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A Result Related To The Value Distribution Of Gamma Functions.

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***Abstract**: In this paper we have extended a result of Nevanlinna theory to Euler's gamma function which is known to be a meromorphic function.*
***Key Words**: Nevanlinna theory, Euler's gamma function.*

I. Introduction And Main Results

Let $\Gamma(z)$ be the Euler's gamma function defined by,

$$\Gamma(z) = \frac{e^{-vz}}{z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}}$$

Where $\gamma = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k} - \log n\right)$, the Euler's constant .

Clearly $\Gamma(z)$ is a meromorphic function with simple poles $\{-k\}_{k=0}^{\infty}$ and $\Gamma(z) \neq 0$ for any $z \in C$.

In 1999, Zhuan Ye[2] has proved the following result.

Theorem (A) : with usual notations,

i) $T(r, \Gamma) = (1 + o(1))\ \frac{r}{\pi}\log r$

ii) $\delta(o,\Gamma) = 1$, $\delta(\infty,\Gamma) = 1$

iii) $\delta(a,\Gamma) = 0$ for $a \neq 0, \infty$

Proceeding on the same lines,we can observe the following.

since $\delta(a,\Gamma) = 0$ for $a \neq 0, \infty$ and $\delta(a,\Gamma) = 1$ and $\delta(\infty,\Gamma) = 1$,

Using the basics of Nevalinna theory, we can easily prove that,

i) $\Theta(a, \Gamma) = 0$ for $a \neq 0, \infty$

ii) $\Theta(0, \Gamma) = 0$ and $\Theta(\infty,\Gamma) = 1$.

We wish to establish the following result.

Theorem : Let $\Gamma(z)$ be the Euler's gamma function.Then

$T(r, \Gamma^{(n)}) \sim (n+1)\, T(r, \Gamma)$ where n is any positive integer, as $r \to \infty$ outside a set of finite linear measure.

Proof: Clearly $\Gamma(z)$ is a meromorphic function with simple poles

$\{-k\}_{k=0}^{\infty}$ and $\Gamma(z) \neq 0$ for any $z \in C$.

Therefore, $$\overline{N}\left(r, \frac{1}{\Gamma}\right) = 0$$

Using the basics of Nevanlinna theory, we have

$m\left(r, \frac{\Gamma^{|}}{\Gamma}\right) \leq 0$ { $\log T(r, \Gamma) + O(\log r)$ as $r \to \infty$ outside a set of finite linear measure.

By induction on n, we can prove that,

$$m\left(r, \frac{\Gamma^{(n)|}}{\Gamma}\right) \leq 0 \ \{\log T(r, \Gamma) + O(\log r) \text{ for all finite n.}$$

Since $N\left(r, \frac{\Gamma^{(n)|}}{\Gamma}\right) = n\,\overline{N}(r,\Gamma) + \overline{N}\left(r, \frac{1}{\Gamma}\right)$

$= n\,\overline{N}(r,\Gamma)$, since $\overline{N}\left(r, \frac{1}{\Gamma}\right) = 0$.

We have, $T\left(r, \frac{\Gamma^{(n)}}{\Gamma}\right) \leq n\,\overline{N}(r,\Gamma) + O\,\{\log T(r, \Gamma)\} + O(\log r)\}$

Then, $T(r, \Gamma^{(n)}) = T\left(r, \frac{\Gamma^{(n)}}{\Gamma}, \Gamma\right)$

$\leq T\left(r, \frac{\Gamma^{(n)}}{\Gamma}\right) + T(r, \Gamma)$

$\leq T(r, \Gamma) + n\,\overline{N}(r,\Gamma) + O\,\{\{\log T(r, \Gamma)\} + O(\log r)\}$ ----------(1)

Conversely, $T(r, \Gamma) = T\left(r, \frac{\Gamma^{(n)}}{\Gamma}, \Gamma^{(n)}\right)$

$\leq T\left(r, \frac{\Gamma}{\Gamma^{(n)}}\right) + T(r, \Gamma^{(n)}) + O(1)$

$\leq T(r, \Gamma^{(n)}) + n\,\overline{N}(r,\Gamma) + O\{\log T(r, \Gamma)\} + O(\log r)$ ----------(2)

From (1) and (2), We have

$| T(r, \Gamma^{(n)}) - T(r, \Gamma) | \leq n\,\overline{N}(r,\Gamma) + O\{\log T(r, \Gamma)\} + O(\log r)$

On simplification, we get $\lim_{n\to\infty} \frac{T(r,\Gamma^{(n)})}{T(r,\Gamma)} = n+1$

Or **$T(r, \Gamma^{(n)}) \sim (n+1)\, T(r, \Gamma)$**

Hence the result.

References

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