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Total Dominating Sets and Total Domination Polynomials of Square of Cycles

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Abstract: *Let G= (V , E) be a simple connected graph. A set $S \subseteq V$ is a total dominating set of G if every vertex is adjacent to an element of S. Let $D_t(C_n^2,i)$ be the family of all total dominating sets of the graph C_n^2, $n \geq 6$ with cardinality i, and let $d_t(C_n^2,i) = |D_t(C_n^2, i)|$. In this paper we construct $d_t(C_n^2,i)$, and obtain the polynomial $D_t(c_n^2, x) = \sum_{i=\gamma_t(C_n^2)}^{n} d_t(C_n^2,i)x^i$ which we call total domination polynomial of C_n^2, $n \geq 6$ and obtain some properties of this polynomial.*

Keywords*: square of cycle, total domination set, total domination polynomial*

I. Introduction

Let G = (V,E) be a simple connected graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in V-S is adjacent to atleast one vertex in S. A set $S \subseteq V$ is a total dominating set if every vertex of the graph is adjacent to an element of S. The total domination number of a graph G is the minimum cardinality of a total dominating set in G, and it is denoted by $\gamma_t(G)$. Obviously $\gamma_t(G) < |V|$. A cycle on n vertices is denoted by C_n is a path which originates and concludes at the same vertex.The length of a cycle is the number of edges in the cycle. The square of a simple connected graph G is a graph with same set of vertices of G and an edge between two vertices if and only if there is a path of length atmost two between them. It is denoted by G^2. We use the notation $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also we denote the set { 1,2,…………,n} by [n], throughout this paper.

Let C_n^2, $n \geq 6$ be the square of the cycle C_n, $n \geq 6$ and let $D_t(C_n^2, i)$ be the family of total dominating sets of the graph C_n^2, $n \geq 6$ with cardinality i and let $d_t(C_n^2,i) = |D_t(C_n^2,i)|$. The total domination polynomial $D_t(C_n^2, x)$ of C_n^2, $n \geq 6$ is defined as
$D_t(C_n^2, x) = \sum_{i=\gamma_t(C_n^2)}^{n} d_t(C_n^2,i)x^i$, where $\gamma_t(C_n^2)$ is the total domination number of C_n^2.

II. Total Dominating Sets Of Square Of Cycles

Let $D_t(C_n^2,i)$ be the family of total dominating sets of C_n^2, $n \geq 6$ with cardinality i. We will investigate total dominating sets of C_n^2, $n \geq 6$.

Lemma 2.1

$$\gamma_t(C_n^2) = \begin{cases} \lceil \frac{n}{5} \rceil + 2 & \text{if } n \equiv 0 \pmod 5 \\ \lceil \frac{n}{5} \rceil + 1 & \text{if } n \not\equiv 0 \pmod 5 \end{cases}$$

Lemma 2.2

Let C_n^2, $n \geq 6$ be the square of cycle C_n with $|V(C_n^2)| = n$. Then $d_t(C_n^2,i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 1$ or $i > n$ and $d_t(C_n^2,i) > 0$ if $\lceil \frac{n}{5} \rceil + 1 \leq i \leq n$.

Proof:

If $n \equiv 0 \pmod 5$, then the total domination number of the square of cycle C_n^2 is $\gamma_t(C_n^2) = \lceil \frac{n}{5} \rceil + 2$. Therefore $d_t(C_n^2,i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 1$ or $i > n$.

Also $d_t(C_n^2,i) > 0$ if $\lceil \frac{n}{5} \rceil + 1 \leq i \leq n$.

On the other hand, if $n \not\equiv 0 \pmod 5$, then the total domination number of C_n^2 is $\gamma_t(C_n^2) = \lceil \frac{n}{5} \rceil + 1$. Therefore $d_t(C_n^2,i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 1$ or $i > n$ and $d_t(C_n^2,i) > 0$ if $\lceil \frac{n}{5} \rceil + 1 \leq i \leq n$.

Hence, in general, we have $d_t(C_n^2,i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 1$ or $i > n$ and
$d_t(C_n^2,i) > 0$ if $\lceil \frac{n}{5} \rceil + 1 \leq i \leq n$.

Lemma 2.3

Let C_n^2 ,$n \geq 6$ be the square of cycle with $|V(C_n^2)| = n$. Then we have

(i) $D_t(C_n^2, i) = \varphi$ if $i < \gamma_t(C_n^2)$ or $i > n$.

(ii) $D_t(C_n^2, x)$ has no constant term and first degree terms.

(iii) $D_t(C_n^2, x)$ is a strictly increasing function on $[0,\infty)$.

Proof of (i)

Since C_n^2 has n vertices, there is only one way to choose all these vertices.Therefore $d_t(C_n^2,n) = 1$.

Out of these n vertices, every combination of n-1 vertices can dominate totally only if $\delta(C_n^2) > 1$.

Therefore $d_t(C_n^2,n-1) = n$ if $\delta(C_n^2) > 1$.

Therefore $D_t(C_n^2, i) = \varphi$ if $i < \gamma_t(C_n^2)$ and $D_t(C_n^2, n+k) = \varphi$, $k = 1,2,3,\ldots\ldots\ldots$..

Thus we have $d_t(C_n^2, i) = 0$ for $i < \gamma_t(C_n^2)$ and $d_t(C_n^2, n+i) = 0$,for

$i = 1,2,3,\ldots\ldots\ldots$. ■

Proof of (ii)

A single vertex of C_n^2 cannot totally dominate all the vertices of C_n^2,$n \geq 6$.So the set of all vertices of C_n^2 is totally dominated by atleast two of the vertices of C_n^2 Hence the total domination polynomial has no constant term as well as first degree term. ■

Proof of (iii)

By the definition of total domination, every vertex of C_n^2 is adjacent to an element of total dominating set.

That is $D_t(C_n^2, x) = \sum_{i=\gamma_t(C_n^2)}^{n} |D_t(C_n^2,i)|\, x^i$

Therefore $D_t(C_n^2, x)$ is a strictly increasing function on $[0, \infty)$. ■

Lemma 2.4

Let C_n^2 , $n \geq 6$ be the square of cycle with $|V(C_n^2)| = n$. Then we have

(i) If $D_t(C_{n-1}^2, i-1) = D_t(C_{n-3}^2, i-1) = \varphi$, then $D_t(C_{n-2}^2, i-1) = \varphi$

(ii) If $D_t(C_{n-1}^2, i-1) \neq \varphi$ and $D_t(C_{n-3}^2, i-1) \neq \varphi$, then $D_t(C_{n-2}^2, i-1) \neq \varphi$

(iii) If $D_t(C_{n-1}^2, i-1) = \varphi$, $D_t(C_{n-2}^2, i-1) = \varphi$, $D_t(C_{n-3}^2, i-1) = \varphi$, $D_t(C_{n-4}^2, i-1) = \varphi$ and $D_t(C_{n-5}^2, i-1) = \varphi$, then $D_t(C_n^2,i) = \varphi$

Proof of (i)

Since $D_t(C_{n-1}^2,i-1) = \varphi$ and $D_t(C_{n-3}^2, i-1) = \varphi$

$\Rightarrow d_t(C_{n-1}^2,i-1) = 0$ and $d_t(C_{n-3}^2, i-1) = 0$

Then $i-1 < \lceil\frac{n-1}{5}\rceil+1$ or $i-1 > n-1$ and

$i-1 < \lceil\frac{n-3}{5}\rceil+1$ or $i-1 > n-3$.

If $i-1 < \lceil\frac{n-3}{5}\rceil+1$, then $i-1 < \lceil\frac{n-2}{5}\rceil+1$.

Therefore $D_t(C_{n-2}^2, i-1) = \varphi$.

If $i-1 > n-1$, then $i-1 > n-2$

Therefore $D_t(C_{n-2}^2, i-1) = \varphi$.

Hence in all the cases $D_t(C_{n-2}^2,i-1) = \varphi$ ■

Proof of (ii)

Since $D_t(C_{n-1}^2,i-1) \neq \varphi$ and $D_t(C_{n-3}^2, i-1) \neq \varphi$

$d_t(C_{n-1}^2,i-1) \neq 0$ and $d_t(C_{n-3}^2, i-1) \neq 0$

Then $\lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-1$ and

$\lceil\frac{n-3}{5}\rceil+1 \leq i-1 \leq n-3$

$\Longrightarrow \lceil\frac{n-3}{5}\rceil+1 \leq \lceil\frac{n-2}{5}\rceil+1 \leq i-1 \leq n-3 < n-2$, since $\lceil\frac{n-1}{5}\rceil+1 \leq i-1$

$\Longrightarrow \boldsymbol{\lceil\frac{n-2}{5}\rceil+1 \leq i-1 \leq n-2}$

$\Rightarrow d_t(C_{n-2}^2,i-1) \neq 0$

$\Rightarrow D_t(C_{n-2}^2,i-1) \neq \varphi$ ■

Proof of (iii)

Since,

$D_t(C_{n-1}^2,i-1) = \varphi$, $D_t(C_{n-2}^2,i-1) = \varphi$, $D_t(C_{n-3}^2,i-1) = \varphi$,$D_t(C_{n-4}^2,i-1) = \varphi$ and $D_t(C_{n-5}^2,i-1) = \varphi$

$\therefore d_t(C_{n-1}^2,i-1) = 0$,$d_t(C_{n-2}^2,i-1) = 0$, $d_t(C_{n-3}^2,i-1) = 0$, $d_t(C_{n-4}^2, i-1) = 0$ and $d_t(C_{n-5}^2, i-1) = 0$

$\Longrightarrow$ $i-1 < \lceil\frac{n-1}{5}\rceil+1$ or $i-1 > n-1$;

$i-1 < \lceil\frac{n-2}{5}\rceil+1$ or $i-1 > n-2$;

i-1 < $\lceil\frac{n-3}{5}\rceil$+1 or i-1 > n-3;

i-1 < $\lceil\frac{n-4}{5}\rceil$+1 or i-1 > n-4 and

i-1 < $\lceil\frac{n-5}{5}\rceil$+1 or i-1 > n-5

If i-1 < $\lceil\frac{n-1}{5}\rceil$+1, i-1 < $\lceil\frac{n-2}{5}\rceil$+1, i-1 < $\lceil\frac{n-3}{5}\rceil$+1, i-1 < $\lceil\frac{n-4}{5}\rceil$+1 and i-1 < $\lceil\frac{n-5}{5}\rceil$+1

$\Longrightarrow$ i < $\lceil\frac{n-1}{5}\rceil$+2; i < $\lceil\frac{n-2}{5}\rceil$+2 ; i < $\lceil\frac{n-3}{5}\rceil$+2, i < $\lceil\frac{n-4}{5}\rceil$+2and i < $\lceil\frac{n-5}{5}\rceil$+2

$\Longrightarrow$ i < $\lceil\frac{n-5}{5}\rceil$+2 ≤ $\lceil\frac{n}{5}\rceil$+2

$\Longrightarrow$ i < $\lceil\frac{n}{5}\rceil$+1

$\Longrightarrow$ $d_t(C_n^2, i) = 0$.

Therefore $D_t(C_n^2, i) = \varphi$.

If i-1 > n-1 , i-1 > n-2 , i-1 > n-3 ,i-1 > n-4 and i-1 > n-5

then i-1 > n-1 > n-2 > n-3>n-4.n-5

$\Longrightarrow$ i > n > n-1 > n-2 > n-3.n-4

$\Longrightarrow$ i > n

$\Longrightarrow$ $d_t(C_n^2, i) = 0$

$\Longrightarrow$ $D_t(C_n^2, i) = \varphi$ ■

Lemma 2.5

. Let C_n^2 ,n ≥ 6 be the square of cycle with $|V(C_n^2)|$ = n. Suppose that $D_t(C_n^2, i) \neq \varphi$, then we have

(i) $D_t(C_{n-2}^2, i-1) = \varphi$, $D_t(C_{n-3}^2, i-1) = \varphi$, $D_t(C_{n-4}^2, i-1) = \varphi$ and $D_t(C_{n-1}^2, i-1) \neq \varphi$ if and only if n = i.

(ii) $D_t(C_{n-1}^2, i-1) \neq \varphi$, $D_t(C_{n-2}^2, i-1) \neq \varphi$, $D_t(C_{n-3}^2, i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and $D_t(C_{n-5}^2, i-1) = \varphi$ if only if i = n-3.

(iii) $D_t(C_{n-1}^2, i-1) \neq \varphi$, $D_t(C_{n-2}^2, i-1) \neq \varphi$,$D_t(C_{n-3}^2, i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and $D_t(C_{n-5}^2, i-1) \neq \varphi$ if and only if $\lceil\frac{n-1}{5}\rceil+2 \leq i \leq n-4$

Proof of (i)

Suppose ,

$D_t(C_{n-2}^2, i-1) = \varphi$,$D_t(C_{n-3}^2, i-1) = \varphi$, $D_t(C_{n-4}^2, i-1) = \varphi$ and $D_t(C_{n-1}^2, i-1) \neq \varphi$

$\Longrightarrow$ $d_t(C_{n-2}^2, i-1) = 0$, $d_t(C_{n-3}^2, i-1) = 0$, $d_t(C_{n-4}^2, i-1) = 0$ and $d_t(C_{n-1}^2, i-1) \neq 0$

$\Longrightarrow$ i-1 < $\lceil\frac{n-2}{5}\rceil$+1 or i-1 > n-2 ; i-1 < $\lceil\frac{n-3}{5}\rceil$+1 or i-1 > n-2 and

i-1 < $\lceil\frac{n-4}{5}\rceil$+1 or i-1 > n-2

If i-1 < $\lceil\frac{n-4}{5}\rceil$+1 ≤ $\lceil\frac{n-1}{5}\rceil$+1 , then i-1 < $\lceil\frac{n-1}{5}\rceil$+1

$\Rightarrow$ $d_t(C_{n-1}^2, i-1) = 0$ which is a contradiction, since $d_t(C_{n-1}^2, i-1) \neq 0$

Therefore i-1 > n-2

$\Longrightarrow$ i -1 ≥ n-1

$\Longrightarrow$ i ≥ n

Also $d_t(C_{n-1}^2, i-1) \neq 0$

$\Longrightarrow$ $\lceil\frac{n-1}{5}\rceil$+1 ≤ i-1 ≤ n-1

$\Longrightarrow$ i-1 ≤ n-1

$\Longrightarrow$ i ≤ n

Hence n = i

Conversely, if i = n , then,

$D_t(C_{n-2}^2, i-1) = D_t(C_{n-2}^2, n-1) = \varphi$,

$D_t(C_{n-3}^2, i-1) = D_t(C_{n-3}^2, n-1) = \varphi$,

$D_t(C_{n-4}^2, i-1) = D_t(C_{n-4}^2, n-1) = \varphi$ and

$D_t(C_{n-1}^2, i-1) = D_t(C_{n-1}^2, n-1) \neq \varphi$, since $D_t(C_n^2, n) \neq \varphi$

Proof of (ii)

Suppose,

$D_t(C_{n-1}^2, i-1) \neq \varphi$, $D_t(C_{n-2}^2, i-1) \neq \varphi$, $D_t(C_{n-3}^2, i-1) \neq \varphi$,

$D_t(C_{n-4}^2,i-1) \neq \varphi$ and $D_t(C_{n-5}^2,i-1) = \varphi$

$\Longrightarrow d_t(C_{n-1}^2,i-1) \neq 0$, $d_t(C_{n-2}^2,i-1) \neq 0$, $d_t(C_{n-3}^2,i-1) \neq 0$, $d_t(C_{n-4}^2,i-1) \neq 0$ and $d_t(C_{n-5}^2,i-1) = 0$

$\Rightarrow \lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-1$; $\lceil\frac{n-2}{5}\rceil+1 \leq i-1 \leq n-2$; $\lceil\frac{n-3}{5}\rceil+1 \leq i-1 \leq n-3$; $\lceil\frac{n-4}{5}\rceil+1 \leq i-1 \leq n-4$ and $i-1 < \lceil\frac{n-5}{5}\rceil+1$ or $i-1 > n-5$

$\Rightarrow \lceil\frac{n-5}{5}\rceil+1 \leq \lceil\frac{n-4}{5}\rceil+1 \leq i-1$

$\Rightarrow \lceil\frac{n-5}{5}\rceil+1 \leq i-1$

$\Longrightarrow d_t(C_{n-5}^2,i-1) \neq o$ which is a contradiction , since $d_t(C_{n-5}^2,i-1) = o$.

Therefore $i-1 < \lceil\frac{n-5}{5}\rceil+1$ is not possible, so $i-1 > n-5$

$\Longrightarrow i > n-4$

$\Longrightarrow i \geq n-3$

since $d_t(C_{n-4}^2,i-1) \neq o$

$\Longrightarrow \lceil\frac{n-4}{5}\rceil+1 \leq i-1 \leq n-4$

$\Rightarrow i-1 \leq n-4$

$\Rightarrow i \leq n-3$

Hence $i = n-3$

Conversely, if $i = n-3$, then

$D_t(C_{n-1}^2,i-1) = D_t(C_{n-1}^2, n-4) \neq \emptyset$

$D_t(C_{n-2}^2,i-1) = D_t(C_{n-2}^2, n-4) \neq \emptyset$

$D_t(C_{n-3}^2,i-1) = D_t(C_{n-3}^2, n-4) \neq \emptyset$

$D_t(C_{n-4}^2,i-1) = D_t(C_{n-4}^2, n-4) \neq \emptyset$

But $D_t(C_{n-5}^2,i-1) = D_t(C_{n-5}^2, n-4) = \emptyset$ ∎

Proof of (iii)

Suppose,

$D_t(C_{n-1}^2, i-1) \neq \varphi$, $D_t(C_{n-2}^2, i-1) \neq \varphi$,$D_t(C_{n-3}^2, i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and $D_t(C_{n-5}^2, i-1) \neq \varphi$

$\Rightarrow d_t(C_{n-1}^2, i-1) \neq 0$, $d_t(C_{n-2}^2, i-1) \neq 0$,$d_t(C_{n-3}^2, i-1) \neq 0$, $d_t(C_{n-4}^2, i-1) \neq 0$ and $d_t(C_{n-5}^2, i-1) \neq 0$

$\Rightarrow \lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-1$; $\lceil\frac{n-2}{5}\rceil+1 \leq i-1 \leq n-2$; $\lceil\frac{n-3}{5}\rceil+1 \leq i-1 \leq n-3$; $\lceil\frac{n-4}{5}\rceil+1 \leq i-1 \leq n-4$ and $\lceil\frac{n-5}{5}\rceil+1 \leq i-1 \leq n-5$

$\Longrightarrow \lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-5$

$\Longrightarrow \lceil\frac{n-1}{5}\rceil+2 \leq i \leq n-4$.

Conversely, if $\lceil\frac{n-1}{5}\rceil+2 \leq i \leq n-4$.

$\Longrightarrow \lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-5$

$\Longrightarrow \lceil\frac{n-5}{5}\rceil+1 \leq \left\lceil\frac{n-4}{5}\right\rceil+1 \leq \left\lceil\frac{n-3}{5}\right\rceil+1 \leq \left\lceil\frac{n-2}{5}\right\rceil+1 \leq \left\lceil\frac{n-1}{5}\right\rceil \leq i-1 \leq n-5 < n-4 < n-3 < n-2 < n-1$

$\Longrightarrow \lceil\frac{n-5}{5}\rceil+1 \leq i-1 \leq n-5$; $\lceil\frac{n-4}{5}\rceil+1 \leq i-1 \leq n-4$; $\lceil\frac{n-3}{5}\rceil+1 \leq i-1 \leq n-3$; $\lceil\frac{n-2}{5}\rceil+1 \leq i-1 \leq n-2$ and $\lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-1$

$\Rightarrow D_t(C_{n-1}^2, i-1) \neq \varphi$, $D_t(C_{n-2}^2, i-1) \neq \varphi$,$D_t(C_{n-3}^2, i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and $D_t(C_{n-5}^2, i-1) \neq \varphi$.

Theorem 2.6

For every $n \geq 8$ and $i > \lceil\frac{n}{5}\rceil+1$, then we have

(i) $D_t(C_{7k,}^2{}_{2k}) = \{\{1,3, ,,\ldots\ldots 7k-6,7k-4\}\ \{1,6, \ldots\ldots\ldots 7k-6,7k-1\}, \{2,4, \ldots\ldots 7k-5,7k-3\}, \{2,7, ,,\ldots\ldots 7k-5,7k\}, \{3.5,\ldots\ldots\ldots 7k-4,7k-2\}, \{4,6, \ldots\ldots 7k-3,7k-1\}, \{5,7, ,,\ldots\ldots 7k-2,7k\}\}$ for $k \geq 1$.

.

(ii) If $D_t(C_{n-2}^2, i-1) = D_t(C_{n-3}^2, i-1) = D_t(C_{n-4}^2, i-1) = D_t(C_{n-5}^2,i-1) = \varphi$ and

$D_t(C_{n-1}^2,i-1) \neq \varphi$ then $D_t(C_n^2,i) = \{[n]\}$

(iii) If $D_t(C_{n-1}^2,i-1) \neq \varphi$, $D_t(C_{n-2}^2,i-1) \neq \varphi$ and $D_t(C_{n-3}^2,i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and, $D_t(C_{n-5}^2, i-1) = \varphi$ then $D_t(C_n^2,i) = \{[n]-\{x\}/x \in [n]\}$

Proof of (i)

For any $k \geq 1$, split the vertices of $C_{7k,}^2$ into k number of sets of the form {1,2,3,4,5,6,7},{8,9,10,11,12,13,14}………{7k-6,7k-5,7k-4,7k-3,7k-2,7k-1,7k}. The seven total dominating sets of cardinality 2k are constructed by choosing first or third or first or sixth or second and fourth or second and seventh or third and fifth or fourth and sixth or fifth and seventh from each set. Hence $D_t(C_{7k,}^2{}_{2k})$ has the only seven total dominating sets such as {{1,3, ,,……7k-6,7k-4} {1,6, ………7k-6,7k-1},{2,4, ……7k-5,7k-3},{2,7, ,,……7k-5,7k}, {3.5,………7k-4,7k-2}, {4,6, ……7k-3,7k-1},{5,7, ,,……7k-2,7k}} .

Proof of (ii)

Since $D_t(C_{n-2}^2,i-1) = D_t(C_{n-3}^2, i-1) = D_t(C_{n-4}^2, i-1) = D_t(C_{n-5}^2, i-1) = \varphi$ and $D_t(C_{n-1}^2,i-1) \neq \varphi$ and

By lemma 2.5 (i) , we have,

$i = n$

Therefore $D_t(C_n^2,i) = D_t(C_n^2,n) = \{\{1,2,3………n\}\} = \{[n]\}$

Proof of (iii)

Since $D_t(C_{n-1}^2, i-1) \neq \varphi$, $D_t(C_{n-2}^2, i-1) \neq \varphi$ and $D_t(C_{n-3}^2, i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and, $D_t(C_{n-5}^2, i-1) = \varphi$

By Lemma 2.5(ii) , we have,

$i = n-3$

Therefore $D_t(C_n^2,i) = D_t(C_n^2,n-3) = \{[n]-\{x\}/x \in [n]\}$.

Theorem 2.7

For every $n \geq 6$ and $i > [\frac{n}{5}]+1$ if $D_t(C_{n-1}^2, i-1) \neq \varphi$, $D_t(C_{n-2}^2, i-1) \neq \varphi$, $D_t(C_{n-3}^2, i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and, $D_t(C_{n-5}^2, i-1) \neq \varphi$, then $D_t(C_n^2,i) = \{\{X_1 \cup \{n\}/ X_1 \in D_t(C_{n-1}^2, i-1)\} \cup \{X_2 \cup \{n-1\}/ X_2 \in D_t(C_{n-2}^2, i-1)\} \cup \{X_3 \cup \{n-3\}/ X_3 \in D_t(C_{n-3}^2, i-1)\} \cup \{X_4 \cup \{n-4\}/ X_4 \in D_t(C_{n-4}^2, i-1)\}\}$

Proof :

The construction of $D_t(C_n^2,i)$ from $D_t(C_{n-1}^2, i-1)$,$D_t(C_{n-2}^2, i-1)$,$D_t(C_{n-3}^2, i-1)$ and $D_t(C_{n-4}^2, i-1)$ is as follows: Let X_1 be the total dominating set of C_{n-1}^2 with cardinality i-1. All the elements of $D_t(C_{n-1}^2, i-1)$ end with n-1 or n-2 or n-3.Therefore adjoin n with X_1. Hence if $X_1 \in D_t(C_{n-1}^2, i-1)$ then $X_1 \cup \{n\} \in D_t(C_n^2,i)$. Next let us consider $D_t(C_{n-2}^2, i-1)$.Here all the elements of $D_t(C_{n-2}^2, i-1)$ end with n-2 or n-3or n-4. Therefore adjoin n-1 with X_2 where X_2 be the total dominating set of C_{n-2}^2 with cardinality i-1. Hence if $X_2 \in D_t(C_{n-2}^2, i-1)$ then $X_2 \cup \{n-1\} \in D_t(C_n^2,i)$. Let X_3 be the total dominating set of C_{n-3}^2 with cardinality i-1. Let us consider $D_t(C_{n-3}^2, i-1)$. Here all the elements of $D_t(C_{n-3}^2, i-1)$ end with n-3 or n-4 or n-5. Therefore adjoin n-2 with X_3. Hence if $X_3 \in D_t(C_{n-3}^2, i-1)$ then $X_3 \cup \{n-2\} \in D_t(C_n^2,i)$. Let us consider $D_t(C_{n-4}^2, i-1)$. Here all the elements of $D_t(C_{n-4}^2, i-1)$ end with n-4 or n-5 or n-6. Let X_4 be the total dominating set of C_{n-4}^2 with cardinality i-1.Therefore adjoin n-3 with X_4. Hence if $X_4 \in D_t(C_{n-4}^2, i-1)$ then $X_4 \cup \{n-4\} \in D_t(C_n^2,i)$.

Hence $D_t(C_n^2,i) = \{\{X_1 \cup \{n\}/ X_1 \in D_t(C_{n-1}^2, i-1)\} \cup \{X_2 \cup \{n-1\}/ X_2 \in D_t(C_{n-2}^2, i-1)\} \cup \{X_3 \cup \{n-3\}/ X_3 \in D_t(C_{n-3}^2, i-1)\} \cup \{X_4 \cup \{n-4\}/ X_4 \in D_t(C_{n-4}^2, i-1)\}\}$

Theorem(2.8)

If $D_t(C_n^2,i)$ is the family of the total dominating sets of C_n^2 with cardinality i,where $i > [\frac{n}{5}]+1$, then $d_t(C_n^2,i) = d_t(C_{n-1}^2,i-1) + d_t(C_{n-2}^2,i-1) + d_t(C_{n-3}^2,i-1) + d_t(C_{n-4}^2,i-1)$

Proof:

By theorem (2.6) and (2.7).

(i)If $D_t(C_{n-2}^2, i-1) = D_t(C_{n-3}^2, i-1) = D_t(C_{n-4}^2, i-1) = D_t(C_{n-5}^2,i-1) = \varphi$ and $D_t(C_{n-1}^2,i-1) \neq \varphi$ then $D_t(C_n^2,i) = \{[n]\}$

(ii) If $D_t(C_{n-1}^2,i-1) \neq \varphi$, $D_t(C_{n-2}^2,i-1) \neq \varphi$ and $D_t(C_{n-3}^2,i-1) \neq \varphi$, $D_t(C_{n-4}^2, i-1) \neq \varphi$ and, $D_t(C_{n-5}^2, i-1) = \varphi$ then $D_t(C_n^2,i) = \{[n]-\{x\}/x \in [n]\}$

(iii)$D_t(C_n^2,i) = \{\{X_1 \cup \{n\}/ X_1 \in D_t(C_{n-1}^2, i-1)\} \cup \{X_2 \cup \{n-1\}/ X_2 \in D_t(C_{n-2}^2, i-1)\} \cup \{X_3 \cup \{n-3\}/ X_3 \in D_t(C_{n-3}^2, i-1)\} \cup \{X_4 \cup \{n-4\}/ X_4 \in D_t(C_{n-4}^2, i-1)\}\}$

By this construction in each case, we obtain that

$|D_t(C_n^2,i)| = |D_t(C_{n-1}^2,i-1)| + |D_t(C_{n-2}^2,i-1)| + |D_t(C_{n-3}^2,i-)| + |D_t(C_{n-4}^2,i-1)|$

Therefore $d_t(C_n^2,i) = d_t(C_{n-1}^2,i-1) + d_t(C_{n-2}^2,i-1) + d_t(C_{n-3}^2,i-1) + d_t(C_{n-4}^2,i-1)$

Table 2.1: $d_t(C_n^2,i)$ the number of total dominating sets of C_n^2 with cardinality i

i	2	3	4	5	6	7	8	9	10	11	12	13	14
n													
6	12	20	15	6	1								
7	7	28	35	21	7	1							
8	0	24	62	56	28	8	1						
9	0	9	81	117	84	36	9	1					
10	0	0	81	193	200	120	45	10	1				
11	0	0	66	249	387	319	165	55	11	1			
12	0	0	48	280	615	699	483	220	66	12	1		
13	0	0	26	276	839	1286	1174	702	286	78	13	1	
14	0	0	7	221	998	2041	2424	1867	987	364	91	14	1

III. Total Domination Polynomial Of Square Of Cycles

Let $D_t(C_n^2,x) = \sum_{i=\gamma_t(C_n^2)}^{n} d_t(C_n^2,i)x^i$, be the total domination polynomial of the C_n^2, $n \geq 6$.

In this section we study this polynomial.

Theroem 3.1

For every $n \geq 10$ and $i > \lceil \frac{n}{5} \rceil + 1$, we have

$D_t(C_n^2,x) = x[D_t(C_{n-1}^2,x) + D_t(C_{n-2}^2,x) + D_t(C_{n-3}^2,x) + D_t(C_{n-4}^2,x]$ with the initial values

$D_t(C_6^2,x) = 12x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$,

$D_t(C_7^2,x) = 7x^2 + 28x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

$D_t(C_8^2,x) = 24x^3 + 62x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$

$D_t(C_9^2,x) = 9x^3 + 81x^4 + 117x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$

Proof

If $i > \lceil \frac{n}{5} \rceil + 1$ and by theorem 2.5 and 2.6, we have

We have $d_t(C_n^2,i) = d_t(C_{n-1}^2, i-1) + d_t(C_{n-2}^2, i-1) + d_t(C_{n-3}^2, i-1) + d_t(C_{n-4}^2, i-1)$

$d_t(C_n^2,i)x^i = d_t(C_{n-1}^2, i-1)x^i + d_t(C_{n-2}^2, i-1)x^i + d_t(C_{n-3}^2, i-1)x^i + d_t(C_{n-4}^2, i-1)x^i$

$\Sigma d_t(C_n^2,i)x^i = \Sigma d_t(C_{n-1}^2, i-1)x^i + \Sigma d_t(C_{n-2}^2, i-1)x^i + \Sigma d_t(C_{n-3}^2, i-1)x^i + \Sigma d_t(C_{n-4}^2, i-1)x^i$

$D_t(C_n^2,x) = x[D_t(C_{n-1}^2,x) + D_t(C_{n-2}^2,x) + D_t(C_{n-3}^2,x) + D_t(C_{n-4}^2,x]$ with the initial values

$D_t(C_6^2,x) = 12x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$,

$D_t(C_7^2,x) = 7x^2 + 28x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

$D_t(C_8^2,x) = 24x^3 + 62x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$

$D_t(C_9^2,x) = 9x^3 + 81x^4 + 117x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$

By theorem 3.1 , we obtain $d_t(C_n^2,i)$, for $6 \leq n \leq 14$ as shown in Table 2.1. There are interesting relationship between numbers in this Table. In the following theorem we obtain some properties of $d_t(C_n^2,i)$.

Theroem 3.2

The following properties hold for the coefficients of $D_t(C_n^2,x)$:

(i) $d_t(C_{n,}^2{}_n) = 1$, for every $n \geq 6$

(ii) $d_t(C_n^2,n-1) = n$, for every $n \geq 6$

(iii) $d_t(C_n^2,n-2) = \frac{1}{2}[n(n-1)]$, for every $n \geq 6$

(iv) $d_t(C_n^2,n-3) = \frac{1}{6}[n(n-1)(n-2)]$ for every $n \geq 6$

(v) $d_t(C_n^2,n-4) = \frac{1}{24}[n(n-1)(n-2)(n-3)]-n$, for every $n \geq 7$

(vi) $d_t(C_{7k}^2,2k) = 7$, for every $k \geq 1$

Proof:

(i) Since for any graph G with n vertices, $d_t(G,n) = 1$, then $d_t(C_n^2,n) = 1$.

(ii) Since $D_t(C_n^2,n-1) = \{[n]-\{x\}/ x \in [n]\}$

$\mid D_t(C_n^2,n-1) \mid = nC_1 = n$, $d_t(C_n^2,n-1) = n$

(iii) To prove $d_t(C_n^2,n-2) = \frac{1}{2}[n(n-1)]$

We apply induction on n.

When n = 6

L.H.S = $d_t(C_6^2,4) = 15$ (from table)

R.H.S = $\frac{1}{2}[6(6-1)] = 15$

Therefore the result is true for n = 6.

Suppose that the result is true for all natural numbers less than n, and we prove it for n.

We have $d_t(C_n^2,n-2) = d_t(C_{n-1}^2, n-3) + d_t(C_{n-2}^2, n-3) + d_t(C_{n-3}^2, n-3) + d_t(C_{n-4}^2, n-3)$

$= \frac{1}{2}\{(n-1)(n-2)\} + n - 2 + 1 + 0$

$= \frac{1}{2}[n^2 - 3n+2+ 2n - 2]$

$= \frac{1}{2}(n^2 - n)$

$= \frac{1}{2}[n(n-1)]$, for every $n \geq 6$

Hence the result is true for all' n'.

Hence by induction hypothesis , we have

$d_t(C_n^2,n-2) = \frac{1}{2}[n(n-1)]$, for every $n \geq 6$

(iv) To prove $d_t(C_n^2,n-3) = \frac{1}{6}[n(n-1)(n-2)]$, for every $n \geq 6$

We apply induction on n.

When n = 6

L.H.S = $d_t(C_6^2,3) = 20$ (from table)

R.H.S = $\frac{1}{6}[6(6-1)6-2)] =20$

Therefore the result is true for n = 6.

Suppose that the result is true for all natural numbers less than n, and we prove it for n.

We have $d_t(C_n^2,n-3) = d_t(C_{n-1}^2, n-4) + d_t(C_{n-2}^2, n-4) + d_t(C_{n-3}^2, n-4) + d_t(C_{n-4}^2, n-4)$

$= \frac{1}{6}[(n-1)(n-2)(n-3)] + \frac{1}{2}\{(n-2)(n-3)\} +n-2$

$= \frac{1}{6}[(n-1)(n^2-5n+6)] + \frac{1}{2}\{n^2-5n+6\}+n-2$

$= \frac{1}{6}[n^3-5n^2+6n-n^2+5n-6+3n^2-15n+18+6n-12]$

$= \frac{1}{6}[n^3-3n^2+2n]$

$= \frac{1}{6}[n(n-1)(n-2)]$ for $n \geq 6$

Hence the result is true for all' n'.

Hence by induction hypothesis , we have

$d_t(C_n^2,n-3) = \frac{1}{6}[n(n-1)(n-2)]$, for every $n \geq 6$

(v) To prove $d_t(C_n^2,n-4) = \frac{1}{24}[n(n-1)(n-2)(n-3)]-n$, for every $n \geq 7$

We apply induction on n.

When n = 7

L.H.S = $d_t(C_7^2,3) = 28$ (from table)

R.H.S = $\frac{1}{24}[7(7-1)(7-2)(7-3)]-7=28$

Therefore the result is true for n = 6.

Suppose that the result is true for all natural numbers less than n, and we prove it for n.

We have $d_t(C_n^2,n-4) = d_t(C_{n-1}^2, n-5) + d_t(C_{n-2}^2, n-5) + d_t(C_{n-3}^2, n-5) + d_t(C_{n-4}^2, n-5)$

$= \frac{1}{24}[(n-1)(n-2)(n-3)(n-4)]-(n-1) + \frac{1}{6}[(n-2)(n-3)(n-4)] + \frac{1}{2}[(n-3)(n-4)] +(n-4)$

$= \frac{1}{24}\{(n^2 -3n+2)(n^2-7n+12)\}-(n-1) + \frac{1}{6}(n-2)(n^2-7n+12) + \frac{1}{2}(n^2-7n+12) +(n-4)$

$= \frac{1}{24}\{n^4-7n^3+12n^2-3n^3+21n^2-36n+2n^2-14n+24-24n+24+4n^3-28n^2+48n-8n^2+56n-96+1284n+144+24n-96\}$

$= \frac{1}{24}\{n^4-6n^3+11n^2-30n\}$

$= \frac{1}{24}\{n^4-6n^3+11n^2-6n-24n\}$

$= \frac{1}{24}\{n^4-6n^3+11n^2-6n\} - \frac{1}{24}\{24n\}$

$= \frac{1}{24}[n(n-1)(n-2)(n-3)]-n$

Hence the result is true for all' n'.

Hence by induction hypothesis , we have

$d_t(C_n^2, n-4) = \frac{1}{24}[n(n-1)(n-2)(n-3)]-n$, for every $n \geq 7$

(vi) To prove $d_t(C_{7k}^2, 2k) = 7$, for every $k \geq 1$

By theorem (2.6)(i)

$D_t(C_{7k,}^2{}_{2k}) = \{\{1,3, ,,\ldots\ldots 7k-6,7k-4\}\ \{1,6, \ldots\ldots\ldots 7k-6,7k-1\},\{2,4, \ldots\ldots 7k-5,7k-3\},$
$\{2,7, ,,\ldots\ldots 7k-5,7k\},\ \{3.5,\ldots\ldots\ldots 7k-4,7k-2\},\ \{4,6, \ldots\ldots 7k-3,7k-1\},$
$\{5,7, ,,\ldots\ldots 7k-2,7k\}\}$ for $k \geq 1$.

Therefore $|D_t(C_{7k}^2, 2k)| = 7$

Hence $d_t(C_{7k}^2, 2k) = 7$, for every $k \geq 1$

Theorem 3.3

(i) For every $n \geq 10$ and $i > \lceil \frac{n}{5} \rceil +1$,we have

$d_t(C_{n+1}^2, j+1) - d_t(C_n^2, j+1) = d_t(C_n^2, j) - d_t(C_{n-4}^2, j)$

Proof:

By theorem (2.8)

We have $d_t(C_{n+1}^2, j+1) = d_t(C_n^2, j) + d_t(C_{n-1}^2, j) + d_t(C_{n-2}^2, j) + d_t(C_{n-3}^2, j)$

$d_t(C_n^2, j+1) = d_t(C_{n-1}^2, j) + d_t(C_{n-2}^2, j) + d_t(C_{n-3}^2, j) + d_t(C_{n-4}^2, j)$

$d_t(C_{n+1}^2, j+1) - d_t(C_n^2, j+1) = \{d_t(C_n^2, j) + d_t(C_{n-1}^2, j) + d_t(C_{n-2}^2, j) + d_t(C_{n-3}^2, j)\} -$
$\{d_t(C_{n-1}^2, j) + d_t(C_{n-2}^2, j) + d_t(C_{n-3}^2, j) + d_t(C_{n-4}^2, j)\}$
$= d_t(C_n^2, j) - d_t(C_{n-4}^2, j)$

IV. Conclusion

We obtain total domination sets and total domination polynomial square of cycles. Similarly we can find total domination sets and total domination polynomial of specified graphs.

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