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Analytical Aproach on the Dynamics of the Quadratic Map $F\mu(X) = \mu x(1-X)$, For $1 < \mu < 3$ And $0 < X < 1$

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Abstract: *This research work is an exhaustic survey of some basic results and ideas in the theory of one dimensional dynamical systems. We have looked at the eventual behaviour of the iterates of the quadratic family $f_\mu(x) = \mu x\ (1-x)$, where μ is parameter value, $0 < x < 1$ and $1 < \mu < 3$. In contrast to the graphical approach outlined in Devaney, we gave an analytical approach as means of justifying the convergence of the sequence of iterates to the fixed points.*

Keywords: *Quadratic Map, Dynamic Systems, Quadratic Family and Fixed Points*

I. Introduction

In modern times, there is no shortage of information of any kind no matter how simple or complex a system might be. Mathematical science has remained the backbone of modern technological and scientific advancement. In particular discrete dynamics and by extension the analysis of discrete dynamical systems have provided a gate way to both graphical and analytical solution to real-life problems. It is against this back drop that informed the need to holistically isolate and analyze the dynamics of the quadratic family, $f_\mu(x) = \mu x\ (1-x)$.

Proportion 5.1 Given $f_\mu(x) = \mu x(1-x)$, $\mu > 1$ then
i. $f_\mu(0) = f_\mu(1)=0$ and
ii. there exists p_μ such that $f_\mu(p_\mu)=p_\mu$,
Where $p_\mu = \frac{\mu-1}{\mu}$ and $0 < p_\mu < 1$ Devaney, (1986)

Proof: conditions i, essentially gives the zeros of $f_\mu(x) = \mu x\ (1-x)$ obtained by setting $\mu x(1-x) = 0$,that is $x = 0$ or $x= 1$ since $\mu > 1$. Thus $f_\mu(0) = f_\mu(1) = 0$ implies, by Rolle's theorem, the existence of a critical points $0 < c < 1$ Such that $f_\mu'(c) = 0$. Consequently $f'(c) = \mu - 2\mu c = 0$ i.e, $x = c = ½$. May (1976) and Yuguda (1998).

Hence the graph of $f_\mu(x)$ being a quadratic, opens down wards and so f_μ increases on $(0,½)$ and decreases on$(½, 1)$.

For the fixed points, we have $f_\mu(p_\mu) = p_\mu \Rightarrow \mu p_\mu(1-p_\mu) - p_\mu= 0$
$\Rightarrow p_\mu\ (\mu\ (1-p_\mu) -1) = 0$ that is
$p_\mu = 0$ or $\mu(1-p_\mu) - 1 = 0$ or $- \mu p_\mu = 1 - \mu$
$\Rightarrow p_\mu = \frac{\mu-1}{\mu}$

ii. if $\mu > 1$ then $p_\mu = \frac{\mu-1}{\mu} > 0$, i.e $p_\mu = 1 - \frac{1}{\mu} < 1$

Thus $0< p_\mu < 1$. This establishes the result.

Proposition 5.2: The sequence of iterates $f_\mu^n(x) \to -\infty$ as $n \to \infty$ whenever $x < 0$ or $x > 1$, and for $\mu >1$. Devaney, (1986)
Proof: suppose $x < 0$,
then $f_\mu(x) - x = \mu x(1- x) - x = x\ (\mu-1) - \mu\ x^2$.
Now $x\ (\mu-1) < 0$ since $x < 0$ and $\mu -1 > 0$.
Also $\mu x^2 > 0$ since $x^2 > 0$. Hence $x\ (\mu-1) - \mu\ x^2 < 0$ that is $\mu x(1-x) < x$.
Hence $f_\mu(x) < x$ and $f_\mu^2\ (x) = f_\mu(f_\mu(x)) < f_\mu(x) < x$
Thus inductively $f_\mu^n\ (x) = f_\mu\ \ (f_\mu^{n-1}(x))\ , \ldots\ , f_\mu(f_\mu(x)) < f_\mu(x) < x$.

This shows that $f_\mu^n(x)$ is a decreasing sequence of n which cannot converge to p otherwise we would have $f_\mu^{n+1}(x) \to f_\mu(p) = k < p$ where as $f_\mu^n(x) \to p$
this is a contradiction and $f_\mu(x)$ cannot be bounded below.
Hence $f_\mu^n(x) \to -\infty$ as $n \to \infty$.

Now if $x > 1$, then $f_\mu(x) = \mu x(1-x) < 0$, since $1 - x < 0$.

Hence $\mu x(1-x) < x$ that is $f_\mu(x) < x$ and $f_\mu^2(x) = f_\mu(f_\mu(x)) < f_\mu(x) < x$ and by the previous argument, we deduce that $f_\mu^n(x) \to -\infty$ as $n \to \infty$ as well. We shall now look at the dynamics proper of $f_\mu(x)$ as the parameter value μ is varied accordingly in the next proposition.

Proposition 5.3 suppose $1 < \mu < 3$ then

i. f_μ has an attracting fixed point at $p_\mu = \frac{\mu-1}{\mu}$ and a repelling fixed point at 0.

ii. if $0 < x < 1$, then $\lim_{n\to\infty} f_\mu^n(x) = p_\mu$ Devaney, (1986)

Proof: Under the analysis of the proof of proposition 5.3, the fixed points were actually found to be $p_\mu = 0$ and $p_\mu = \frac{\mu-1}{\mu}$ it remains to prove that $p_\mu = \frac{\mu-1}{\mu}$ is attracting and $p_\mu = 0$ is repelling.

Using the notion of hyperbolicity we see that

$$f_\mu'(p_\mu) = \mu - 2\mu\frac{(\mu-1)}{\mu} = \mu - 2\mu + 2 = 2 - \mu.$$

And $|f_\mu'(p_\mu)| = |2-\mu|$ and for $1 < \mu < 3$, we have $-1 > -\mu > -3$ if and only if $2-1 > 2-\mu > 2-3$ that is $1 > 2-\mu > -1$ that is $|2-\mu| < 1$. Hence we conclude that $|f_\mu'(p_\mu)| = |2-\mu| < 1$ and p_μ is an attracting fixed point.

In case of $p_\mu = 0$, we have $f_\mu'(0) = \mu - 2\mu.0 = \mu > 0$ since $1 < \mu < 3$ and $|f_\mu'(0)| = |\mu| = \mu > 1$ and so 0 is a repelling fixed point.

In order to prove (ii), we first consider the case $1 < \mu < 2$ in which $1/\mu > ½$ and $1 - 1/\mu < 1 - ½$ that is $\frac{\mu-1}{\mu} = p_\mu < ½$, and since $1/\mu > 0$, $0 < p_\mu < ½$

Now for $x \in (0, p_\mu)$, $f_\mu'(x) = \mu(1-2x) > 0$ and so f_μ is increasing $0 < x < p_\mu < ½$, thus $0 < f_\mu(x) < p_\mu$. We shall show that $f_\mu^n(x)$ is an increasing sequence of n on $(0, p_\mu)$ that is $f_\mu(x) - x > 0$.

But

$$\begin{aligned} f_\mu(x) - x &= \mu x(1-x) - x \\ &= x(\mu-1) - \mu x^2 \\ &= \frac{\mu}{\mu}[x(\mu-1) - \mu x^2] \\ &= \mu\left[x\left(\frac{\mu-1}{\mu}\right) - x^2\right] \\ &= \mu\left[x\left(\frac{\mu-1}{\mu}\right) - x^2\right] \\ &= \mu x(p_\mu - x) > 0 \text{ since } x < p_\mu, \end{aligned}$$

Therefore $\mu x(1-x) > x$. Hence $f_\mu(x) > x$
and so $f_\mu^2(x) = f_\mu(f_\mu(x)) > f(x) > x > 0$.

And inductively $p_\mu > f_\mu^n(x) > \ldots > f_\mu(f_\mu(x)) > f(x) > 0$. Thus $f_\mu^n(x)$is an increasing sequence of n. Now suppose there is a least upper bound (lub)
$p_a < p_\mu$ for the sequence. Then we have $f_\mu^n(x) \to p_a$ and
$f_\mu^{n+1}(x) = f_\mu\left(f_\mu^n(x)\right) \to f_\mu(p_a) > p_a$
This is a contradiction since we cannot have $f_\mu^n(x) \to k > p_a$ if p_a is as defined. So there exists no such lub. p_a. Hence p_μ must be the least upper bound. Consequently $f_\mu^n(x) \to p_\mu$ as $n \to \infty$ for all $x \in (0, p_\mu)$

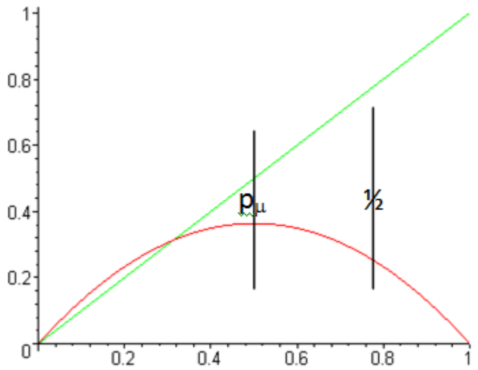

Fig 1: The graph of $f(x) = \mu x(1-x)$ for $1 < \mu < 2$

For $x \in (p_\mu\ \tfrac{1}{2})$, $f\mu(x) - x = \mu x(p_\mu - x) < 0$

Since $x > p_\mu$ so that $\mu x(1-x) < x$. Hence $f_\mu(x) < x$. Note that in this case $f_\mu'(x) = \mu(1-2x) > 0$ therefore $f_\mu(x)$ is still increasing on $(p_\mu, \tfrac{1}{2})$ with
$f_\mu(p_\mu) < f_\mu(x) < f_\mu(\tfrac{1}{2})$ that is $p_\mu < f_\mu(x) < \mu/4 < 1/2$. But $f_\mu(x) < x$
implies $f_\mu^2(x) = f_\mu(f_\mu(x)) < f_\mu(x) < \tfrac{1}{2}$
Hence $p_\mu < f_\mu^n(x) < \ldots < f_\mu(f_\mu(x)) < f_\mu(x) < \tfrac{1}{2}$.

Now suppose there exists a greatest lower bound (glb) $p_a > p_\mu$ for this sequence Then we have $f_\mu^n \to p_a$ and $f_\mu^{n+1}(x) = f_\mu(f_\mu^n(x)) \to f_\mu(p_a) < p_a$
A clear contradiction. Since we cannot have $f_\mu^n(x) \to k < p_a$ if p_a is as defined.

So there exists no such greatest lower bound (glb) p_a. Therefore p_μ is the glb. Consequently $f_\mu^n(x) \to p_\mu$ as $n \to \infty$ for all $x \in (p_\mu\ \tfrac{1}{2})$. But then
$(0, \tfrac{1}{2}) = (0, p_\mu] \cup [p_\mu\ \tfrac{1}{2})$. Therefore $f_\mu^n(x) \to p_\mu$ for all $x \in (0, \tfrac{1}{2})$ as $n \to \infty$.

When $x \in (\tfrac{1}{2}, 1)$, $f_\mu'(x) = \mu - 2\mu x = \mu(1-2x) < 0$.

Therefore $f_\mu(x)$ is decreasing on $(\tfrac{1}{2}, 1)$ so $0 = f_\mu(1) < f_\mu(x) < f_\mu(\tfrac{1}{2}) = \mu/4 < 1/2$ Therefore $0 < f_\mu(x) < \tfrac{1}{2}$. Thus $f_\mu(x)$ maps the interval $f_\mu(\tfrac{1}{2}, 1)$ into

the interval $(0, \tfrac{1}{2})$. But $f_\mu^{n+1}(x) = f_\mu^n(f_\mu(x))$ $\therefore f_\mu^{n+1}(x) = f_\mu^n(x^1)$ where
$x^1 = f_\mu(x) \in (0, \tfrac{1}{2})$ and so $\lim_{n\to\infty} f_\mu^n(x) = p_\mu$. thus for $1 < \mu < 2$,

$$\lim_{n\to\infty} f_\mu^n(x) = p_\mu,$$

If $\mu = 2$, then $p_\mu = \dfrac{\mu-1}{\mu} = \tfrac{1}{2}$ and $f_2(\tfrac{1}{2}) = \tfrac{1}{2}$.

Therefore for $x \in (0, \tfrac{1}{2}]$, $f_2(x)$ is increasing with x so $0 < f_2(x) < 1/2$.

Also $f_2(x) - x = x(1-2x) > 0$, so $f_2(x) > x$ i.e $f_2^2(x) > f_2(x) > x > 0$. Hence

$\tfrac{1}{2} > f_2^n(x) > \ . \ . \ . > f_2(x) > x > 0$. We claim that this sequence is converging to ½ since its bounded above by ½ . For otherwise suppose there exist a lub (least upper bound) $p_a < \tfrac{1}{2}$, then $f_2^n(x) \to p_a$ but $f_\mu^{n+1}(x) = f_2(f_\mu^n(x) \to f_\mu(p_a) = k > p_a$. this is a contradiction. Thus ½ must be the lub.

Therefore $f_\mu^n(x) \to \tfrac{1}{2}$, for all $x \in (0, \tfrac{1}{2}]$.

For $x \in [\tfrac{1}{2}, 1)$, $f_2(x)$ is decreasing with x
Since $f_2(x) = 2(1-2x) < 0$. Therefore $0 = f_2(1) < f_2(x) < f_2(\tfrac{1}{2}) = \tfrac{1}{2}$
i.e $0 < f_2(x) < \tfrac{1}{2}$. thus $f_2(x)$ map the interval $(\tfrac{1}{2}, 1)$ on to $(0, \tfrac{1}{2}]$, but on
$(0, \tfrac{1}{2}]$, $f_\mu^n(x) \to \tfrac{1}{2}$. Therefore $x \in [\tfrac{1}{2}\ 1)$, $x' = f_2(x) \in (0, \tfrac{1}{2}]$ and

$f_\mu^n(x') = f_2^{n-1}(x') \to \tfrac{1}{2}$ and since $[0, 1) = (0, \tfrac{1}{2}) \cup ({}^1/_2, 1) \cup [\tfrac{1}{2}]$, the result follows for all $x \in (0, 1)$. Consequently $f_2^n(x) \to \tfrac{1}{2}$ as $n \to \infty$

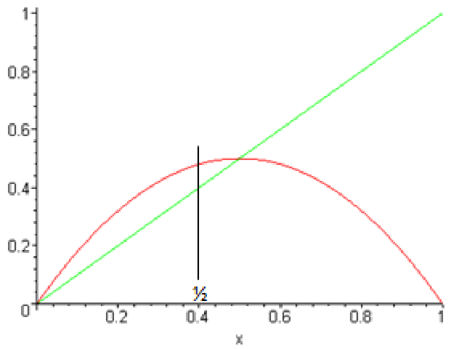

Fig 3: The graph of $f(x) = \mu x(1-x)$ *for* $\mu = 2$

In the case $2 < \mu < 3$, we have ${}^1/\mu > {}^1/_3$ if and only if $-{}^1/\mu < -{}^1/_3$ that is

$$1 - {}^1/\mu < 1 - {}^1/_3 \Leftrightarrow \frac{\mu-1}{\mu} = p_\mu < {}^2/_3$$

Also, $${}^1/\mu < {}^1/_2 \Leftrightarrow \frac{-1}{\mu} > -{}^1/_2 \text{ i.e}$$

$$\text{i.e } 1 - {}^1/_\mu > 1 - \tfrac{1}{2} \Rightarrow \frac{\mu-1}{\mu} = p_\mu > \tfrac{1}{2} \therefore \tfrac{1}{2} < p_\mu < {}^2/_3 < 1.$$

Now let p_μ be the unique point in $(0, \tfrac{1}{2})$ which is mapped on to p_μ by f_μ that is $f_\mu(\hat{p}_\mu) = p_\mu$. As shown on the diagram below

This is possible since $f_\mu(x)$ is a quadratic polynomial. Now $(\hat{p}_\mu, p_\mu)$, contains the critical point ½ at which $f_\mu(x)$ is maximum. Therefore $f(x)$ is monotonically increasing on $(\hat{p}_\mu, \tfrac{1}{2})$. Since $\hat{f}_\mu(x) = \mu(1-2x) > 0$ for all
$x \in [\hat{p}_\mu, \tfrac{1}{2}]$.

So that $f_\mu[\hat{p}_\mu, p_\mu] = f_\mu\Big[[\hat{p}_\mu, \tfrac{1}{2}] \cup [\tfrac{1}{2}, p_\mu]\Big]$ *and*
We can specifically write
$f_\mu[\hat{p}_\mu, \tfrac{1}{2}] = [f_\mu(\hat{p}_\mu), f_\mu(\tfrac{1}{2})] = [p_\mu, {}^\mu/_4]$

Also f_μ is decreasing on (½, p_μ] since $f'_\mu(x) = \mu\,(1 - 2x) < 0$ for all

$x \in [½, p_\mu]$

So that we can also write

$f_\mu\,[½, p_\mu] = [f_\mu\,(p_\mu), f_\mu\,(½)] = [p_\mu, {}^{\mu}/_{4}]$

Therefore $f\,[\hat{p}_\mu, p_\mu] = [\hat{p}_\mu, {}^{\mu}/_{4}] \cup [p_\mu, {}^{\mu}/_{4}] = [p_\mu, {}^{\mu}/_{4}]$ with $p_\mu < f_\mu\,(x) < {}^{\mu}/_{4}$. But since $f_\mu(x)$ is decreasing on $[p_\mu\ \frac{\mu}{4}]$, we have $f_\mu^2\left[p_\mu\ \frac{1}{2}\right] = f_\mu\left[p_\mu\ \frac{\mu}{4}\right] = \left[f_\mu\left(\frac{\mu}{4}\right), [f_\mu(p_\mu)]\right] = \left[f_\mu\left(\frac{\mu}{4}\right), p_\mu\right]$

However, if $x < p_\mu$, $f_\mu\,(x) - x = \mu x\,(p_\mu - x) > 0$

And so $f_\mu(x) > x$ so that $f_\mu^2(x) > f_\mu(x) > x,\ \ x < p_\mu$.

therefore $f_\mu\,({}^{\mu}/_{4}) = f_\mu^2\,(½) > f_\mu(½) > ½$

and $½ < p_\mu$. Consequently $f_\mu^2(x)$ maps the interval $[p_\mu, \hat{p}_\mu]$ inside $[½, p_\mu]$. Now $f_\mu^n(x)$ is increasing with n on $[½, \hat{P}_\mu]$ and

$½ < {}^{\mu}/_{4} = f_\mu(½) < f_\mu(x) < p_\mu$. That is $½, < f_\mu\,(x) < p_\mu$, and for the sequence of interates of n, $p_\mu > f_\mu^n(x) > \ .\ .\ >$ $f_\mu(x) > x\ > ½$. Therefore p_μ is an upper bound (*ub*) of the sequence.

We now suppose there exists a least upper bound (*lub*) $p_a < p_\mu$ such that $f_\mu^n(x) \rightarrow p_a$. But then $f_\mu^{n+1}(x) = f_\mu(f^n(x)) \rightarrow f_\mu(p_a) = k > p_a$. A clear contradiction of the assertion that p_a is a *lub*. And so p_μ is the *lub*

Thus $f_\mu^n(x) \rightarrow p_\mu$ as $n \rightarrow \infty$ for $x \in [½, p_\mu]$ and consequently for all $x \in [p_\mu\ \hat{p}_\mu]$.

However, if $x < \hat{p}_\mu$, then $\hat{f}_\mu\ (x)\ - x = \mu x\,(p_\mu - x) > 0$ i.e $f_\mu^n(x)$ is an increasing sequence of n which cannot be bounded above by $\hat{p}_\mu$ since

$f_\mu\left(\hat{p}_\mu\right) = \hat{p}_\mu > p_\mu$ and since $\hat{p}_\mu$ is the lub for the sequence $f_\mu^n(x)$. when $x < \hat{p}_\mu$, there exist $k > 0$ such that $f_\mu^k(x) \in [p_\mu\ \hat{p}_\mu]$.

And $f_\mu^{n+k}\ (x) \rightarrow p_\mu$ as $n \rightarrow \infty$.

Finally if $x \in (p_\mu, 1)$, then $f_\mu(x)$ is decreasing with respect to x on $(p_\mu, 1)$, so, $0 = f_\mu(1) < f_\mu(x) < p_\mu$. i.e $0 < f_\mu\,(x) < p_\mu$ and so $f_\mu(x)$ maps the interval $(p_\mu\ 1)$ on to $(0, p_\mu)$. Thus $f_\mu^n(x) = f_\mu^{n-1}(x^1) \rightarrow p_\mu$ as $n \rightarrow \infty$, where

$x^1 = f_\mu(x) \in (0, p_\mu)$ and $x \in (p_\mu, 1)$.

Hence in all cases, we have $f_\mu^n(x)\ \rightarrow p_\mu$ as $n \rightarrow \infty$, since

$(0, 1) = (0, \hat{p}_\mu) \cup [\hat{p}_\mu\ \ p_\mu] \cup (\hat{p}_\mu\ 1)$.

Now putting the three cases together $1 < \mu < 2$, $\mu = 2$ and $2 < \mu < 3$ f_μ has only two fixed points and all other points in (0, 1) are asymptotic to p_μ.

II. Summary

We used the quadratic family to study in details one dimensional dynamics. We begin in earnest by proposition 5.3 which essentially gives the zeros and fixed points of the map leading to proposition 5.2 which shows what happens when iterating the quadratic family out side the unit interval. Then of course proposition 5.3 is a true analysis of the behaviour of the map when $1 < \mu < 3$.

III. Conclusion

In proposition 5.3 we established that for $1 < \mu < 3$ and $x \in [0, 1]$, $f_\mu\,(x)$ has only two fixed points, one at $x = 0$ and the other at $x = p_\mu$, where $p_\mu = \frac{\mu - 1}{\mu}$, and all other points in the interval [0,1] are asymptotic to the fixed point p_μ

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