^{\pi}\frac{R^{\beta}(u)}{(2\sin u/2)^{\delta+1}}\sin\left[\left(n+1/2+\delta/2\right)u-\frac{\pi\delta}{2}+\phi\beta\right]du$$

$$+\frac{O(1)}{n}\int_t^{\pi}\frac{du}{(2\sin u/2)^2}du$$

$=J_1+J_2$ (say)

$$J_1 = \frac{1}{A_n^{\delta,\beta}}\int_t^{\pi}\frac{R^{\beta}(u)}{(2\sin u/2)^{\delta+1}}\sin\left[\left(n+1/2+\delta/2\right)u-\frac{\pi\delta}{2}+\phi\beta\right]du$$

letting $\theta(u) = \frac{R^{\beta}(u)}{(2\sin u/2)^{\delta+1}}$

and

$$t(u) = \beta\phi = \beta\tan^{-1}\left(\frac{u-\pi}{2\log\left(\frac{a}{2\sin u/2}\right)}\right)$$

We have

$$J_1 = \frac{1}{A_n^{\delta,\beta}}\int_t^{\pi}\left\{\theta(u)\sin\left[\left(n+1/2+\delta/2\right)u-\frac{\pi\delta}{2}\right]\cos t(u)+\theta(u)\cos\left[\left(n+1/2+\delta/2\right)u-\frac{\pi\delta}{2}\right]\sin t(u)\right\}du$$

$$= \frac{1}{A_n^{\delta,\beta}}(\wedge_1(t)+\wedge_2(t)) \quad \text{(say)}$$

As $\theta(u)$ is monotonic non-increasing by Mean value theorem we have for some ξ with $t < \xi < \pi$

$$\wedge_1(t)=\int_t^{\pi}\theta(u)\sin\left[\left(n+\tfrac{1}{2}+\tfrac{\delta}{2}\right)u-\frac{\pi\delta}{2}\right]\cos t(u)du$$

$$=\theta(t)\int_t^{\xi}\sin\left[\left(n+\tfrac{1}{2}+\tfrac{\delta}{2}\right)u-\frac{\pi\delta}{2}\right]\cos t(u)du \qquad (t<\xi<\pi)$$

$$=\theta(t)\left\{\left[\frac{-\cos\left[\left(n+\tfrac{1}{2}+\tfrac{\delta}{2}\right)u-\frac{\pi\delta}{2}\right]}{\left(n+\tfrac{1}{2}+\tfrac{\delta}{2}\right)}\cos t(u)\right]_{u=t}^{\xi}-\int_t^{\xi}\frac{\cos\left[\left(n+\tfrac{1}{2}+\tfrac{\delta}{2}\right)u-\frac{\pi\delta}{2}\right]\sin t(u)}{\left(n+\tfrac{1}{2}+\tfrac{\delta}{2}\right)}t'(u)du\right\}$$

$$=\frac{O(1)\log^{\beta}\left(\frac{a\pi}{2t}\right)}{nt^{\delta+1}}$$

since

$$\theta(t)=O(1)\frac{\log^{\beta}\left(\frac{a\pi}{2t}\right)}{t^{\delta+1}},\ t'(u)=\frac{O(1)}{\left(\log\frac{2\pi}{u}\right)^{2}}$$

Similarly

$$\wedge_2(t)=O(1)\frac{\log\left(\frac{a\pi}{2t}\right)}{nt^{\delta+1}}$$

Combining the estimates of $\wedge_1(t)$ and $\wedge_2(t)$

$$J_1=O(1)\frac{\log^{\beta}\left(\frac{a\pi}{2t}\right)}{n^{\delta+1}t^{\delta+1}(\log n)^{\beta}}$$

Now

$$J_2=\frac{O(1)}{n}\int_t^{\pi}\frac{du}{(2\sin\tfrac{u}{2})^2}=\frac{O(1)}{n}\int_t^{\pi}\frac{du}{u^2}=\frac{O(1)}{nt}$$

Now using the estimates of J_1 and J_2 in $\wedge(t)=J_1+J_2$ the first result of Lemma 3(ii) is ensured.

Case II

For $\delta=-1,-2...,\beta>0$ the proof of second estimates of Lemma 3(ii) can be verified parallel to the proof of Case I.

Lemma 4

(i) $V_0^{\pi/n}(\phi)=\dfrac{O(1)}{n^{\delta+1}(\log n)^{\beta}}\displaystyle\sum_{k=1}^{n}k^{\delta}\log^{\beta}\left(\frac{ak}{2}\right)V_0^{\pi/k}(\phi)$

(ii) $V_0^{\pi}(\phi)=O(1)\displaystyle\sum_{k=1}^{n}k^{\delta}\log^{\beta}\left(\frac{ak}{2}\right)V_0^{\pi/k}(\phi)$

(iii) $\dfrac{1}{n^{\delta+1}(\log n)^{\beta}}\displaystyle\int_{\pi/n}^{\pi}\frac{\log^{\beta}(\frac{a\pi}{2t})}{t^{\delta+1}}|d\phi(t)|$

$$=\frac{1}{n^{\delta+1}(\log n)^{\beta}}\sum_{k=1}^{n}\log^{\beta}(\frac{ak}{2})k^{\delta}V_0^{\pi/k}(\phi)$$

(iv) $\frac{1}{n}\int_{\pi/n}^{\pi}\frac{|d\phi(t)|}{t}=\frac{O(1)}{n}\sum_{k=1}^{n}V_0^{\pi/k}(\phi)$

Proof of Lemma 4(i)

Applying Lemma 1, we get

$$\sum_{k=1}^{n}k^{\delta}\log^{\beta}\left(\frac{ak}{2}\right)V_0^{\pi/k}(\phi)\geq V_0^{\pi/n}(\phi)\sum_{k=1}^{n}k^{\delta}\log^{\beta}\left(\frac{ak}{2}\right)\approx\frac{n^{\delta+1}}{\delta+1}\left(\log\frac{an}{2}\right)^{\beta}V_0^{\pi/n}(\phi)$$

from which the result follows

Proof of 4(ii) is trivial

Proof of 4(iii)

Integrating by parts, we get

$$\int_{\pi/n}^{\pi}\log^{\beta}\left(\frac{a\pi}{2t}\right)\frac{|d\phi(t)|}{t^{\delta+1}}$$

$$\leq\int_{\pi/n}^{\pi}\log^{\beta}\left(\frac{a\pi}{2t}\right)\frac{dV_0^t\phi}{t^{\delta+1}}$$

$$=\frac{\log^{\beta}\left(\frac{a}{2}\right)V_0^{\pi}(\phi)}{\pi^{\delta+1}}-\frac{\log^{\beta}\left(\frac{an}{2}\right)V_0^{\pi/n}(\phi)n^{\delta+1}}{\pi^{\delta+1}}$$

$$+\int_{\pi/n}^{\pi}\frac{\left[\beta\log^{\beta-1}\left(\frac{a\pi}{2t}\right)t^{-1}t^{\delta+1}-(\delta+1)t^{\delta}\log^{\beta}\left(\frac{a\pi}{2t}\right)\right]V_0^t(\phi)dt}{t^{2\delta+2}}$$

$$=\frac{\log^{\beta}\left(\frac{a}{2}\right)V_0^{\pi}(\phi)}{\pi^{\delta+1}}-\frac{\log^{\beta}\left(\frac{an}{2}\right)V_0^{\pi/n}(\phi)}{\pi^{\delta+1}}n^{\delta+1}+O(1)\int_{\pi/n}^{\pi}\frac{\log^{\beta}\left(\frac{a\pi}{2t}\right)V_0^t(\phi)dt}{t^{\delta+2}}\qquad(7)$$

Putting $u=\pi/t$ in the last integral, we have

$$\int_{\pi/n}^{\pi}\frac{\log^{\beta}\left(\frac{a\pi}{2t}\right)}{t^{\delta+2}}V_0^t(\phi)dt=\frac{1}{\pi^{\delta+1}}\int_{1}^{n}\log^{\beta}\left(\frac{au}{2}\right)V_0^{\pi/u}(\phi)u^{\delta}du$$

$$=\frac{1}{\pi^{\delta+1}}\sum_{k=1}^{n-1}\int_{k}^{k+1}\log^{\beta}\left(\frac{au}{2}\right)V_0^{\pi/u}(\phi)u^{\delta}du$$

$$\leq\frac{1}{\pi^{\delta+1}}\sum_{k=1}^{n-1}V_0^{\pi/k}(\phi)\int_{k}^{k+1}u^{\delta}\log^{\beta}\left(\frac{au}{2}\right)du$$

$$=O(1)\sum_{k=1}^{n}\log^{\beta}\left(\frac{ak}{2}\right)V_0^{\pi/k}(\phi)k^{\delta}\qquad(8)$$

Collecting the results from (7) and (8) and applying Lemma 4(i) and Lemma 4(ii), we obtain the desired estimate. We omit the proof of (iv) as its proof is similar to that of 4(iii).

4. Proof of the Theorem

From (2)

$$W_n(f,x)=\frac{1}{P_n}\sum_{k=0}^{n}P_{n-k}s_k(f,x)$$

$$=\frac{1}{P_n}\sum_{k=0}^{n}P_{n-k}\begin{bmatrix} s_k(f,x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\} \\ +\tfrac{1}{2}\{f(x+0)+f(x-0)\} \end{bmatrix}$$

$$=> W_n(f,x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

$$=\frac{1}{P_n}\sum_{k=0}^{n}P_{n-k}\left\{s_k(f,x)-\tfrac{1}{2}\left(f(x+0)+f(x-0)\right)\right\} \qquad (9)$$

Substituting the integral representation of $s_k(f,x)$ (see [6], Vol.I p 92 – 95) that is

$$s_k(f,x)=\frac{1}{\pi}\int_0^{\pi}D_k(t)\{f(x+t)+f(x-t)\}dt$$

in (3.1) and using $\frac{2}{\pi}\int_0^{\pi}D_k(t)dt=1$ we have

$$A_n(f,x)=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\left[\frac{1}{\pi}\int_0^{\pi}D_k(t)\{f(x+t)+f(x-t)\}dt-\frac{1}{\pi}\int_0^{\pi}D_k(t)\{f(x+0)+f(x-0)\}dt\right]$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}D_k(t)\phi(t)dt \qquad (10)$$

$$=\frac{1}{\pi}\int_0^{\pi}K_n(t)\phi(t)dt$$

$$=\frac{1}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right)K_n(t)\phi(t)dt$$

In the special case when $p_n=A_n^{\delta-1,\beta}$, (4.2) reduces to

$$A_n^{\delta,\beta}(f,x)=K_n^{\delta,\beta}(f,x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

$$=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right]K_n^{\delta,\beta}(t)\phi(t)dt$$

$$=\frac{1}{\pi}[I_1+I_2] \qquad (11)$$

$$I_1=\int_0^{\pi/n}K_n^{\delta,\beta}(t)\phi(t)dt$$

$$=O(1)n\int_0^{\pi/n}|\phi(t)|dt$$

$$=O(1)n\int_0^{\pi/n}V_0^t(\phi)dt$$

$$=O(1)V_0^{\pi/n}(\phi) \qquad (12)$$

By using Lemma 2(i)

Further Integrating by parts and using Lemma 3(ii), we get

$$I_2=\int_{\pi/n}^{\pi} K_n^{\delta,\beta}(t)\phi(t)dt$$

$$=-\int_{\pi/n}^{\pi} \phi(t)d\wedge(t)$$

$$=\phi(\frac{\pi}{n})\wedge(\frac{\pi}{n})+\int_{\pi/n}^{\pi}\wedge(t)d\phi(t)$$

$$=O(1)V_0^{\pi/n}(\phi)+O(1)\int_{\pi/n}^{\pi}\left\{\frac{\left(\log\frac{a\pi}{2t}\right)^{\beta}}{n^{\delta+1}(\log n)^{\beta}t^{\delta+1}}+\frac{1}{nt}\right\}\left|d\phi(t)\right|$$

$$=O(1)V_0^{\pi/n}(\phi)+O(1)\frac{1}{n^{\delta+1}(\log n)^{\beta}}\int_{\pi/n}^{\pi}\frac{\left(\log\frac{a\pi}{2t}\right)^{\beta}}{t^{\delta+1}}\left|d\phi(t)\right|+O(1)\frac{1}{n}\int_{\pi/n}^{\pi}\frac{\left|d\phi(t)\right|}{t} \quad (13)$$

Lastly using (11), (12) and (13) and applying (i), (iii) and (iv) part of Lemma4, we get for $-1<\delta<0,\beta\in R$

$$A_n^{\delta,\beta}(f,x)=O(1)\frac{1}{n^{\delta+1}(\log n)^{\beta}}\sum_{k=1}^{n}k^{\delta}(\log\frac{ak}{2})^{\beta}V_0^{\pi/k}(\phi)+O(1)\frac{1}{n}\sum_{k=1}^{n}V_0^{\pi/k}(\phi)$$

and this completes part(a) of the theorem.

Part (b): For $\delta=-1,\beta>0$ integrating by parts and using Lemma 3(ii), we have

$$I_2=O(1)(\log n)V_0^{\pi/n}(\phi)+O(1)\int_{\pi/n}^{\pi}\frac{\left(\log\frac{a\pi}{2t}\right)^{\beta}}{(\log n)^{\beta-1}}dV_0^{t}(\phi)+O(1)\frac{1}{n}\int_{\pi/n}^{\pi}t^{-1}dV_0^{t}(\phi) \quad (14)$$

Using (4.3), (4.4) and (4.6), we get

$$A_n^{\delta,\beta}(f,x)=O(1)(\log n)V_0^{\pi/n}(\phi)+\frac{O(1)}{(\log n)^{\beta-1}}\int_{\pi/n}^{\pi}(\log\frac{a\pi}{2t})^{\beta}dV_0^{t}(\phi)+\frac{O(1)}{n}\int_{\pi/n}^{\pi}\frac{dV_0^{t}(\phi)}{t}$$

(15)

Now, integrating by parts

$$\int_{\pi/n}^{\pi}t^{-1}dV_0^{t}(\phi)$$

$$=\frac{V_0^{\pi}(\phi)}{\pi}-\frac{V_0^{\pi/n}(\phi)}{\pi/n}+\int_{\pi/n}^{\pi}t^{-2}V_0^{t}(\phi)dt$$

$$\leq\frac{V_0^{\pi}(\phi)}{\pi}+\int_{\pi/n}^{\pi}t^{-2}V_0^{t}(\phi)dt$$

$$=\frac{V_0^{\pi}(\phi)}{\pi}+\frac{1}{\pi}\int_{1}^{n}V_0^{\pi/t}(\phi)dt$$

$$\leq\frac{V_0^{\pi}(\phi)}{\pi}+\frac{1}{\pi}\sum_{k=1}^{n}V_0^{\pi/k}(\phi)$$

$$\leq\frac{2}{\pi}\sum_{k=1}^{n}V_0^{\pi/k}(\phi) \quad (16)$$

Since $V_0^{\pi/k}(\phi) < \sum_{k=1}^{n} V_0^{\pi/k}(\phi)$

Next, integrating by parts

$$\int_{\pi/n}^{\pi} (\log\frac{a\pi}{2t})^{\beta} dV_0^{t}(\phi)$$

$$= (\log\frac{a\pi}{2})^{\beta} V_0^{\pi}(\phi) - (\log\frac{an}{2})^{\beta} V_0^{\pi/n}(\phi) + \beta \int_{\pi/n}^{\pi} \frac{(\log\frac{a\pi}{2t})^{\beta-1}}{t} V_0^{t}(\phi) dt$$

$$\leq (\log\frac{a\pi}{2})^{\beta} V_0^{\pi}(\phi) + \beta \int_{\pi/n}^{\pi} \frac{(\log\frac{a\pi}{2t})^{\beta-1}}{t} V_0^{t}(\phi) dt$$

$$= (\frac{a\pi}{2})^{\beta} V_0^{\pi}(\phi) + \beta \int_{1}^{n} (\log\frac{at}{2})^{\beta-1} V_0^{\pi/t}(\phi) \frac{dt}{t}$$

$$\leq (\frac{a\pi}{2})^{\beta} V_0^{\pi}(\phi) + \beta \sum_{k=1}^{n} \int_{k}^{k+1} V_0^{\pi/t}(\phi) (\log\frac{at}{2})^{\beta-1} \frac{dt}{t}$$

$$\leq (\frac{a\pi}{2})^{\beta} V_0^{\pi}(\phi) + \beta \sum_{k=1}^{n} V_0^{\pi/k}(\phi) \frac{(\log\frac{ak}{2})^{\beta-1}}{k}$$

$$= O(1) \sum_{k=1}^{n} V_0^{\pi/k}(\phi) \frac{(\log\frac{ak}{2})^{\beta-1}}{k} \tag{17}$$

Collecting the results from (4.7), (4.8) and (4.9), we get

$$A_n^{\delta,\beta}(f,x) = O(1)(\log n) V_0^{\pi/n}(\phi) + O(1) \frac{1}{(\log n)^{\beta-1}} \sum_{k=1}^{n} V_0^{\pi/k}(\phi) \frac{(\log\frac{ak}{2})^{\beta-1}}{k} + \frac{O(1)}{n} \sum_{k=1}^{n} V_0^{\pi/k}(\phi) \tag{18}$$

We have

$$\sum_{k=1}^{n} V_0^{\pi/k}(\phi) \frac{(\log\frac{ak}{2})^{\beta-1}}{k} \geq V_0^{\pi/n}(\phi) \sum_{k=1}^{n} \frac{(\log\frac{ak}{2})^{\beta-1}}{k},$$

which ensures that

$$V_0^{\pi/n}(\phi) = \frac{O(1)}{(\log n)^{\beta}} \sum_{k=1}^{n} V_0^{\pi/k}(\phi) \frac{(\log\frac{ak}{2})^{\beta-1}}{k} \tag{19}$$

As $\frac{(\log\frac{ak}{2})^{\beta-1}}{k}$ is monotonic increasing for $1 \leq k \leq n_0 = \left[\frac{2e^{\beta-1}}{a}\right]$ and monotonic decreasing for $k > n_0$, we have

$$\sum_{k=1}^{n} V_0^{\pi/k}(\phi) \frac{(\log\frac{ak}{2})^{\beta-1}}{k} \geq \frac{(\log\frac{an}{2})^{\beta-1}}{n} \sum_{k=1}^{n} V_0^{\pi/k}(\phi)$$

which ensures

$$\sum_{k=1}^{n} V_0^{\pi/k}(\phi) = O(1)\frac{n}{(\log\frac{an}{2})^{\beta-1}}\sum_{k=1}^{n} V_0^{\pi/k}(\phi)\frac{(\log\frac{ak}{2})^{\beta-1}}{k} \tag{20}$$

Finally collecting the results from (4.10), (4.11) and (4.12), we get

$$A_n^{\delta,\beta}(f,x) = O(1)\frac{1}{(\log n)^{\beta-1}}\sum_{k=1}^{n} V_0^{\pi/k}(\phi)\frac{(\log\frac{ak}{2})^{\beta-1}}{k}$$

and this completes the proof of part(b) of the Theorem.

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