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$m(\alpha)-$ Series To Circular Functions Using Power Set Notation

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Abstrct: *In this paper, the authors investigate the summation-complete relation to certain type of generalized higher order $\alpha-$ difference equation to find the value of $m(\alpha)-$series to circular functions in the field of finite difference methods. We provide an example to illustrate the $m(\alpha)-$ series to circular functions.*

Key words: *Generalized α -difference equation, summation solution, complete solution, circular functions.*

I. Introduction

In 1984, Jerzy Popenda [1] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989 Miller and Rose [6] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The general fractional h-difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in [2, 7]. As application of $\Delta_h^{-\nu}$, by taking $\nu = m$(positive integer) and $h=\ell$, the sum of m^{th} partial sums on n^{th} powers of arithmetic, arithmetic-geometric progressions and products of n consecutive terms of arithmetic progression have been derived using $\Delta_\ell^{-m} u(k)$ [3].

In 2011, M.Maria Susai Manuel, et.al, [4] have extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ which is defined as $\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$, for the real valued function $u(k)$, $\ell \in (0,\infty)$ is fixed. In [5], the authors have used the generalized α -difference equation;

$$v(k+\ell) - \alpha v(k) = u(k), k \in [0,\infty), \ell \in (0,\infty) \quad (1)$$

and obtained a summation solution of the above equation in the form

$$v(k) = \sum_{r=1}^{[\frac{k}{\ell}]} \alpha^{r-1} u(k - r\ell), \quad j = k - [\frac{k}{\ell}]\ell. \quad (2)$$

There are two types of solutions for the equation (1): one is summation another one is closed form solution. If we are able to find a closed form solution of equation (1), which is coinciding with the summation solution of that equation, then we can obtain formula for finding the values of several finite series. In this paper, we extend the theory of generalized m^{th} order difference equation developed in [8] to generalized m^{th} order α -difference equation.

In [9], the authors have defined the $m-$series of $u(k)$. Here we define corressponding $m(\alpha)-$ series as and obtain several results on $m(\alpha)-$ series

For $m \in \mathrm{N}(1)$, the $m(\alpha)-$ series of $u(k)$ with respect to ℓ is defined as below:

$$1(\alpha)-series: u_{1\alpha(\ell)}(k) = u(k-\ell) + \alpha u(k-2\ell) + \cdots + \alpha^{[\frac{k}{\ell}]-1} u\left(k - [\frac{k}{\ell}]\ell\right),$$

$$2(\alpha)-series: u_{2\alpha(\ell)}(k) = u_{1\alpha(\ell)}(k-\ell) + \alpha u_{1\alpha(\ell)}(k-2\ell) + \cdots + \alpha^{[\frac{k}{\ell}]-1} u_{1\alpha(\ell)}\left(k - [\frac{k}{\ell}]\ell\right)$$

and in general $m(\alpha)-$ series:

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$$u_{m\alpha(\ell)}(k)=u_{(m-1)\alpha(\ell)}(k-\ell)+\alpha u_{(m-1)\alpha(\ell)}(k-2\ell)+\cdots+\alpha^{[\frac{k}{\ell}]-1}u_{(m-1)\alpha(\ell)}\left(k-[\frac{k}{\ell}]\ell\right).$$

We find that the $m(\alpha)-$series of $u(k)$ is the summation solution of the m^{th} order $\alpha-$difference equation

$$\Delta^m_{\alpha(\ell)}v(k)=u(k), k\in[0,\infty), \ell>0. \quad (3)$$

where $\Delta^m_{\alpha(\ell)}u(k)=\Delta_{\alpha(\ell)}(\Delta^{m-1}_{\alpha(\ell)}u(k))$. Hence in this paper, we obtain $m(\alpha)-$ series to $\sin pk$ and $\cos qk$ by equating summation and closed form solution of equation (3).

II. Preliminaries

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be useful for further subsequent discussions. Let $\ell>0$ be fixed, $k\in[0,\infty), j=k-[\frac{k}{\ell}]\ell$ where $[\frac{k}{\ell}]$ denotes the integer part of $\frac{k}{\ell}$. Throughout this paper, $\alpha\neq 0$ and 1, m is positive integer, $u(k)$ defined on $[0,\infty)$ and $u(k)=0$, $k\in(-\infty,0)$. Consider the power set notations; $L_{m-1}=\{1,2,\ldots,m-1\}$, $0(L_{m-1})=\{\phi\}$, where ϕ is empty set, $1(L_{m-1})=\{\{1\},\{2\},\{3\},\cdots,\{m-1\}\}$, $2(L_{m-1})=\{\{1,2\},\{1,3\},\cdots,\{1,m-1\}, \{2,3\},\cdots,\{2,m-1\}, \cdots,\{m-2,m-1\}\}$. In general, $t(L_{m-1})$ = set of all subsets of size t from the set L_{m-1}, $t(L_{m-1})=\{\{1,2,\cdots,m-1\}\}$, $\wp(L_{m-1})=\bigcup_{t=0}^{m-1}t(L_{m-1})$, power set of L_{m-1}, $\sum_{t=1}^{m-1}f(t)=0$ for $m\leq 1$, and $\prod_{i=2}^{t}f(i)=1$ for $t\leq 1$.

Definition 2.1 *[10] Let $u(k)$, $k\in[0.\infty)$ be a real valued function and $\ell\in(0,\infty)$ be fixed. Then the generalized α difference operator on $u(k)$ is defined as:*

$$\Delta_{\alpha(\ell)}u(k)=u(k+\ell)-\alpha u(k). \quad (4)$$

Lemma 2.2 *[10] The inverse of the generalized $\alpha-$difference operator denoted by $\Delta^{-1}_{\alpha(\ell)}$ is defined as follows. If $\Delta_{\alpha(\ell)}v(k)=u(k)$, then*

$$\Delta^{-1}_{\alpha(\ell)}u(k)=v(k)-\alpha^{[\frac{k}{\ell}]}v(j) \quad (5)$$

is solution of equation (3) when m=1.

Lemma 2.3 *Let p be any real number such that p ℓ is not integer multiple of 2 π. Then, when $m=1$, equation (3) has solutions*

$$\Delta^{-1}_{\alpha(\ell)}\sin pk=\frac{\sin p(k-\ell)-\alpha\sin pk}{1-2\alpha\cos p\ell+\alpha^2}+c_j \quad (6)$$

and

$$\Delta^{-1}_{\alpha(\ell)}\cos pk=\frac{\cos p(k-\ell)-\alpha\cos pk}{1-2\alpha\cos p\ell+\alpha^2}+c_j \quad (7)$$

Proof. Replacing u(k) by $\sin pk$ and $\cos pk$ in (4), we find that

$$\Delta_{\alpha(\ell)}\sin pk=\sin pk(\cos p\ell-\alpha)+\cos pk\sin p\ell \quad (8)$$

and

$$\Delta_{\alpha(\ell)}\cos pk=\cos pk(\cos p\ell-\alpha)-\sin pk\sin p\ell. \quad (9)$$

Now, multiplying (8) by $(\cos p\ell-\alpha)$, (9) by $\sin p\ell$ and then subtracting the second resultant from the first one, we find that

$$\Delta_{\alpha(\ell)}\{(\cos p\ell-\alpha)\sin pk-\sin p\ell\cos pk\}=(1-2\alpha\cos p\ell+\alpha^2)\sin pk. \qquad (10)$$

Now, (6) follows from (6) and dividing (10) by $(1-2\alpha\cos p\ell+\alpha^2)$.

Similarly multiplying (8) by $\sin p\ell$, (9) by $(\cos p\ell-\alpha)$ and then adding them, we find that

$$\Delta_{\alpha(\ell)}\{\sin pk\sin p\ell-(\cos p\ell-\alpha)\cos pk\}=(1-2\alpha\cos p\ell+\alpha^2)\cos pk \qquad (11)$$

Now (7) follows from Definition (2.2) and dividing (11) by $(1-2\alpha\cos p\ell+\alpha^2)$.

Lemma 2.4 *If* $p\ell$ *and* $q\ell$ *are not multiple of* 2π *, then*

$$\Delta_{\alpha(\ell)}^{-m}\sin pk=\sum_{t=0}^{m}(-1)^t\frac{m^{(t)}}{t!}\alpha^t\frac{\sin p(k-(m-t)\ell)}{(1-2\alpha\cos p\ell+\alpha^2)^m}+c_j, \qquad (12)$$

$$\Delta_{\alpha(\ell)}^{-m}\cos qk=\sum_{t=0}^{m}(-1)^t\frac{m^{(t)}}{t!}\alpha^t\frac{\cos q(k-(m-t)\ell)}{(1-2\alpha\cos q\ell+\alpha^2)^m}+c_j \qquad (13)$$

are closed form solutions of equation (3) when $u(k)=\sin pk$, $\cos qk$ respectively .

Proof. When $m=1$, (12) and (13) are obtained from (6) and (7). By induction on m , $m\geq 2$, we assume that,

$$\Delta_{\alpha(\ell)}^{-(m-1)}\sin pk=\sum_{t=0}^{m-1}(-1)^t\frac{(m-1)^{(t)}}{t!}\alpha^t\frac{\sin p(k-(m-1-t)\ell)}{(1-2\alpha\cos p\ell+\alpha^2)^{(m-1)}}+c_j. \qquad (14)$$

From (6), we have

$$\Delta_{\alpha(\ell)}^{-1}\sin p(k-(m-1-t)\ell)=\frac{\sin p(k-(m-t)\ell)-\alpha\sin p(k-(m-1-t)\ell)}{(1-2\alpha\cos p\ell+\alpha^2)}. \qquad (15)$$

Since $\frac{(m-1)^{(r-1)}}{(r-1)!}+\frac{(m-1)^{(r)}}{r!}=\frac{m^{(r)}}{r!}$, (12) follows by taking $\Delta_{\alpha(\ell)}^{-1}$, applying (15) and equating coefficients of $\sin p(k-(m-t)\ell)$ for $t=0,1,\cdots,m$.

Similar argument and (7) gives the proof of (13).

Lemma 2.5 *[9] Let* $n\in N(1)$, $k\in[0,\infty)$ *and* p,q *are constants. Then*

$$\sin^n pk=\begin{cases}\dfrac{1}{2^{n-1}}\displaystyle\sum_{r=0}^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}+r}\frac{n^{(r)}}{r!}\sin p(n-2r)k & \text{ifnisodd}\\[2ex] \dfrac{1}{2^{n-1}}\left[\displaystyle\sum_{r=0}^{\frac{n-2}{2}}(-1)^{\frac{n}{2}+r}\frac{n^{(r)}}{r!}\cos p(n-2r)k+\frac{n^{\left(\frac{n}{2}\right)}}{2\left(\frac{n}{2}\right)!}\right] & \text{ifniseven.}\end{cases} \qquad (16)$$

and

$$\cos^n qk = \begin{cases} \dfrac{1}{2^{n-1}} \displaystyle\sum_{r=0}^{\frac{n-1}{2}} \frac{n^{(r)}}{r!} \cos q(n-2r)k & \text{ifnisodd} \\ \dfrac{1}{2^{n-1}} \left[\displaystyle\sum_{r=0}^{\frac{n-2}{2}} \frac{n^{(r)}}{r!} \cos q(n-2r)k + \frac{n^{\left(\frac{n}{2}\right)}}{2\left(\frac{n}{2}\right)!} \right] & \text{ifniseven.} \end{cases} \quad (17)$$

Remark 2.6 *Now consider the following Summation Notation*

(i) If n_1 and n_2 are odd positive integer. Then we denote

$$\sum_{(n_1,n_2)}^{s,c} = \frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}} \sum_{r_2=0}^{\frac{n_2-1}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!}.$$

(ii) If n_1 is odd and n_2 is even positive integer. Then we denote

$$\sum_{(n_1,n_2]}^{s,c} = \frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}} \sum_{r_2=0}^{\frac{n_2-2}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!}.$$

(iii) If n_1 is even and n_2 is odd positive integer. Then we denote

$$\sum_{[n_1,n_2)}^{s,c} = \frac{(-1)^{\frac{n_1}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-2}{2}} \sum_{r_2=0}^{\frac{n_2-1}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!}.$$

(iv) If n_1 and n_2 both are even positive integer. Then we denote

$$\sum_{[n_1,n_2]}^{s,c} = \frac{(-1)^{\frac{n_1}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-2}{2}} \sum_{r_2=0}^{\frac{n_2-2}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!}.$$

(v) we take $P = p(n_1-2r_1)+q(n_2-2r_2)$ and $\overline{P} = p(n_1-2r_1)-q(n_2-2r_2)$ and hence P and $\overline{P}$ are varying with respect to n_1, n_2, r_1, r_2, p and q.

Corollary 2.7 *(i) If n_1 and n_2 are odd positive integers, then*

$$\sin^{n_1} pk \cos^{n_2} qk = \sum_{(n_1,n_2)}^{s,c} (\sin Pk + \sin \overline{P}k). \quad (18)$$

(ii) If n_1 is an odd positive integer and n_2 is an even positive integer, then

$$\sin^{n_1} pk \cos^{n_2} qk = \sum_{(n_1,n_2]}^{s,c} \left\{ (\sin Pk + \sin \overline{P}k) + \frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!} \sin\left(\frac{P+\overline{P}}{2}\right)k \right\}. \quad (19)$$

(iii) If n_1 is an even positive integer and n_2 is an odd positive integer, then

$$\sin^{n_1} pk \cos^{n_2} qk = \sum_{(n_1,n_2]}^{s,c} \left\{ (\cos Pk + \cos \overline{P}k) + \frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!} \cos\left(\frac{P-\overline{P}}{2}\right)k \right\}. \quad (20)$$

(iv) If n_1 and n_2 are even positive integers, then

$$\sin^{n_1} pk\cos^{n_2} qk = \sum_{[n_1,n_2]}^{s,c}\left\{(\cos Pk+\cos\overline{P}k)+\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\cos\left(\frac{P-\overline{P}}{2}\right)k\right.$$

$$\left.+\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n}{2})!}\cos\left(\frac{P+\overline{P}}{2}\right)k+\frac{1}{2}\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\right\}. \qquad (21)$$

III. Main Result

In this section we equate the summation and closed form solutions of equation (3) and obtain formula for $m(\alpha)-$ series to circular functions.

Theorem 3.1 *[11] ($m(\alpha)-$ series formula) If closed form solution $\Delta_{\alpha(\ell)}^{-t}u(k)$ of equation (3) exits, for $t=1,2,\cdots,m$, then*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\alpha^{r-m}u(k-r\ell)=F_{m(\alpha)}u(k)-\alpha^{[\frac{k}{\ell}]-(m-1)}F_{m(\alpha)}u((m-1)\ell+j), \qquad (22)$$

where $$F_{m(\alpha)}u(k)=\Delta_{\alpha(\ell)}^{-m}u(k)+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(m-m_t)}}{(m-m_t)!}\alpha^{[\frac{k}{\ell}]-(m-1)}$$

$$\Delta_{\alpha(\ell)}^{-m_1}u((m_1-1)\ell+j)\prod_{i=2}^{t}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!}\alpha^{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil-(m_i-1)}. \qquad (23)$$

We give the following Theorem which will be used to obtain $m(\alpha)-$ series to circular function.

Theorem 3.2 *Let $k\in[\ell,\infty)$ and $j=k-[\frac{k}{\ell}]\ell$. Then*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\alpha^{r-m}(k-r\ell)^0=F_{m(\alpha)}u(k)-\alpha^{[\frac{k}{\ell}]-(m-1)}F_{m(\alpha)}u((m-1)\ell+j), \qquad (24)$$

where $$F_{m(\alpha)}u(k)=\Delta_{\alpha(\ell)}^{-m}k^0+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\frac{\left([\frac{k}{\ell}]\right)^{(m-m_t)}}{(m-m_t)!}\alpha^{[\frac{k}{\ell}]-(m-1)}$$

$$\Delta_{\alpha(\ell)}^{-m_1}((m_1-1)\ell+j)^0\prod_{i=2}^{t}\frac{[\frac{(m_i-1)\ell+j}{\ell}]^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!}\alpha^{[\frac{(m_i-1)\ell+j}{\ell}]-(m_i-1)}. \qquad (25)$$

Proof. The proof follows by taking $u(k)=k^0$ in Theorem 3.1.

Remark 3.3 *Here after we denote* $\Pi(t)=\prod_{i=2}^{t}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!}\alpha^{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil-(m_i-1)}$

and $P\ell$, $\overline{P}\ell$, $\left[\frac{P+\overline{P}}{2}\right]\ell$, $\left[\frac{P-\overline{P}}{2}\right]\ell$ are not integer multiple of 2π.

Theorem 3.4 *If* n_1 *and* n_2 *are odd positive integers, then* $m(\alpha)-$ *series to* $sin^{n_1}p(k)cos^{n_2}q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\alpha^{r-m}\sin^{n_1}p(k-r\ell)\cos^{n_2}q(k-r\ell)=F_{m(\alpha)}u(k)$$

$$-\alpha^{[\frac{k}{\ell}]-(m-1)}F_{m(\alpha)}u((m-1)\ell+j), \qquad (26)$$

where $F_{m(\alpha)}u(k)=\sum_{(n_1,n_2)}^{s,c}\left\{\sum_{r_3=0}^{m}(-1)^{r_3}\frac{m^{(r_3)}}{r_3!}\alpha^{r_3}\left(\frac{\sin P(k-(m-r_3)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^m}+\right.\right.$

$$\left.\frac{\sin\overline{P}(k-(m-r_3)\ell)}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^m}\right)+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\sum_{r_4=0}^{m_1}(-1)^{r_4}\frac{m_1^{(r_4)}}{r_4!}\alpha^{r_4}$$

$$\left.\left(\frac{\sin P((r_4-1)\ell+j)}{(1-2\alpha\cos P\ell+\alpha^2)^{m_1}}+\frac{\sin\overline{P}((r_4-1)\ell+j}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^{m_1}}\right)\Pi(t)\frac{([\frac{k}{\ell}])^{(m-m_t)}}{(m-m_t)!}\alpha^{[\frac{k}{\ell}]-(m-1)}\right\}.$$

Proof. The proof is obtained by replacing $u(k)$ by $\sin^{n_1}pk\ \cos^{n_2}qk$ in Theorem 3.1 and applying equation (18) on Lemma 2.3.

Remark 3.5 *When* $n_2=0$ *in (26) we will get* $\Delta_{\alpha(\ell)}^{-m}\sin^{n_1}Pk$ *and when* $n_1=0$ *in (26) we will get* $\Delta_{\alpha(\ell)}^{-m}\cos^{n_2}Pk$.

The following example illustrates a 4-series to $\sin^3 6k\cos^3 5k$,

Example 3.6 *Consider the case* $m=4$, $p=6$, $q=5$, $n_1=3$, $n_2=3$, $P=(6(3-2r_1))+5(3-2r_2)$ *and* $\overline{P}=(6(3-2r_1)-5(3-2r_2))$. *In this case,* $L_3=\{1,2,3\},1(L_3)=\{\{1\},\{2\},\{3\}\},2(L_3)=\{\{1,2\},\{2,3\},\{1,3\}\},3(L_3)=\{\{1,2,3\}\}$ *and (26) becomes*

$$\sum_{r=4}^{[\frac{k}{\ell}]}\frac{(r-1)^{(4-1)}}{(4-1)!}\alpha^{r-4}\sin^3 6(k-r\ell)\cos^3 5(k-r\ell)=F_{4(\alpha)}u(k)-\alpha^{[\frac{k}{\ell}]-3}F_{4(\alpha)}u(3\ell+j) \qquad (27)$$

where $F_{4(\alpha)}u(k)=\sum_{(3,3)}^{s,c}\left\{\sum_{r_3=0}^{4}(-1)^{r_3}\frac{4^{(r_3)}}{r_3!}\alpha^{r_3}\left(\frac{\sin P(k-(4-r_3)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^4}+\right.\right.$

$$\left.\frac{\sin\overline{P}(k-(4-r_3)\ell)}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^4}\right)+\sum_{t=1}^{4-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\sum_{r_4=0}^{m_1}(-1)^{r_4}\frac{m_1^{(r_4)}}{r_4!}\alpha^{r_4}$$

$$\left.\left(\frac{\sin P((r_4-1)\ell+j)}{(1-2\alpha\cos P\ell+\alpha^2)^{m_1}}+\frac{\sin\overline{P}((r_4-1)\ell+j}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^{m_1}}\right)\Pi(t)\frac{([\frac{k}{\ell}])^{(4-m_t)}}{(4-m_t)!}\alpha^{[\frac{k}{\ell}]-(4-1)}\right\}.$$

The five summation expression of (27) can be obtained by adding the sums corresponds to

$$\sum_{(3,3)}^{s,c}\sum_{r_4=0}^{1}(-1)^{r_4}\frac{1^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}+\frac{\sin\overline{P}(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}\right)\frac{([\frac{k}{\ell}])^{(3)}}{3!}\alpha^{[\frac{k}{\ell}]-3}$$

$$\sum_{(3,3)}^{s,c}\sum_{r_4=0}^{2}(-1)^{r_4}\frac{2^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P((\ell+j)-(2-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^2}+\frac{\sin\overline{P}((\ell+j)-(2-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^2}\right)\frac{([\frac{k}{\ell}])^{(2)}}{2!}\alpha^{[\frac{k}{\ell}]-3}$$

$$\sum_{(3,3)}^{s,c}\sum_{r_4=0}^{3}(-1)^{r_4}\frac{3^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P((2\ell+j)-(3-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^3}+\frac{\sin\overline{P}((2\ell+j)-(2-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^3}\right)\frac{([\frac{k}{\ell}])^{(1)}}{1!}\alpha^{[\frac{k}{\ell}]-3}$$

Corresponds to $2(L_3)$

$$\sum_{(3,3)}^{s,c}\sum_{r_4=0}^{1}(-1)^{r_4}\frac{1^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}+\frac{\sin\overline{P}(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}\right)$$

$$\frac{[\frac{(2-1)\ell+j}{\ell}]^{(1)}}{1!}\alpha^{[\frac{(2-1)\ell+j}{\ell}]-1}\frac{([\frac{k}{\ell}])^{(2)}}{2!}\alpha^{[\frac{k}{\ell}]-3}$$

$$\sum_{(3,3)}^{s,c}\sum_{r_4=0}^{2}(-1)^{r_4}\frac{2^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P(\ell+j-(2-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^2}+\frac{\sin\overline{P}(\ell+j-(2-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^2}\right)$$

$$\frac{[\frac{(3-1)\ell+j}{\ell}]^{(1)}}{1!}\alpha^{[\frac{(3-1)\ell+j}{\ell}]-2}\frac{([\frac{k}{\ell}])^{(1)}}{1!}\alpha^{[\frac{k}{\ell}]-3}$$

$$\sum_{(3,3)}^{s,c}\sum_{r_4=0}^{1}(-1)^{r_4}\frac{1^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}+\frac{\sin\overline{P}(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}\right)$$

$$\frac{[\frac{(3-1)\ell+j}{\ell}]^{(2)}}{2!}\alpha^{[\frac{(3-1)\ell+j}{\ell}]-2}\frac{([\frac{k}{\ell}])^{(1)}}{1!}\alpha^{[\frac{k}{\ell}]-3}$$

and to $3(L_3)$

$$\sum_{(3,3)}^{s,c}\sum_{r_4=0}^{1}(-1)^{r_4}\frac{1^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}+\frac{\sin\overline{P}(j-(1-r_4)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^1}\right)$$

$$\frac{[\frac{(2-1)\ell+j}{\ell}]^{(1)}}{1!}\alpha^{[\frac{(2-1)\ell+j}{1!}]-3}\frac{[\frac{(3-1)\ell+j}{\ell}]^{(1)}}{1!}\alpha^{[\frac{(3-1)\ell+j}{\ell}]-3}\frac{[\frac{k}{\ell}]^{(1)}}{1!}\alpha^{([\frac{k}{\ell}])-3}.$$

Theorem 3.7 *If* n_1 *is an odd positive integer and* n_2 *is an even positive integer, then the* $m(\alpha)-$ *series to* $\sin^{n_1}p(k)\cos^{n_2}q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\alpha^{r-m}\sin^{n_1}p(k-r\ell)\cos^{n_2}q(k-r\ell)=F_{m(\alpha)}u(k)$$

$$-\alpha^{[\frac{k}{\ell}]-(m-1)}F_{m(\alpha)}u((m-1)\ell+j), \qquad (28)$$

where $$F_{m(\alpha)}u(k)=\sum_{(n_1,n_2]}^{s,c}\left\{\sum_{r_3=0}^{m}(-1)^{r_3}\frac{m^{(r_3)}}{r_3!}\alpha^{r_3}\left(\frac{\sin P(k-(m-r_3)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^m}+\right.\right.$$

$$\left.\frac{\sin\overline{P}(k-(m-r_3)\ell)}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^m}\right)+\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\left(\frac{\sin(\frac{P+\overline{P}}{2})(k-(m-r_3)\ell)}{(1-2\alpha\cos(\frac{P+\overline{P}}{2})\ell+\alpha^2)^m}\right)$$

$$+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\sum_{r_4=0}^{m_1}(-1)^{r_4}\frac{m_1^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\sin P((r_4-1)\ell+j)}{(1-2\alpha\cos P\ell+\alpha^2)^{m_1}}\right.$$

$$\left.+\frac{\sin\overline{P}((r_4-1)\ell+j}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^{m_1}}\right)+\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\left(\frac{\sin(\frac{P+\overline{P}}{2})((r_4-1)\ell+j))}{(1-2\alpha\cos(\frac{P+\overline{P}}{2})\ell+\alpha^2)^{m_1}}\right)$$

$$\left.\Pi(t)\frac{([\frac{k}{\ell}])^{(m-m_t)}}{(m-m_t)!}\alpha^{[\frac{k}{\ell}]-(m-1)}\right\}.$$

Proof. The proof is obtained by replacing $u(k)$ by $\sin^{n_1}pk\ \cos^{n_2}qk$ in Theorem 3.1 and applying equation (19) on Lemma 2.3.

Theorem 3.8 *If* n_1 *is an even positive integer and* n_2 *is an odd positive integer, then the* $m(\alpha)-$ *series to* $\sin^{n_1}p(k)\cos^{n_2}q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\alpha^{r-m}\sin^{n_1}p(k-r\ell)\cos^{n_2}q(k-r\ell)=F_{m(\alpha)}u(k)$$

$$-\alpha^{[\frac{k}{\ell}]-(m-1)}F_{m(\alpha)}u((m-1)\ell+j)\qquad(29)$$

where $$F_{m(\alpha)}u(k)=\sum_{[n_1,n_2)}^{s,c}\left\{\sum_{r_3=0}^{m}(-1)^{r_3}\frac{m^{(r_3)}}{r_3!}\alpha^{r_3}\left(\frac{\cos P(k-(m-r_3)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^m}+\right.\right.$$

$$\left.\frac{\cos\overline{P}(k-(m-r_3)\ell)}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^m}\right)+\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\left(\frac{\cos(\frac{P-\overline{P}}{2})(k-(m-r_3)\ell)}{(1-2\alpha\cos(\frac{P-\overline{P}}{2})\ell+\alpha^2)^m}\right)$$

$$+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\sum_{r_4=0}^{m_1}(-1)^{r_4}\frac{m_1^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\cos P((r_4-1)\ell+j)}{(1-2\alpha\cos P\ell+\alpha^2)^{m_1}}\right.$$

$$+\frac{\cos\overline{P}((r_4-1)\ell+j)}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^{m_1}}\Bigg)+\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\left(\frac{\cos(\frac{P-\overline{P}}{2})((r_4-1)\ell+j)}{(1-2\alpha\cos(\frac{P-\overline{P}}{2})\ell+\alpha^2)^{m_1}}\right)$$

$$\Pi(t)\frac{([\frac{k}{\ell}])^{(m-m_t)}}{(m-m_t)!}\alpha^{[\frac{k}{\ell}]-(m-1)}\Bigg\}.$$

Proof. The proof is obtained by replacing $u(k)$ by $\sin^{n_1} pk\ \cos^{n_2} qk$ in Theorem 3.1 and applying equation (20) on Lemma 2.3.

Theorem 3.9 *If* n_1 *and* n_2 *are even positive integer then the* $m(\alpha)-$*series to* $\sin^{n_1} p(k)\cos^{n_2} q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\alpha^{r-m}\sin^{n_1} p(k-r\ell)\cos^{n_2} q(k-r\ell)=F_{m(\alpha)}u(k)$$

$$-\alpha^{[\frac{k}{\ell}]-(m-1)}F_{m(\alpha)}u((m-1)\ell+j) \qquad (30)$$

where $$F_{m(\alpha)}u(k)=\sum_{[n_1,n_2]}^{s,c}\left\{\sum_{r_3=0}^{m}(-1)^{r_3}\frac{m^{(r_3)}}{r_3!}\alpha^{r_3}\left(\frac{\cos P(k-(m-r_3)\ell)}{(1-2\alpha\cos P\ell+\alpha^2)^m}+\right.\right.$$

$$\left.\frac{\cos\overline{P}(k-(m-r_3)\ell)}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^m}\right)+\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\left(\frac{\cos(\frac{P-\overline{P}}{2})(k-(m-r_3)\ell)}{(1-2\alpha\cos(\frac{P-\overline{P}}{2})\ell+\alpha^2)^m}\right)$$

$$+\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\left(\frac{\cos(\frac{P+\overline{P}}{2})(k-(m-r_3)\ell)}{(1-2\alpha\cos(\frac{P+\overline{P}}{2})\ell+\alpha^2)^m}\right)+\frac{1}{2}\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\Delta_{\alpha(\ell)}^{-m}k^0$$

$$+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\left\{\sum_{r_4=0}^{m_1}(-1)^{r_4}\frac{m_1^{(r_4)}}{r_4!}\alpha^{r_4}\left(\frac{\cos P((r_4-1)\ell+j)}{(1-2\alpha\cos P\ell+\alpha^2)^{m_1}}\right.\right.$$

$$+\frac{\cos\overline{P}((r_4-1)\ell+j}{(1-2\alpha\cos\overline{P}\ell+\alpha^2)^{m_1}}\Bigg)+\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\left(\frac{\cos(\frac{P-\overline{P}}{2})((r_4-1)\ell+j)}{(1-2\alpha\cos(\frac{P-\overline{P}}{2})\ell+\alpha^2)^{m_1}}\right)$$

$$+\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\left(\frac{\cos(\frac{P+\overline{P}}{2})((r_4-1)\ell+j)}{(1-2\alpha\cos(\frac{P+\overline{P}}{2})\ell+\alpha^2)^{m_1}}\right)+\frac{1}{2}\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\Delta_{\alpha(\ell)}^{-m_1}((m-1)\ell+j)^0\Bigg\}$$

$$\Pi(t)\frac{([\frac{k}{\ell}])^{(m-m_t)}}{(m-m_t)!}\alpha^{[\frac{k}{\ell}]-(m-1)}\Bigg\}.$$

Proof. The proof is obtained by replacing $u(k)$ by $\sin^{n_1} pk \ \cos^{n_2} qk$ in Theorem 3.1, applying equation (21) on Lemma 2.3 and using Theorem 3.2.

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References

[1] Jerzy Popenda and Blazej Szmanda, On the Oscillation of Solutions of Certain Difference Equations, Demonstratio Mathematica, XVII(1), (1984), 153 - 164.

[2] R. A. C. Ferreira and D. F. M. Torres, Fractional h-difference equations arising from the calculus of variations, Applicable Analysis and Discrete Mathematics, 5(1) (2011), 110-121.

[3] M. M. Susai Manuel, G.B. A. Xavier, V. Chandrasekar and R. Pugalarasu, Theory and application of the Generalized Difference Operator of the n^{th} kind(Part I), Demonstratio Mathematica,vol.45,no.1,pp.95-106,2012.

[4] M.Maria Susai Manuel, V.Chandrasekar and G.Britto Antony Xavier, Solutions and Applications of Certain Class of α -Difference Equations, International Journal of Applied Mathematics, 24(6) (2011), 943-954.

[5] M.Maria Susai Manuel, V.Chandrasekar and G.Britto Antony Xavier, Theory of Generalized α -Difference Operator and its Applications in Number Theory, Advances in Differential Equations and Control Processes, 9(2) (2012), 141-155.

[6] K.S.Miller and B.Ross, Fractional Difference Calculus in Univalent Functions, Fractional Calculus and Their Applications(Koriyama, 1988),pp . 139-152, Horwood, Chichester, UK, 1989.

[7] N. R. O. Bastos, R. A. C. Ferreira, and D. F. M. Torres, Discrete-Time Fractional Variational Problems, Signal Processing, vol.91,no. 3,pp. 513-524, 2011.

[8] G. Britto Antony Xavier, S. U. Vasanthakumar, and B. Govindan, Discrete Beta-Factorial Function and Its Series in Terms of Closed form solution of Generalized Difference Equation International journal of Mathematical Archive-4(10), Oct.-2013

[9] G. Britto Antony Xavier, S. Sathya and S. U. Vasanthakumar, m-Series Of The Generalized Difference Equation To Circular Functions International journal of Mathematical Archive-4(7), 2013, 200-209.

[10] M.Maria Susai Manuel, G.Britto Antony Xavier, D.S.Dilip and G.Dominic Babu. Oscillation, nonoscillation and growth of solutions of generalized second order nonlinear α difference equations. Global J. Math. Sci. Theory and practical,4(1):121:132, 2012.

[11] G. Britto Antony Xavier, H.Nasira Begum, $m(\alpha)-$ Series Of The Generalized $\alpha-$ Difference Equation To Polynomial Factorial International journal of Mathematics Trends and Technology Volume 11 Number 2 Jul 2014