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Fixed Point Theorem For ∅ - Wekaly Expansive Mappings And R-Wekaly Commuting Mappings In Metric Spaces

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Abstract: *In this paper, we prove common fixed point theorems for φ-weakly expansive mappings, which generalize and extend the results of S. M. Kang[10] using the concept of weak reciprocal continuity in metric spaces. we introduce the concept of φ-weakly expansive mappings.*
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I. Introduction

In 1997, Alber and Guerre-Delabriere [11] introduced the notion of φ-weakly contraction. We introduce the notion of φ-weakly expansive mappings in metric space, In 1986, Jungck [2] introduced the notion of compatible mappings, In 1994, Pant [4] introduced the notion of R-weak commuativity in metric spaces to extend the scope of the study of common fixed point theorems from the class of weakly commuting mappings to wider class of R-weakly commuting mappings. in 1997, Pathak et al. [3] improved the notion of R-weakly commuting mappings to R-weakly commuting mappings of type (A_f) and of type (A_g). In 1998 and 1999, Pant [5], [6] introduced a new notion of continuity, known as reciprocal continuity, Recently, Pant et al. [7] generalized the notion of reciprocal continuity to weak reciprocal continuity, In 2012, Manro and Kuman [9] proved the following fixed point theorem in complete metric spaces: In 1922, Banach proved a common fixed point theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This result of Banach is known as Banachs fixed point theorem or Banach contraction principle.

II. Preliminaries

Definition: Let F be a self mapping of a metric space (X, d). Then F is said to be expansive if there exists a real number h > 1 such that d(Fx, Fy) ≥ hd(x, y) for all x, y ∈ X.

Definition: Let F be a self mapping of a metric space (X, d). Then F is said to be ϕ-weakly contraction if there exists a continuous mapping $\emptyset : [0, \infty) \to [0, \infty)$ with $\emptyset(0) = 0$ and $\emptyset(t) < t$ for all t > 0 such that

$$d(Fx, Fy) \leq d(x, y) - \emptyset(d(x, y)), \text{ for all } x, y \in X.$$

Definition: Let F be a self mapping of a metric space (X, d). Then F is said to be ϕ-weakly expansive if there exists a continuous mapping $\emptyset : [0, \infty) \to [0, \infty)$ with $\emptyset(0) = 0$ and $\emptyset(t) > t$ for all t > 0 such that $d(Fx, Fy) \geq d(x, y) + \emptyset(d(x, y))$, for all x, y ∈ X.

Definition: Let F and G be two self mappings of a metric space (X, d). Then F is said to be ϕ-weakly expansive with respect to G : X → X if there exists a continuous mapping $\emptyset : [0, \infty) \to [0, \infty)$ with $\emptyset(0) = 0$ and $\emptyset(t) > t$ for all t > 0 such that $d(Fx, Fy) \geq d(Gx, Gy) + \emptyset(d(Gx, Gy))$, for all x, y ∈ X.

Definition: Let F and G be two self mappings of a metric space (X, d). Then F is said to be compatible if $d(FGx_n, GFx_n) = 0$, whenever$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = t$ for some t ∈ X. An immediate consequence is that if F and G are compatible and Fz = Gz, z is called a coincidence point of F and G, then FGz = GFz.

Definition: Let F and G be two self mapping of a metric space (X, d). Then F and G are said to be R-weakly commuting if there exists R > 0 such that d(FGx, GFx) ≤ Rd(Fx, Gx) for all x ∈ X.

Definition: Let F and G be two self mapping of a metric space (X, d). Then F and G are said to be
1. R-weakly commuting of type (A_G) if there exists R > 0 such that d(FFx, GFx) ≤ Rd(Fx, Gx) for all x ∈ X.

1. R-weakly commuting of type (A_F) if there exists some $R > 0$ such that $d(FGx, GGx) \leq Rd(Fx, Gx)$ for all $x \in X$.

Definition: Let F and G be two self mapping of a metric space (X, d). Then F and G are said to be R-weakly commuting of type (P) if there exists $R > 0$ such that $d(FFx, GGx) \leq Rd(Fx, Gx)$ for all $x \in X$.

Definition: Let F and G be two self mappings of a metric space (X, d). Then F and G are said to be reciprocally continuous if $\lim_{n\to\infty} FGx_n = Ft$ and $\lim_{n\to\infty} GFx_n = Gt$, whenever$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = t$ for some $t \in X$.
If F and G are both continuous, then they are obviously continuous, but the converse need not be true.

Definition: Let F and G be two self mappings of a metric space (X, d). Then F and G are said to be weakly reciprocally continuous if $\lim_{n\to\infty} FGx_n = Ft$ or $\lim_{n\to\infty} GFx_n = Gt$, whenever$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = t$ for some $t \in X$.
If F and G are both reciprocally continuous, then they are obviously weakly reciprocally continuous, but the converse need not be true.

III. Main Result

Fixed Point Theorem For ∅ - Weakly Expansive Mapping

Theorem 3.1: Let M and D be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying
1. $D(X) \subset M(X)$;
2. There exists a continuous mapping $\emptyset : [0, \infty) \to [0, \infty)$ with $\emptyset(0) = 0$ and $\emptyset(t) > t$ for all $t > 0$ such that

$$d(Mx, My) \geq N(Dx, Dy) + \emptyset(N(Dx, Dy))$$

Where,

$$N(Dx, Dy) = \min\{d(Dx, Dy), d(Mx, Dx), d(My, Dy), d(Mx, My)\ d(Mx, Dy)\}$$

For all $x, y \in X$.
If M and D are compatible, then M and D have a unique common fixed point in X.
Proof: Let x_0 be any point in X. Since $D(X) \subset M(X)$, there exists a sequence $\{x_n\}$ such that $Dx_n = Mx_{n+1}$. Define a sequence $\{y_n\}$ in X by

$$y_{n+1} = Dx_n = Mx_{n+1} \quad (3.1)$$

Case I : We assume that if $y_n = y_{n+1}$ for some $n \in N$, there is nothing to prove.
Case I : We assume that $y_n \neq y_{n+1}$ for all $n \in N$, we have

$$\begin{aligned} d(y_n, y_{n-1}) &= d(Mx_{n+1}, Mx_n) \\ &\geq \min\{d(Dx_{n+1}, Dx_n), d(Mx_{n+1}, Dx_{n+1}), d(Mx_n, Dx_n), d(Mx_{n+1}, Mx_n), d(Mx_{n+1}, Dx_n)\} + \\ &\quad \emptyset[\min\{d(Dx_{n+1}, Dx_n), d(Mx_{n+1}, Dx_{n+1}), d(Mx_n, Dx_n), d(Mx_{n+1}, Mx_n), d(Mx_{n+1}, Dx_n)\}] \quad (3.2) \\ &\geq \min\{d(y_{n+2}, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1}), d(y_{n+1}, y_n), d(y_{n+1}, y_{n+1})\} + \\ &\quad \emptyset[\min\{d(y_{n+2}, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1}), d(y_{n+1}, y_n), d(y_{n+1}, y_{n+1})\}] \\ &\geq d(y_{n+1}, y_n) + \emptyset(d(y_{n+1}, y_n)) \end{aligned}$$

That is,

$$d(y_n, y_{n-1}) \geq d(y_{n+1}, y_n)$$

Hence the sequence $\{d(y_{n+1}, y_n)\}$ is strictly decreasing and bounded below. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(y_{n+1}, y_n) = r$. Letting $n\to\infty$ in (3.2) we get $r \geq r + \emptyset(r)$, which is a contradiction. Hence we have $r = 0$. Therefore

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0 \quad (3.3)$$

Now we will show that $\{y_n\}$ is a Cauchy sequence.
Let $\{y_n\}$ is not a Cauchy sequence. So there exists an $\varepsilon > 0$ and the subsequence $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that minimal n(k) in the sense that $n(k) > m(k) > k$ and $d(y_{m(k)}, y_{n(k)}) > \varepsilon$. Therefore $d(y_{m(k)}, y_{n(k)-1}) \geq \varepsilon$.
By the triangular inequality, we have

$$\begin{aligned} \varepsilon &< d(y_{m(k)}, y_{n(k)}) \\ &\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)} \\ &\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}) \\ &\leq d(y_{m(k)}, y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)}) \end{aligned}$$

Letting k→∞ in the above inequality and using (3.3) we get,

$$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = \lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = \varepsilon \quad (3.4)$$

From (2), we have

$$d(y_{m(k)-1}, y_{n(k)-1}) = d(Mx_{m(k)}, Mx_{n(k)})$$
$$\geq \min\{ d(Dx_{m(k)}, Dx_{n(k)}), d(Mx_{m(k)}, Dx_{m(k)}), d(Mx_{n(k)}, Dx_{n(k)}), d(Mx_{m(k)}, Mx_{n(k)}), d(Mx_{m(k)}, Dx_{n(k)}) \}$$
$$+ \emptyset[\min\{ d(Dx_{m(k)}, Dx_{n(k)}), d(Mx_{m(k)}, Dx_{m(k)}), d(Mx_{n(k)}, Dx_{n(k)}), d(Mx_{m(k)}, Mx_{n(k)}), d(Mx_{m(k)}, Dx_{n(k)}) \}]$$

$$\geq \min\{ d(y_{m(k)+1}, y_{n(k)+1}), d(y_{m(k)}, y_{n(k)+1}), d(y_{n(k)}, y_{n(k)+1}), d(y_{m(k)}, y_{n(k)}), d(y_{m(k)}, y_{n(k)+1})\}$$
$$+\emptyset [\min\{ d(y_{m(k)+1}, y_{n(k)+1}), d(y_{m(k)}, y_{n(k)+1}), d(y_{n(k)}, y_{n(k)+1}), d(y_{m(k)}, y_{n(k)}), d(y_{m(k)}, y_{n(k)+1})\}]$$

$$\geq d(y_{m(k)}, y_{n(k)}) +\emptyset [d(y_{m(k)}, y_{n(k)})]$$

Letting k→∞, and using (3.4) we get $\varepsilon \geq \varepsilon + \emptyset(\varepsilon)$, which is contradiction, since $\emptyset(\varepsilon) > \varepsilon$. Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists a point z ϵ X such that $\lim_{n\to\infty} y_n = z$. Therefore by (3.1) we have

$$\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} Dx_n = \lim_{n\to\infty} Mx_{n+1} = z$$

Suppose that M and D are compatible mappings. Now, by weak reciprocal continuity of M and D, we obtain

$$lim_{n\to\infty} MDx_n = Mz \text{ or } lim_{n\to\infty} DMx_n = Dz.$$

Let $lim_{n\to\infty} MDx_n = Mz$. Then the compatibility of M and D gives

$$\lim_{n\to\infty} d(MDx_n, DMx_n) = 0$$

Hence ,

$$lim_{n\to\infty} DMx_n = Mz$$

Now we claim that Mz = Dz. Let Mz ≠ Dz. Fro (3.1), we get $lim_{n\to\infty} DMx_{n+1} = lim_{n\to\infty} DDx_n = Mz$. Therefore from (2), we get

$$d(Mz, MDx_n) \geq min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, MDx_n), d(Mz, DDx_n)\}$$
$$+\emptyset[min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, MDx_n), d(Mz, DDx_n)\}]$$

Letting n→∞, we get

$$\geq min\{d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Mz), d(Mz, Mz)\} +$$
$$\emptyset[min\{d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Mz), d(Mz, Mz)\}]$$
$$\geq d(Mz, Dz) + \emptyset[d(Mz, Dz)]$$
$$> 2\, d(Mz, Dz)$$

Which is a contradiction. Hence $Mz = Dz$. Again the compatibility of M and D implies that commutativity at a coincidence point. Hence $DMz = MDz = MMz = DDz$.
Using (2), we obtain

$$d(Dz, DDz) = d(Mz, MDz)$$
$$\geq min\{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)\} +$$
$$\emptyset[min\{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)\}]$$

$$\geq min\{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, DDz), d(Dz, DDz)\} +$$
$$\emptyset[min\{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, DDz), d(Dz, DDz)\}]$$

$$\geq d(Dz, DDz) + \emptyset[d(Dz, DDz)]$$

Which implies that $Dz = DDz$. Also we get $Dz = DDz = MDz$ and so Dz is a common fixed point of M and D.
Next, suppose that $lim_{n\to\infty} DMx_n = Dz$. Since $D(X) \subset M(X)$ there exists u ∈ X such that $Dz = Mu$ and therefore $lim_{n\to\infty} DMx_n = Mu$. The compatibility of M and D implies that $lim_{n\to\infty} MDx_n = Mu$. Now, we prove that Mu = Du. Let $Mu \neq Du$. By (3.1), we have

$$\lim_{n\to\infty} DMx_{n+1} = \lim_{n\to\infty} DDx_n = Mu$$

From (2), we have

$$d(Mu, MDx_n) \geq min\{d(Du, DDx_n), d(Mu, Du), d(MDx_n, DDx_n), d(Mu, MDx_n), d(Mu, DDx_n)\}$$
$$+\emptyset[min\{d(Du, DDx_n), d(Mu, Du), d(MDx_n, DDx_n), d(Mu, MDx_n), d(Mu, DDx_n)\}]$$

Letting n→∞, we get

$$d(Mu, Mu) \geq min\{d(Du, Mu), d(Mu, Du), d(Mu, Mu), d(Mu, Mu), d(Mu, Mu)\} +$$
$$\emptyset[min\ \{d(Du, Mu), d(Mu, Du), d(Mu, Mu), d(Mu, Mu), d(Mu, Mu)\}]$$
$$\geq d(Mu, Du) + \emptyset[d(Mu, Du)]$$

$$> 2\ d(Mu, Du)$$

Which is a contradiction. Hence $Mu = Du$. Again the compatibility of M and D implies that commutativity at a coincidence point. Hence $DMu = MDu = MMu = DDu$. Finally Using (2), we obtain

$$d(Du, DDu) = d(Mu, MDu)$$

$$\geq min\{d(Du, DDu), d(Mu, Du), d(MDu, DDu), d(Mu, MDu), d(Mu, DDu)\} +$$
$$\emptyset[min\ \{d(Du, DDu), d(Mu, Du), d(MDu, DDu), d(Mu, MDu), d(Mu, DDu)\}]$$
$$\geq min\{d(Du, DDu), d(Du, Du), d(DDu, DDu), d(Du, DDu), d(Du, DDu)\} +$$
$$\emptyset[min\ \{d(Du, DDu), d(Du, Du), d(DDu, DDu), d(Du, DDu), d(Du, DDu)\}]$$

$$\geq d(Du, DDu) + \emptyset[d(Du, DDu)]$$

Which implies that $Du = DDu$. Also we get $Du = DDu = MDu$ and so Du is a common fixed point of M and D.

Uniqueness: Let v and w(v ≠ w) be two common fixed point M and D. From (2), we have

$$d(v, w) = d(Mv, Mw)$$
$$\geq min\{d(Dv, Dw), d(Mv, Dv), d(Mw, Dw), d(Mv, Mw), d(Mv, Dw)\} +$$
$$\emptyset[min\ \{d(Dv, Dw), d(Mv, Dv), d(Mw, Dw), d(Mv, Mw), d(Mv, Dw)\}]$$

$$\geq min\{d(v, w), d(v, v), d(w, w), d(v, w), d(v, w)\} + \emptyset[min\ \{d(v, w), d(v, v), d(w, w), d(v, w), d(v, w)\}]$$

$$\geq d(v, w) + \emptyset(d(v, w))$$

Which implies that $v = w$. Hence M and D have a unique common fixed point.

Fixed Point Theorem For R-Weakly Commuting of Type (A_g) and Type (A_f)

Theorem 3.2: Let M and D be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying

1. D(X) ⊂ M(X);
2. There exists a continuous mapping ∅ : [0, ∞) ⟶ [0, ∞) with ∅(0) = 0 and ∅ (t) > t for all t > 0 such that

$$d(Mx, My) \geq N\ (Dx, Dy) + \emptyset(\ N\ (Dx, Dy)\ )$$

Where,

$$N(Dx, Dy) = min\{d(Dx, Dy), d(Mx, Dx), d(My, Dy), d(Mx, My)\ d(Mx, Dy)\}$$

For all x, y ∈ X. If M and D are R-weakly commuting of type (A_g) and type (A_f), then M and D have a unique common fixed point in X.

Proof: From above theorem $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists a point z ∈ X such that $lim_{n\to\infty} y_n = z$. Therefore by (3.1) we have

$$\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} Dx_n = \lim_{n\to\infty} Mx_{n+1} = z$$

Now, suppose that M and D are R-weakly commuting of type (A_f) . The weak reciprocal continuity of M and D, implies that $lim_{n\to\infty} MDx_n = Mz$ or $lim_{n\to\infty} DMx_n = Dz$.

Let $lim_{n\to\infty} MDx_n = Mz$. Then the R-weakly commuting of type (A_f) of M and D yields, $d(DDx_n, MDx_n) \leq Rd(Mx_n, Dx_n)$ and therefore $lim_{n\to\infty} d(DDx_n, Mz) \leq Rd(z,z) = 0$, that is $lim_{n\to\infty} DDx_n = Mz$.

Now we claim that Mz = Dz. Let Mz ≠ Dz. From (2), we get

$$d(Mz, MDx_n) \geq min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, MDx_n), d(Mz, DDx_n)\}$$
$$+\emptyset[min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, MDx_n), d(Mz, DDx_n)\}]$$

Letting n→∞, we get

$$\geq min\{d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Mz), d(Mz, Mz)\} +$$
$$\emptyset[min\,\{d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Mz), d(Mz, Mz)\}]$$
$$\geq\ d(Mz, Dz) +\ \emptyset[d(Mz, Dz)]$$

Which is a contradiction. Hence $Mz = Dz$.
Again by R-weakly commutativity of type (A_f) $d(DDz, MDz) \leq Rd(Dz, Mz) = Rd(z,z) = 0$ that is $DDz = MDz$.
Therefore $DMz = MDz = MMz = DDz$. Using (2), we obtain

$$d(Dz, DDz) = d(Mz, MDz)$$
$$\geq min\{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)\} +$$
$$\emptyset[min\,\{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)\}]$$

$$\geq min\{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, DDz), d(Dz, DDz)\} +$$
$$\emptyset[min\,\{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, DDz), d(Dz, DDz)\}]$$
$$\geq\ d(Dz, DDz) +\ \emptyset[d(Dz, DDz)]$$

Which implies that $Dz = DDz$. Then we also get $Dz = DDz = MDz$ and so Dz is a common fixed point of M and D.Similarly, if $lim_{n\to\infty} DMx_n = Dz$, we can easily prove.
Suppose that M and D are R-weakly commuting of type (A_g) . Again, as done above, we can easily prove that Mz is a common fixed point of M and D.

Uniqueness: From theorem 3.1, we can easily prove the uniqueness of the theorem. Hence M and D have a unique common fixed point.

Fixed Point Theorem For R-Weakly Commuting of Type (P)

Theorem 3.3: Let M and D be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying
1. D(X) ⊂ M(X);
2. There exists a continuous mapping ∅ : [0, ∞) → [0, ∞) with ∅(0) = 0 and ∅ (t) > t for all t > 0 such that

$$d(Mx, My)\ \geq\ N\,(Dx, Dy)\ +\ \emptyset(\,N\,(Dx, Dy)\,)$$

Where,

$$N(Dx, Dy)\ =\ min\{d(Dx, Dy), d(Mx, Dx), d(My, Dy), d(Mx, My)\ d(Mx, Dy)\}$$

For all x, y ∈ X.
If M and D are R-weakly commuting of type (P), then M and D have a unique common fixed point in X.
Proof: From above theorem $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists a point z ∈ X such that $lim_{n\to\infty} y_n = z$. Therefore by (3.1) we have

$$\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} Dx_n = \lim_{n\to\infty} Mx_{n+1} = z$$

Now, suppose that M and D are R-weakly commuting of type (P) . The weak reciprocal continuity of M and D, implies that $lim_{n\to\infty} MDx_n = Mz$ or $lim_{n\to\infty} DMx_n = Dz$. Let $lim_{n\to\infty} MDx_n = Mz$. Then the R-weakly commutativity of type (P) of M and D yields, $d(MMx_n, DDx_n) \leq Rd(Mx_n, Dx_n)$ and therefore $lim_{n\to\infty} d(MMx_n, DDx_n) \leq Rd(z,z) = 0$ That is $lim_{n\to\infty}(MMx_n, DDx_n) = 0$. Using (3.1), we have $MDx_{n-1} = MMx_n \to Mz$ and $DDx_n \to Mz\ an\ n \to \infty$.

Now we claim that $Mz = Dz. Let\ Mz \neq Dz$. From (2), we get

$d(Mz, MDx_n) \geq min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, MDx_n), d(Mz, DDx_n)\}$
$+\emptyset[min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, MDx_n), d(Mz, DDx_n)\}]$
Letting n→∞, we get
$\geq min\{d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Mz), d(Mz, Mz)\} +$
$\emptyset[min\ \{d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Mz), d(Mz, Mz)\}]$
$\geq\ d(Mz, Dz) +\ \emptyset[d(Mz, Dz)]$
Which is a contradiction. Hence $Mz\ =\ Dz$. Again by using the R-weakly commutativity of type (P), we have $d(MMz, DDz) \leq Rd(Mz, Dz) = 0$ that is DDz = MMz.
Therefore $DMz\ =\ MDz\ =\ MMz\ =\ DDz$.
Using (2), we obtain

$d(Dz, DDz)\ =\ d(Mz, MDz)$
$\geq min\{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)\} +$
$\emptyset[min\ \{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)\}]$

$\geq min\{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, DDz), d(Dz, DDz)\} +$
$\emptyset[min\ \{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, DDz), d(Dz, DDz)\}]$

$\geq\ d(Dz, DDz) +\ \emptyset[d(Dz, DDz)]$

Which implies that $Dz\ =\ DDz$. Then we also get $Dz\ =\ DDz\ =\ MDz$ and so Dz is a common fixed point of M and D. Similarly, if $lim_{n\to\infty} DMx_n = Dz$, we can easily prove.

Uniqueness: From theorem 3.1, we can easily prove the uniqueness of the theorem. Hence M and D have a unique common fixed point.

Corollary: Let M be surjective self mappings of a complete metric space (X, d) satisfying
1. there exists a continuous mapping ∅ : [0, ∞) → [0, ∞) with ∅(0) = 0 and ∅ (t) > t for all t > 0 such that

$$d(Mx, My)\ \geq\ N\ (x, y)\ +\ \emptyset(\ N\ (x, y)\ )$$

Where,
$N(x, y)\ =\ min\{d(x, y), d(Mx, x), d(My, y), d(Mx, My)\ d(Mx, y)\}$ For all x, y ∈ X.
Then M have a unique fixed point in X.

Example : Let X = [0,1] be equipped with the Euclidean metric d(x,y) = $|x - y|$ for all x,y ∈ X. define M,D : X → X by Mx = 8x and Dx = 2x. so DX = [0,2] ⊂ [0, 8] = MX.
Let $\{x_n\}$ be a sequence in X such that $x_n\ = \frac{1}{n}$ for each n. Also ,let ∅ : [0, ∞) → [0, ∞) be defined by ∅(t) = 2t for all t ∈ [0, ∞). Here, $Mx_n = \frac{1}{n} = \frac{8}{n}$, so $lim_{n\to\infty} Mx_n\ = 0$.
Also $lim_{n\to\infty} MDx_n = lim_{n\to\infty} M\frac{2}{n} = lim_{n\to\infty}\frac{16}{n} = 0 = M(0)$, so we can say that M and D are weakly reciprocally continuous. Also, d(Mx, My) = 8$|x - y|$, d(Dx, Dy) = 2$|x - y|$ and

$$\emptyset\big(d(Dx\ Dy)\big) = 4|x - y|$$

Clearly,
d(Mx, My)= $8|x - y|$
$\geq 2|x - y| + \emptyset(2|x - y|)$
$\geq 2|x - y| + 4|x - y|)$
$\geq 6|x - y|$.
Again, $d(DDx_n FDx_n) = \left(D\frac{2}{n}, M\frac{2}{n}\right)$
$= d\left(\frac{4}{n}, \frac{16}{n}\right) = \frac{8}{n}$
$= d(Mx_n, Dx_n) = d\left(\frac{8}{n}, \frac{2}{n}\right) = \frac{6}{n}$
Clearly,
$d(DDx_n, MDx_n) < Rd(Mx_n, Dx_n)$, where R > 4.
Hence M and D are R-weakly commuting mappings of type (A_f). Also M and D are compatible. So all the conditions of Theorem 3.1 and 3.2 are satisfied and 0 is the unique fixed point of M and D.

IV. Conclusion

In this paper, we have presented common fixed point theorems in metric spaces through concept of ∅ -weakly expansive mappings and R – weakly commuting mappings.

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