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Counting subgroups of finite nonmetacyclic 2-groups having no elementary abelian subgroup of order 8

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Abstract: *The aim of this note is to give an explicit formula for the number of subgroups of finite nonmetacyclic 2-groups having no elementary abelian subgroup of order 8.*
Keywords: *Centralproducts, cyclic subgroups, dihedral groups, finite nonmetacyclic 2-groups, number of subgroups.*

I. Introduction

Counting subgroups of finite groups solves one of the most important problems of combinatorial finite group theory. For example, in [1] are determined an explicit expression for the number of subgroups of finite nonabelianp-groups having a cyclic subgroup.

Recall that the problem was completely solved in the abelian case, by establishing an explicit expression of the number of subgroups of a finite abelian group [2]. Unfortunately, in the nonabeliannonmetacyclic casea such expression can be given only for certain finite groups [3]. In this note we prove a counting theorem for a class of finite nonmetacyclic 2-groups having no elementary abelian subgroup of order 8.

A group G is said to be metacyclic if it contains a normal cyclic subgroup$C$ such that G/C is cyclic, otherwise it is said to be nonmetacyclic. Let A and B be groups, a central product of groups A and B is denoted by $A * B$, that is, $A * B = AB$with $[A, B] = \{1\}$, where $[A, B]$ is a commutator subgroup generated by groups A and B.

For basic definitions and results on groups we refer the reader to [4], [5] and [6]. More precisely, we prove the following result in the next section.

Theorem 2.1. Let $G = D * Z$, where $*$ is a central product, $D \cong D_{2^n}$, a dihedral group of order 2^n, $n \geq 3$, $Z \cong C_4$, a cyclic group of order 4 and $D \cap Z = Z(D)$, $Z(D)$ is the center of D. Then the number of subgroups of the group G is given by the following equality:

$$|L(G)| = \begin{cases} 23 & ; \text{if} n = 3 \\ 3\left(2 + n + \sum_{k=2}^{n-2} 2^{n-k}\right) + 2^n & ; \text{if} n \geq 4 \end{cases}$$

where $L(G)$, the set consisting of all subgroups of G forms a complete lattice with respect to set inclusion, called the subgroup lattice of G.

II. Proof of Theorem 2.1

Proof. Let $D \cong D_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{2^{n-1}-1}\rangle, n \geq 3$, a dihedral group of order 2^n, $Z \cong C_4 = \langle a\rangle$, a cyclic group of order 4 and $D \cap Z = Z(D)$. Then $G = D * Z := \frac{D \times Z}{H}$, where $H = \langle (x^{2^{n-2}}, a^2)\rangle$, $n \geq 3$. That is:

$$G := \langle (x, 1)H, (y, 1)H, (1, a)H\rangle$$

such that:

$$(x^{2^{n-1}}, 1)H = (y^2, 1)H = (1, a^4)H = H, \quad (yxy^{-1}, 1)H = \left(x^{2^{n-1}-1}, 1\right)H$$

An important property of this group is that its characteristic subgroup defined by: $\mho_{n-2}(G) :=$ $\langle (x^q, 1)H\rangle$, where $q = 2^{n-2}$, for all $n \geq 3$, is of order 2. Also, for $n \geq 3$, we obtain an epimorphism$\delta: G \longrightarrow D_{2^{n-1}} \times C_2$ defined by:
$\delta(kH) := (kH)\langle (x^{2^{n-2}}, 1)H\rangle, n \geq 3$, where $kH \in G$, $k \in D \times Z$and$(kH)\langle (x^{2^{n-2}}, 1)H\rangle \in D_{2^{n-1}} \times C_2, n \geq 3$.
Clearly, the kernel of $\delta$is
$\mho_{n-2}(G) := \langle (x^{2^{n-2}}, 1)H\rangle$and by the first isomorphism theorem for groups, we obtain that:

$$\frac{G}{\mho_{n-2}(G)} \cong D_{2^{n-1}} \times C_2 \text{ for all } n \geq 3 \qquad (1)$$

Being isomorphic, the groups $\frac{G}{\mho_{n-2}(G)}$ and $D_{2^{n-1}} \times C_2$ have isomorphic lattices of subgroups.

Moreover, since the number of subgroups G which not contain $\mho_{n-2}(G)$ are the trivial subgroup as well as all minimal subgroups of G excepting $\mho_{n-2}(G)$ and since the distinct subgroups generated by the join of any two distinct such subgroups includes $\mho_{n-2}(G)$.
One obtains:

$$|L(G)| = \left|L\left(\frac{G}{\mho_{n-2}(G)}\right)\right| + 2^{n-1} + 3, \quad \text{for all } n \geq 3 \qquad (2)$$

Thus, we need to determine the number of subgroups of $D_{2^{n-1}} \times C_2$ using the following auxiliary result established in [3].
Lemma 2.2: For all $n \geq 3$, the number of all subgroups of order 2^n in the finite 2-group $D_{2^{n-1}} \times C_2$ is:

$$\begin{cases} 16 & ; \text{if } n = 3 \\ 2^{n-1} + 3\left(n + 1 + \sum_{i=1}^{n-2} 2^{n-i}\right) & ; \text{if } n \geq 4 \end{cases} \qquad (3)$$

Hence, the relations (1), (2) and (3) give the explicit expression of

$$|L(G)| = \begin{cases} 23 & ; \text{if } n = 3 \\ 2^n + 3\left(2 + n + \sum_{k=2}^{n-2} 2^{n-k}\right) & ; \text{if } n \geq 4 \end{cases}$$

■

III. Conclusion

In this short note we had worked on minimal subgroups and used a previous result (Lemma 2.2) to obtain a counting theorem for a class of finite nonmetacyclic 2-groups having no elementary abelian subgroup of order 8. It is desirable to consider arbitrary nonabeliannonmetacyclic 2-groups.

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