IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 5 Ver. IV (Sep-Oct. 2014), PP 48-52
www.iosrjournals.org

On The Fastness of the Convergence between Mann and Noor Iteration for the Class of Zamfirescu Operators

Lokesh Kumar
Department of Mathematics, Indian Military Academy, Dehradun, India

Abstract: *Operator theory owes its origin in the book titled "Theorie des Operators Lineaires" by Stefan Banach before the middle of 20th century. The convergence of a class of operators is important since a number of iterations have been developed. The aim of this paper is to established that the Mann iteration converges faster than the Noor Iteration for the class of Zamfirescu operators of an arbitrary closed convex subset of a Banach Space.*

I. Introduction

Let E be a normed linear space, K be an arbitrary closed convex subset of E and T is a self map on K.

Let $x_0 \in K$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence of real numbers. The sequence $\{x_n\}_{n=0}^{\infty} \subset K$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, ... \tag{1.1}$$

is called Mann iteration.

Let $y_0 \in K$ be arbitrary and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences of real numbers in [0, 1]. The sequence $\{y_n\}_{n=0}^{\infty} \subset K$ defined by

$$\begin{aligned} y_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Tz_n, \quad n = 0, 1, 2, ... \\ z_n &= (1 - \beta_n)y_n + \beta_n Tw_n, \quad n = 0, 1, 2, ... \\ w_n &= (1 - \gamma_n)y_n + \gamma_n Ty_n, \quad n = 0, 1, 2, ... \end{aligned} \tag{1.2}$$

is called Noor iteration.

For $\gamma_n = 0$, $\beta_n = 0$, the Noor iteration (1.2) reduces to the Mann iteration (1.1).

Definition 1.1[4]: Let $T : K \to K$ be a map for which there exist real numbers p, q, r satisfying $0 < p < 1$, $0 < q < 1/2$, $0 < r < 1/2$, then T is called a Zamfirescu operator [4] if, for each pair x, y in K, T satisfies at least one of the following conditions given in $(z_1) - (z_3)$;

(z_1) $\quad \|Tx - Ty\| \leq p\|x - y\|$;
(z_2) $\quad \|Tx - Ty\| \leq q\|x - Tx\| + \|x - Ty\|$;
(z_3) $\quad \|Tx - Ty\| \leq r(\|x - y\| + \|y - Tx\|)$

Then T has a unique fixed point s and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n \qquad n = 0, 1, 2, ... \tag{1.3}$$

converges to s, for any $x_0 \in K$.

Definition 1.2 : Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b respectively, and assume that there exists

$$e = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} \tag{1.4}$$

If e = 0, then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.

Definition 1.3: Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p with the error estimates

$$\begin{aligned} \|u_n - p\| &\leq a_n; \qquad n = 0, 1, 2, ... \\ \|v_n - p\| &\leq b_n; \qquad n = 0, 1, 2, ... \end{aligned} \tag{1.5}$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then we say that $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.

We use Definitions 1.2 and 1.3 to prove our main results.

Based on Definition 1.3 G.V.R. Babu and K.N.V.V. Vara Prasad [8] compared the Mann and Ishikawa iteration of the class of Zamfirescu operators defined on a closed convex subset of a uniformly convex Banach space and concluded the Mann iteration always converges faster than the Ishikawa iteration.

Theorem 1.4: Le E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ be a Zamfirescu operator. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.1) and $x_0 \in K$, with $\{a_n\} \subset [0, 1]$ satisfying

(i) $\alpha_0 = 1$

(ii) $0 \le \alpha_n < 1$, for $n \ge 1$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T and, moreover, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.3) for $x_0 \in K$ converges faster than the Mann iteration.

The aim of this paper is to show that the Mann iteration converges faster than the Noor iteration.

For this we use the following theorem

II. Main Theorem

Theorem 2.1: Let E be an arbitrary Banach space, K a closed convex subset of E, and $T:K\to K$ an operator satisfying condition Z. Let $\{y_n\}_{n=0}^{\infty}$ be the Noor iteration defined by (1.2) and $y_0 \in K$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0, 1] with $\sum_{n=0}^{\infty} \alpha_n = \infty = \sum_{n=0}^{\infty} \beta_n$.

Then $\{y_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof: By theorem z, we know that T has a unique fixed point in k, say p consider x, y $\in$ K. Since T is a Zamfirescu operator, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied. If (z_1) holds, then

$$\|Tx - Ty\| \le b\|x - Tx\| + \|y - Ty\|$$

$$\le b\{\|x - Tx\| + [\|y - x\| + \|x - Tx\| + \|Tx - Ty\|]\}$$

So $(1 - b)\|Tx - Ty\| \le b\|x - y\| + 2b\|x - Tx\|$

which yields (using the fact that $0 \le b < 1$)

$$\|Tx - Ty\| \le \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\| \qquad (2.1)$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \le \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Tx\| \qquad (2.2)$$

Denote $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$ (2.3)

Then $0 \le \delta < 1$, in view of (z_1), (2.2) and (2.4)

it results that the inequality

$$\|Tx - Ty\| \le \delta\|x - y\| + 2\delta\|x - Tx\| \qquad (2.5)$$

holds for all x, y $\in$ K.

Now let $\{y_n\}_{n=0}^{\infty}$ be the Noor iteration defined by (1.2) and $y_0 \in K$ arbitrary

Then

$$\|y_{n+1} - p\| = \|(1-\alpha_n)y_n + \alpha_n Tz_n - p\| \qquad (2.6)$$

with x = p and y = z_n from (1.9) we obtain

$$\|Tz_n - p\| \le \delta\|z_n - p\| \qquad (2.7)$$

where δ is given by (2.4)

Further we have

$$\|z_n - p\| = \|(1-\beta_n)y_n + \beta_n Tw_n - p\|$$

$$\le (1 - \beta_n)\|y_n - p\| + \beta_n\|Tw_n - p\| \qquad (2.8)$$

with x = p and y = w_n

$$\|Tw_n - p\| \leq \delta\|w_n - p\| \tag{2.9}$$

Further

$$\|w_n - p\| = \|(1-\gamma_n)y_n + \gamma_n Ty_n - p\|$$
$$\|w_n - p\| \leq (1-\gamma_n)\|y_n - p\| + \gamma_n\|Ty_n - p\| \tag{2.10}$$

with $x = p$ and $y = y_n$ from (2.5) we obtain

$$\|Ty_n - p\| \leq \delta\|y_n - p\| \tag{2.11}$$

and Hence by (2.6) – (2.11) we obtain

$$\|y_{n+1} - p\| \leq [1-\alpha_n(1-\delta)(1+\beta_n\delta + \beta_n\gamma_n\delta^2)]\|y_n - p\|$$

which by the inequality

$$1-\alpha_n(1-\delta)(1+\beta_n\delta + \beta_n\gamma_n\delta^2) \leq 1-\alpha_n(1-\delta)^2$$

implies

$$\|y_{n+1} - p\| \leq [1-(1-\delta)^2\alpha_n]\|y_n - p\|, \quad n = 0, 1, 2, ... \tag{2.12}$$

By (1.16) we inductively obtain

$$\|y_{n+1} - p\| \leq \prod_{k=1}^{n}[1-(1-\delta)^2\alpha_k]\|y_1 - p\|, \quad n = 0, 1, 2, ... \tag{2.13}$$

using the fact that $0 \leq \delta < 1$, $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$

$$\lim_{n\to\infty}\prod_{k=1}^{n}[1-(1-\delta)^2\alpha_k] = 0$$

which by (2.13) implies

$$\lim_{n\to\infty}\|y_{n+1} - p\| = 0$$

i.e. $\{y_n\}_{n=0}^{\infty}$ converges strongly to p.

III. Supplimentary Theorem

On the comparison of fastness of the convergence.

Theorem 3.1: Let E be an arbitrary Banach space, K be an arbitrary closed convex subset of E, and $T:K\to K$ be Zamfirescue operator. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.1) for $x_0 \in K$ and $\{y_n\}_{n=0}^{\infty}$ be defined by (1.2) for $y_0 \in K$ with $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences satisfying

(a) $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$

(b) $\sum_{n=0}^{\infty}\alpha_n = \sum_{n=0}^{\infty}\beta_n = \infty$

Then $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T and moreover, the Mann iteration converges faster than the Noor iteration to the fixed point of T.

Proof: By [2, Theorem 1] established in [1], for $x_0 \in K$, the Mann iteration defined by (1.1) converges strongly to the unique fixed point of T.

By Theorem 2.1, for $y_0 \in K$, the Noor iteration defined by (1.2) converges strongly to the unique fixed point of T. By the uniqueness of fixed point for Zamfirescu operators, the Mann and Noor iteration must converges to the same unique fixed point p(say) of T.

Since T is a Zamfirescu operator, it satisfies the inequalities

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\| \tag{3.1}$$
$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|y - Tx\| \tag{3.2}$$

for all $x, y \in K$, where $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$ and $0 \leq \delta < 1$ see [ ]

Suppose that $x_0 \in K$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration associated with T, and $\{\alpha_n\}_{n=0}^{\infty}$. Now by using Mann iteration (1.1), we have

$$\|x_{n+1} - p\| \leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|Tx_n - p\| \tag{3.3}$$

on using (3.1) with $x = p$ and $y = x_n$, we get

$$\|Tx_n - p\| \leq \delta\|x_n - p\| \tag{3.4}$$

Therefore from (3.3)

$$\|x_{n+1} - p\| \le (1-\alpha_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\|$$
$$= [1-\alpha_n(1-\delta)]\|x_n - p\| \qquad (3.5)$$

and thus

$$\|x_{n+1} - p\| \le \prod_{k=1}^{n} [1-\alpha_k(1-\delta)]\|x_n - p\| \quad n = 0, 1, 2, ... \qquad (3.6)$$

Here we observe that

$$1 - \alpha_k(1-\delta) > 0 \quad \forall k = 0, 1, 2, ... \qquad (3.7)$$

Now let $\{y_n\}_{n=0}^{\infty}$ be the sequence defined by V or iteration (1.2) for $y_0 \in K$. Then we have

$$\|y_{n+1} - p\| \le (1-\alpha_n)\|y_n - p\| + \alpha_n\|Tz_n - p\| \qquad (3.8)$$

on using (3.2) with x = p and y = z_n

$$\|Tz_n - p\| \le 3\delta\|z_n - p\| \qquad (3.9)$$

Again using (3.2) with x = p and y = w_n

$$\|Tw_n - p\| \le 3\delta\|w_n - p\| \qquad (3.10)$$

Again using (3.2) with x = p and y = y_n

$$\|Ty_n - p\| \le 3\delta\|y_n - p\| \qquad (3.11)$$

Now $\quad \|z_n - p\| \le (1-\beta_n)\|y_n - p\| + \beta_n\|Tw_n - p\| \qquad (3.12)$

and $\quad \|w_n - p\| \le (1-\gamma_n)\|y_n - p\| + \gamma_n\|Ty_n - p\| \qquad (3.13)$

and hence by (3.8) – (3.13) we get

$$\|y_{n+1} - p\| \le (1-\alpha_n)\|y_n - p\| + 3\delta\alpha_n\|y_n - p\|$$
$$\le [(1-\alpha_n) + 3\delta\alpha_n(1-\beta_n)]\|y_n - p\| + 9\delta^2\alpha_n\beta_n\|w_n - p\|$$
$$\le [1-\alpha_n + 3\delta\alpha_n(1-\beta_n) + 9\delta^2\alpha_n\beta_n(1-\gamma_n)$$
$$+ 27\delta^2\alpha_n\beta_n\gamma_n]\|y_n - p\|$$

$$\|y_{n+1} - p\| \le [1-\alpha_n(1-3\delta)(1+3\delta\beta_n + 9\delta^2\beta_n\gamma_n)]\|y_n - p\| \qquad (3.14)$$

$\because \quad 0 \le \alpha_n \le 1,\ 0 \le \beta_n \le 1,\ 0 \le \gamma_n \le 1$ and $0 \le \delta < 1$

$\Rightarrow \quad 1-\alpha_n(1-3\delta)(1+3\delta\beta_n + 9\delta^2\beta_n\gamma_n) > 0, \quad \forall k = 0, 1, 2, ... \qquad (3.15)$

we have the following two cases

Case I: Let $\delta \in (0, \frac{1}{3}]$

$$1-\alpha_n(1-3\delta)(1+3\delta\beta_n + 9\delta^2\beta_n\gamma_n) \le 1, \quad n = 0, 1, 2, ... \qquad (3.16)$$

and Hence the inequality (3.14) becomes

$$\|y_{n+1} - p\| \le \|y_n - p\| \quad \forall n \qquad (3.17)$$

and Thus

$$\|y_{n+1} - p\| \le \|y_1 - p\| \qquad (3.18)$$

We now compare the coefficient of the in equalities (3.6) and (3.18) using definition (1.3)

$$a_n = \prod_{k=1}^{n} [1-\alpha_k(1-\delta)],\ b_n = 1 \qquad (3.19)$$

since $\quad \sum_{n=0}^{\infty} \alpha_n = \infty$

So we have $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Case II: Let $\delta \in (\frac{1}{3}, 1)$. In this case

$$1 \le 1-\alpha_n(1-3\delta)(1+3\delta\beta_n + 9\delta^2\beta_n\gamma_n) \le 1-\alpha(1-27\delta^3) \qquad (3.20)$$

so the inequality (3.14) becomes

$$\|y_{n+1} - p_1\| \le [1-\alpha_n(1-27\delta^3)]\|y_n - p\| \quad \forall n \qquad (3.21)$$

Therefore

$$\|y_{n+1} - p_1\| \leq \prod_{k=1}^{n} [1 - \alpha_k(1 - 27\delta^3)]\|y_1 - p\| \qquad (3.22)$$

We compare (3.6) and (3.20) using definition (1.3)

$$a_n = \prod_{k=1}^{n} [1 - \alpha_k(1 - \delta)]$$

$$b_n = \prod_{k=1}^{n} [1 - \alpha_k(1 - 27\delta^3)]$$

We observe that

$$\frac{1 - \alpha_k(1 - \delta)}{1 - \alpha_k(1 - 27\delta^3)} \leq 1 - \alpha_k(1 - \delta)$$

so that

$$\frac{a_n}{b_n} = \prod_{k=1}^{n} [1 - \alpha_k(1 - \delta)], \quad n = 0, 1, 2, ...$$

Thus $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0$

Note that a_n, $b_n \to 0$ as $n \to \infty$

Hence from case I and II it follows that $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ so that the Mann iteration $\{x_n\}$ converges faster than the Ishikawa iteration to the fixed point p of T.

Corollary 3.2: Under the hypothesis of theorem (3.1), Picard iteration defined by (1.3) converges faster than the Noor iteration defined by (1.4) to the fixed point of Zamfirescu operator.

References

[1]. V. Berinde, Comparing Krasnoselskij and Mann iterations for Lipschitzian generalized pseudocontractive operators, Proc. Int. Conf. Fixed Point Theory and Applications (Valencia, 2003), Yokohama Publishers, Yokohama, in press.

[2]. V. Berinde, On the convergence of Mann iteration for a class of quasi contractive operators, in preparation, 2004.

[3]. H. K. Xu, A note on the Ishikawa iteration scheme, J. Math. Anal. Appl. 167 (1992), no. 2, 582 – 587.

[4]. T. Zamfirescu, Fix point theorems in metric spaces, Arch. Math. (Basel) 23 (1972), 292 – 298.

[5]. V. Berinde, On the convergence of the Ishikawa iteration in the class of quasi contractive operators, Acta Mathematica Universitatics comeniance. New Series 73 (2004), no. 1, 119 – 126.

[6]. V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, Fixed Point Theory and Application (2004), no. 2, 97 – 105.

[7]. S. M. Soltuz, The equivalence of Picard, Mann and Ishikawa iteration dealing with quasi-contractive operators, Mathematical communications, vol. 10, (2005), no. 1, 81 – 88.

[8]. G. V. R. Babu and K. N. V. V. Vara Prasad, Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators, Fixed Point Theory and Applications, Vol. 2006, Articla ID 49615, 6 pages, 2006.

[9]. Xue Zhiqun, hemarks of Equivalence among Picard, Mann, and Ishikawa Iteration in Normed space, volume 2007, Article ID 61434, 5 pages