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Flow through an Oscillating Rectangular Duct for Generalized Oldroyd-B Fluid with Fractional Derivatives

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Abstract: *The analytic solution for the unsteady flow of generalized Oldroyd- B fluid on oscillating rectangular duct is studied. In the absence of the frequency of oscillations, we obtain the problem for the flow of generalized Oldroyd- B fluid in a duct of rectangular cross- section moving parallel to its length. The problem is solved by applying the double finite Fourier sine and discrete Laplace transforms. The solutions for the generalized Maxwell fluids and the ordinary Maxwell fluid appear as limiting cases of the solutions obtained here.*
Finally, the effect of material parameters on the velocity profile spotlighted by means of the graphical illustrations.
Keywords: *Generalized Oldroyd-B fluid, oscillating rectangular duct, velocity field.*

I. Introduction

The study of non-Newtonian fluid plays an important role in technological applications compared with Newtonian fluids because of their behavior. Several models have been proposed and examined to explain this non- linear behavior. One of the most popular subclasses of differential type fluids is the Oldryod- B fluid.

It has been found that the viscoelastic generalized Oldroyd- B fluid can be used to approximate the response of many dilute polymeric liquids successfully, and this approach has been widely applied to flow problems with small relaxation and retardation times, with classical Newtonian and Maxwell fluid being included as special cases [6].

In recent years, the study of non-Newtonian fluid flow has increased dramatically, and many of the researchers involved in obtaining exact solutions of the approach through the introduction of fractional calculus in various rheological problems [2,3,6,7,11].

Among these, Khan et al. [7] constructed the exact solutions for the accelerated flows of generalized Oldroyd- B fluid using the fractional calculus approach established constitutive relationship of a viscoelastic fluid model. Zheng et al. [6] deals with the 3D flow of a generalized Oldroyd- B fluid due to a constant pressure gradient between two side walls perpendicular to a plane. Hyder et al. [11] discussed the exact solutions for a viscoelastic fluid with generalized Oldroyd- B fluid.

In addition, some problems concerning unsteady flows through an oscillating rectangular duct have already been investigated. Johri, A. K. and Singh, M.[1] deals with an oscillating flow of a viscous liquid in a porous rectangular duct. Nazar, M. et al. [8] presented an analysis for the unsteady flow of incompressible Maxwell fluid in an oscillating rectangular cross section. Sultan, Q. et al. [10] discussed the analytic solution for the unsteady magnetohydrodynamic (MHD) flow of Oldroyd- B fluid in long porous rectangular cross- section. Nazar, M. et al. [9] determined the velocity filed and the shear stresses corresponding to the unsteady flow of generalized Maxwell fluid on oscillating rectangular duct. Nadeem et al. [12] discussed the Rayleigh Stokes problem for rectangular pipe in Maxwell and second grade fluids.

The purpose of this work is to present analytic solutions for generalized Oldroyd- B fluid on oscillating rectangular duct by means of double finite Fourier sine and discrete Laplace transforms for fractional calculus approach. Finally, the influences of the various parameters on the motion of generalized Oldroyd- B fluid are underlined by graphical illustrations.

II. Governing Equations

The constitutive equations for an incompressible fractional Oldroyd- B fluid given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad (1+\lambda_1^{\alpha}\tilde{D}_t^{\alpha})\mathbf{S}=\mu(1+\lambda_r^{\beta}\tilde{D}_t^{\beta})\mathbf{A} \qquad (1)$$

where $\mathbf{T}$ denoted the cauchy stress, $-p\mathbf{I}$ is the indeterminate spherical stress, $\mathbf{S}$ is the extra stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ is the first Rivlin- Ericksen tensor with the velocity gradient where $\mathbf{L} = \mathrm{grad}\,\mathbf{V}$, μ is the dynamic viscosity of the fluid, λ_1 and λ_r $(<\lambda_1)$ are the relaxation and retardation times, respectively, α and β the fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$ and $\tilde{D}_t^{p}$ the upper convected fractional derivative define by

$$\widetilde{D}_t^{\alpha}\mathbf{S} = D_t^{\alpha}\mathbf{S} + (\mathbf{V}.\nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}} \tag{2}$$

$$\widetilde{D}_t^{\beta}\mathbf{A} = D_t^{\beta}\mathbf{A} + (\mathbf{V}.\nabla)\mathbf{A} - \mathbf{LA} - \mathbf{AL}^{\mathrm{T}} \tag{3}$$

in which D_t^{α} and D_t^{β} are the fractional differentiation operators of order α and β based on the Caputo's definition, defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau \tag{4}$$

here $\Gamma(.)$ denotes the Gamma function and

If $\alpha = \beta = 1$ the ordinary Oldroyd- B model will be obtained.

Consider an incompressible generalized Oldroyd- B fluid at rest of rectangular cross- section whose sides are at $x = 0, x = d, y = 0$ and $y = h$. At time $t = 0^+$ the duct begin to oscillate along z- axis.

The velocity field is

$$\mathbf{V} = \mathbf{V}(x, y, t) = w(x, y, t)\mathbf{k} \tag{5}$$

and the shear stress as the form

$$\mathbf{S} = \mathbf{S}(x, y, t) \tag{6}$$

where w is the velocity and $\mathbf{k}$ is the unit vector in the z- direction .Substituting equations (5) and (6) into (1) and taking account of the initial condition

$$\mathbf{S}(x, y, 0) = 0 \quad , \quad w(x, y, 0) = 0 \tag{7}$$

we obtain

$$\begin{aligned}
&(1+\lambda_1^{\alpha}\mathrm{D}_t^{\alpha})\tau_1 = \mu(1+\lambda_3^{\beta}\mathrm{D}_t^{\beta})\partial_x w(x, y, t) \\
&(1+\lambda_1^{\alpha}\mathrm{D}_t^{\alpha})\tau_2 = \mu(1+\lambda_r^{\beta}\mathrm{D}_t^{\beta})\partial_y w(x, y, t) \\
&(1+\lambda_1^{\alpha}\mathrm{D}_t^{\alpha})\sigma = 2\lambda_1^{\alpha}\left[\tau_1\frac{\partial w}{\partial x} + \tau_2\frac{\partial w}{\partial y}\right]
\end{aligned} \tag{8}$$

where $\mathrm{S}_{xz} = \tau_1$, $\mathrm{S}_{yz} = \tau_2$, $\mathrm{S}_{zz} = \sigma$, $\mathrm{S}_{xx} = \mathrm{S}_{xy} = \mathrm{S}_{yy} = 0$ and $\mathrm{S}_{xz} = \mathrm{S}_{zx}$, $\mathrm{S}_{yz} = \mathrm{S}_{zy}$. Then the equation of motion yields the following scalar equation:

$$\rho\frac{dw}{dt} = \frac{\partial\tau_1}{\partial x} + \frac{\partial\tau_2}{\partial y} + \frac{\partial\sigma}{\partial z} \tag{9}$$

where ρ is the constant density of the fluid. Eliminating τ_1, τ_2 and σ between Eqs. in (8) and (9), we obtain the following fractional differential equation

$$(1+\lambda_1^{\alpha}\mathrm{D}_t^{\alpha})\frac{\partial w}{\partial \mathrm{t}} = \nu(1+\lambda_r^{\beta}\mathrm{D}_t^{\beta})\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \tag{10}$$

where $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity.

We consider the following initial and boundary conditions

$$w(x, y, 0) = \frac{\partial w(x, y, 0)}{\partial t} = 0 \tag{11a}$$

$$w(0, y, t) = w(d, y, t) = w(x, 0, t) = w(x, h, t) = U\cos(\omega t) \tag{11b}$$

or

$$w(x, y, 0) = \frac{\partial w(x, y, 0)}{\partial t} = 0 \tag{12a}$$

$$w(0, y, t) = w(d, y, t) = w(x, 0, t) = w(x, h, t) = U\sin(\omega t) \tag{12b}$$

We denote by $u(x, y, t)$ the solution of problem (10), (11a),(11b) and by $v(x, y, t)$ the solution of problem (10), (12a),(12b) and define the complex velocity field

$$F(x, y, t) = u(x, y, t) + iv(x, y, t)$$

which is the solution of the following problem:

$$(1+\lambda_1^{\alpha}\mathrm{D}_t^{\alpha})\frac{\partial F(x, y, t)}{\partial \mathrm{t}} = \nu(1+\lambda_r^{\beta}\mathrm{D}_t^{\beta})\left[\frac{\partial^2 F(x, y, t)}{\partial x^2} + \frac{\partial^2 F(x, y, t)}{\partial y^2}\right] \tag{13}$$

$$F(x,y,0)=\frac{\partial F(x,y,0)}{\partial t}=0 \tag{14}$$

$$F(0,y,t)=F(d,y,t)=F(x,0,t)=F(x,h,t)=Ue^{i\omega t} \tag{15}$$

III. Calculation of Velocity Field

Consider an incompressible generalized Oldroyd- B fluid at rest of rectangular cross- section whose sides are at $x=0, x=d, y=0$ and $y=h$. At time $\mathrm{t}=0^{+}$ the duct begins to oscillate along z- axis. The fractional differential Eq. (13) with the initial and boundary conditions (14) and (15) will be solved by means of the double finite Fourier sine and discrete Laplace transforms.

Multiplying both sides of Eq. (13) by $\sin(\alpha_m x)$ and $\sin(\beta_n y)$, integrating with respect to x and y over $[0,d]\times[0,h]$ and using Eq. (15), we find that

$$(1+\lambda_1^{\alpha}\mathrm{D}_t^{\alpha})\frac{\partial F_{mn}(t)}{\partial \mathrm{t}}+\nu(1+\lambda_r^{\beta}\mathrm{D}_t^{\beta})(\alpha_m^2+\beta_n^2)F_{mn}(t)=\nu(1+\lambda_r^{\beta}\mathrm{D}_t^{\beta})\left[1-(-1)^m\right]\left[1-(-1)^n\right]\frac{\alpha_m^2+\beta_n^2}{\alpha_m\beta_n}Ue^{i\omega t} \tag{16}$$

where $\alpha_m=\dfrac{m\pi}{d}$, $\beta_n=\dfrac{n\pi}{h}$ and the double finite Fourier sine transforms

$$F_{mn}(t)=\int_0^d\int_0^h F(x,y,t)\sin(\alpha_m x)\sin(\beta_n y)dxdy \qquad ,m,n=1,2,3,\cdots \tag{17}$$

With the initial condition

$$F_{mn}(0)=0,\quad \frac{\partial F_{mn}(0)}{\partial t}=0,\qquad m,n=1,2,3,\cdots \tag{18}$$

Referring to Eq. (16), the corresponding fractional partial differential equation that described such flow takes the form

$$\begin{aligned}(1+\lambda_1^{\alpha}\mathrm{D}_t^{\alpha})\frac{\partial F_{mn}(t)}{\partial \mathrm{t}}&+\nu(\alpha_m^2+\beta_n^2)(1+\lambda_r^{\beta}\mathrm{D}_t^{\beta})F_{mn}(t)=\\ &\nu\left[1-(-1)^m\right]\left[1-(-1)^n\right]\frac{\alpha_m^2+\beta_n^2}{\alpha_m\beta_n}\left[Ue^{i\omega t}+\lambda_r^{\beta}U(i\omega)^{n_1}t^{n_1-\beta}E_{1,n_1-\beta+1}(i\omega t)\right]\end{aligned} \tag{19}$$

where $E_{\alpha,\beta}^{m}(z)=\sum_{j=0}^{\infty}\dfrac{(j+m)!z!}{j!\Gamma(\alpha j+\alpha m+\beta)}$ is the generalized Mittag- Leffler function [4] and n_1 is integer no.

By applying the discrete Laplace transform of Eq.(19) with the initial conditions (18), we get

$$\begin{aligned}\bar{F}_{mn}(s)&=\nu U\left[1-(-1)^m\right]\left[1-(-1)^n\right]\left(\frac{\alpha_m^2+\beta_n^2}{\alpha_m\beta_n}\right)\frac{(1+\lambda_r^{\beta}(i\omega)^{n_1}s^{\beta-n_1})}{s-\omega i}\frac{1}{s+\lambda_1^{\alpha}s^{\alpha+1}+\nu\left(1+\lambda_r^{\beta}s^{\beta}\right)\left(\alpha_m^2+\beta_n^2\right)}\\ &=\frac{a_{mn}U}{s-\omega i}-a_{mn}U\frac{(\lambda_1^{\alpha}s^{\alpha}+1)}{(s+\lambda_1^{\alpha}s^{\alpha+1}+\nu\lambda_{mn}+\nu\lambda_r^{\beta}s^{\beta}\lambda_{mn})}\left(1+\frac{\omega i}{s-\omega i}\right)+\frac{\left((i\omega)^{n_1}s^{-n_1}-1\right)\nu\, a_{mn}\lambda_{mn}U\lambda_2^{\beta}s^{\beta}}{(s-\omega i)(s+\lambda_1^{\alpha}s^{\alpha+1}+\nu\lambda_{mn}+\nu\lambda_r^{\beta}s^{\beta}\lambda_{mn})}\end{aligned} \tag{20}$$

where $a_{mn}=\dfrac{1}{\alpha_m\beta_n}\left[1-(-1)^m\right]\left[1-(-1)^n\right]$, $\lambda_{mn}=\alpha_m^2+\beta_n^2$ and $\bar{F}_{mn}(s)=\int_0^{\infty}F_{mn}(t)e^{-st}dt$ is the Laplace transform of $F_{mn}(t)$.

Now, rewriting Eq. (19) in series form as

$$\begin{aligned}\bar{F}_{mn}(s)&=\frac{a_{mn}U}{s-\omega i}-a_{mn}U\,H_{mn}(s)\left(1+\frac{\omega i}{s-\omega i}\right)+\\ &\left((i\omega)^{n_1}s^{-n_1}-1\right)\left[a_{mn}U\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{k!\left(\nu\lambda_{mn}\right)^{1+k-p}(-\omega i)^j(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}s^{\delta}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\right]\end{aligned}$$

(21)

where

$$H_{mn}(s)=\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{mn}}{\lambda_1^{\alpha}}\right)^{k}\sum_{l=0}^{k}\frac{k!\left(\lambda_r^{\beta}\right)^{l}}{l!(k-l)!}\frac{s^{-k+\beta l+\alpha-1}+\lambda_1^{-\alpha}s^{-k-1+\beta l}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}} \tag{22}$$

and $\delta=\beta-2-k+\beta l-j-\beta q+p+\alpha b$

Applying double inverse Fourier sine transform, we obtain

$$\bar{F}(x,y,s)=\frac{4U}{d\,h}\frac{1}{s-\omega i}\sum_{m,n=1}^{\infty}a_{mn}\sin(\alpha_m x)\sin(\beta_n y)-\frac{4U}{d\,h}\sum_{m,n=1}^{\infty}a_{mn}\left(1+\frac{\omega i}{s-\omega i}\right)H_{mn}(s)\sin(\alpha_m x)\sin(\beta_n y)+$$
$$\frac{4U}{d\,h}\sum_{m,n=1}^{\infty}a_{mn}\sum_{k=0}^{\infty}(-1)^{k}\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{k!\left(\nu\lambda_{mn}\right)^{1+k-p}(-1)^{j}(\omega i)^{j+n_1}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}s^{\delta-n_1}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\sin(\alpha_m x)\sin(\beta_n y)-$$
$$\frac{4U}{d\,h}\sum_{m,n=1}^{\infty}a_{mn}\sum_{k=0}^{\infty}(-1)^{k}\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{k!\left(\nu\lambda_{mn}\right)^{1+k-p}(-\omega i)^{j}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}s^{\delta}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\sin(\alpha_m x)\sin(\beta_n y) \tag{23}$$

Using the formula [5]

$$1=\frac{4}{d\,h}\sum_{m,n=1}^{\infty}a_{mn}\sin(\alpha_m x)\sin(\beta_n y) \tag{24}$$

we obtain for $\bar{F}(x,y,s)$ the expression

$$\bar{F}(x,y,s)=\frac{U}{s-\omega i}-\frac{4U}{d\,h}\sum_{m,n=1}^{\infty}a_{mn}\left(1+\frac{\omega i}{s-\omega i}\right)H_{mn}(s)\sin(\alpha_m x)\sin(\beta_n y)+$$
$$\frac{4U}{d\,h}\sum_{m,n=1}^{\infty}a_{mn}\sum_{k=0}^{\infty}(-1)^{k}\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{k!\left(\nu\lambda_{mn}\right)^{1+k-p}(-1)^{j}(\omega i)^{j+n_1}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}s^{\delta-n_1}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\sin(\alpha_m x)\sin(\beta_n y)-$$
$$\frac{4U}{d\,h}\sum_{m,n=1}^{\infty}a_{mn}\sum_{k=0}^{\infty}(-1)^{k}\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{k!\left(\nu\lambda_{mn}\right)^{1+k-p}(-\omega i)^{j}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}s^{\delta}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\sin(\alpha_m x)\sin(\beta_n y) \tag{25}$$

or

$$\bar{F}(x,y,s)=\frac{U}{s-\omega i}-\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}a_{MN}\left(1+\frac{\omega i}{s-\omega i}\right)H_{MN}(s)\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}+$$
$$\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}a_{MN}\sum_{k=0}^{\infty}(-1)^{k}\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{k!\left(\nu\lambda_{MN}\right)^{1+k-p}(-1)^{j}(\omega i)^{j+n_1}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}s^{\delta-n_1}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}-$$
$$\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}a_{MN}\sum_{k=0}^{\infty}(-1)^{k}\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{k!\left(\nu\lambda_{MN}\right)^{1+k-p}(-\omega i)^{j}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}s^{\delta}}{\left(s^{\alpha}+\lambda_1^{-\alpha}\right)^{k+1}(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}$$

(26)

where $M=2m-1$, $N=2n-1$, $\alpha_M=\dfrac{(2m-1)\pi}{d}$ and $\beta_N=\dfrac{(2n-1)\pi}{h}$.

By applying the inverse Laplace transform to Eq. (26), using (22) and the formula [4]

$$L^{-1}\left\{\frac{k!s^{\lambda-\mu}}{(s^{\lambda}\pm c)}\right\}=t^{\lambda t+\mu-1}E_{\lambda,\mu}^{k}(\mp c\,t^{\lambda})=t^{\lambda t+\mu-1}\sum_{r=0}^{\infty}\frac{(r+k)!\left(\mp c\,t^{\lambda}\right)^{r}}{\Gamma(\lambda r+\lambda k+\mu)} \tag{27}$$

we obtain for the complex velocity field $F(x,y,t)$, the following expression:

$$F(x,y,t)=Ue^{i\omega t}-\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}\Bigg\{\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\Big[t^{\alpha k+(k-\beta l+1)-1}E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}t^{\alpha})$$

$$+t^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}t^{\alpha})\Big]+\omega i\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\int_0^t e^{i\omega(t-s)}\Big[s^{\alpha k+(k-\beta l+1)-1}E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}s^{\alpha})$$

$$+s^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}s^{\alpha})\Big]ds\Bigg\}+\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-1)^j(\omega i)^{j+n_1}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}$$

$$\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}\Big[t^{\alpha k+(\alpha-\delta+n_1)-1}E^k_{\alpha,\alpha-\delta+n_1}(-\lambda_1^{-\alpha}t^{\alpha})\Big]-\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-\omega i)^j}{(k-l)!(l-j)!}$$

$$\frac{(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(j-q)!(q-p)!(p-b)!}\Big[t^{\alpha k+(\alpha-\delta)-1}E^k_{\alpha,\alpha-\delta}(-\lambda_1^{-\alpha}t^{\alpha})\Big]$$

(28)

Setting $d=2a_1$, $h=2b_1$ and changing the origin of the coordinate system (taking $x=x^*+a_1$, $y=y^*+b_1$ and dropping out the star notation), the complex velocity can be written in the form

$$F(x,y,t)=Ue^{i\omega t}-\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\Bigg\{\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\Big[t^{\alpha k+(k-\beta l+1)-1}E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}t^{\alpha})$$

$$+t^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}t^{\alpha})\Big]+\omega i\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\int_0^t e^{i\omega(t-s)}\Big[s^{\alpha k+(k-\beta l+1)-1}E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}s^{\alpha})$$

$$+s^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}s^{\alpha})\Big]ds\Bigg\}+\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-1)^j(\omega i)^{j+n_1}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}$$

$$\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\Big[t^{\alpha k+(\alpha-\delta+n_1)-1}E^k_{\alpha,\alpha-\delta+n_1}(-\lambda_1^{-\alpha}t^{\alpha})\Big]-\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-\omega i)^j}{(k-l)!(l-j)!}$$

$$\frac{(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(j-q)!(q-p)!(p-b)!}\Big[t^{\alpha k+(\alpha-\delta)-1}E^k_{\alpha,\alpha-\delta}(-\lambda_1^{-\alpha}t^{\alpha})\Big]$$

(29)

The velocity field corresponding to the cosine oscillation of the ducts, respectively to the sine oscillation of the duct is given by

$$u(x,y,t)=\mathrm{Re}\big[F(x,y,t)\big]$$

$$=U\cos(\omega t)-\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\Bigg\{\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\Big[t^{\alpha k+(k-\beta l+1)-1}$$

$$E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}t^{\alpha})+t^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}t^{\alpha})\Big]-\omega\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\int_0^t\sin\big(\omega(t-s)\big)$$

$$\Big[s^{\alpha k+(k-\beta l+1)-1}E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}s^{\alpha})+s^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}s^{\alpha})\Big]ds\Bigg\}+\mathrm{Re}\Bigg[\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\sum_{k=0}^{\infty}(-1)^k$$

$$\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-1)^j(\omega i)^{j+n_1}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\Big[t^{\alpha k+(\alpha-\delta+n_1)-1}E^k_{\alpha,\alpha-\delta+n_1}(-\lambda_1^{-\alpha}t^{\alpha})\Big]$$

$$-\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-\omega i)^j(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\Big[t^{\alpha k+(\alpha-\delta)-1}E^k_{\alpha,\alpha-\delta}(-\lambda_1^{-\alpha}t^{\alpha})\Big]\Bigg]$$

(30)

$$v(x,y,t)=\operatorname{Im}\left[F(x,y,t)\right]$$

$$=U\sin(\omega t)-\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\left\{\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\left[t^{\alpha k+(k-\beta l+1)-1}\right.\right.$$

$$E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}t^{\alpha})+t^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}t^{\alpha})\Big]+\omega\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r^{\beta})^l}{l!(k-l)!}\int_0^t\cos\left(\omega(t-s)\right)$$

$$\left[s^{\alpha k+(k-\beta l+1)-1}E^k_{\alpha,k-\beta l+1}(-\lambda_1^{-\alpha}s^{\alpha})+s^{\alpha k+(k-\beta l+\alpha+1)-1}E^k_{\alpha,k-\beta l+\alpha+1}(-\lambda_1^{-\alpha}s^{\alpha})\right]ds\Big\}+\operatorname{Im}\left[\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\sum_{k=0}^{\infty}(-1)^k\right.$$

$$\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-1)^j(\omega i)^{j+n_1}(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\frac{\cos(\alpha_M x)}{\alpha_M}\frac{\cos(\beta_N y)}{\beta_N}\left[t^{\alpha k+(\alpha-\delta+n_1)-1}E^k_{\alpha,\alpha-\delta+n_1}(-\lambda_1^{-\alpha}t^{\alpha})\right]$$

$$\left.-\frac{4U}{a_1b_1}\sum_{m,n=1}^{\infty}(-1)^{m+n}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-\omega i)^j(\lambda_1^{\alpha})^{-1-k+b}(\lambda_r^{\beta})^{1+l-q}}{(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}\left[t^{\alpha k+(\alpha-\delta)-1}E^k_{\alpha,\alpha-\delta}(-\lambda_1^{-\alpha}t^{\alpha})\right]\right]\tag{31}$$

Special Cases:

1- If $\alpha=\beta=1$, we can get similar solution of complex velocity distribution for unsteady flows of an Ordinary Oldroyd- B fluid, as obtained in Ref[10]. Thus the complex velocity field reduces to

$$F(x,y,t)=Ue^{i\omega t}-\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}\left\{\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r)^l}{l!(k-l)!}\left[t^{k+(k-l+1)-1}E^k_{1,k-l+1}(-\lambda_1^{-1}t\ )\right.\right.$$

$$+t^{k+(k-l+2)-1}E^k_{1,k-l+2}(-\lambda_1^{-1}t\ )\Big]+\omega i\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1}\right)^k\sum_{l=0}^{k}\frac{(\lambda_r)^l}{l!(k-l)!}\int_0^t e^{i\omega(t-s)}\left[s^{k+(k-l+1)-1}E^k_{1,k-l+1}(-\lambda_1^{-1}s\ )\right.$$

$$\left.+s^{k+(k-l+2)-1}E^k_{1,k-l+2}(-\lambda_1^{-1}s\ )\right]ds\Big\}+\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-1)^j(\omega i)^{j+1}(\lambda_1\ )^{-1-k+b}(\lambda_r\ )^{1+l-q}}{(k-l)!(l-j)!(j-q)!(q-p)!(p-b)!}$$

$$\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}\left[t^{k+(-\delta+2)-1}E^k_{1,-\delta+2}(-\lambda_1^{-1}t)\right]-\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\sum_{l+j+q+p+b=k}^{l,j,q,p,b\geq 0}\frac{\left(\nu\lambda_{MN}\right)^{1+k-p}(-\omega i)^j}{(k-l)!(l-j)!}$$

$$\frac{(\lambda_1\ )^{-1-k+b}(\lambda_r\ )^{1+l-q}}{(j-q)!(q-p)!(p-b)!}\left[t^{k+(1-\delta)-1}E^k_{1,1-\delta}(-\lambda_1^{-1}t)\right]\tag{32}$$

2- If $\lambda_r\to 0$, we can get similar solution of complex velocity distribution for unsteady flows of generalized Maxwell fluid with fractional derivatives, as obtained in Ref[9]. Thus the complex velocity field reduces to

$$F(x,y,t)=Ue^{i\omega t}-\frac{16U}{d\,h}\sum_{m,n=1}^{\infty}\frac{\sin(\alpha_M x)}{\alpha_M}\frac{\sin(\beta_N y)}{\beta_N}\left\{\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\left[G^k_{\alpha,\alpha-k-1,k+1}(-\lambda_1^{-\alpha},t)+\right.\right.$$

$$\left.\left.+\lambda_1^{-\alpha}G^k_{\alpha,-k-1,k+1}(-\lambda_1^{-\alpha},t)\right]+\omega i\sum_{k=0}^{\infty}\left(\frac{-\nu\lambda_{MN}}{\lambda_1^{\alpha}}\right)^k\int_0^t e^{i\omega(t-s)}\left[G^k_{\alpha,\alpha-k-1,k+1}(-\lambda_1^{-\alpha},s)+\lambda_1^{-\alpha}G^k_{\alpha,-k-1,k+1}(-\lambda_1^{-\alpha},s)\right]ds\right\}\tag{33}$$

where $G_{a,b,c}(d,t)$ is the generalized G- functions defined by [4]

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma\left[(c+j)a-b\right]}\tag{34}$$

IV. Numerical results and discussion:

In this work, we have discussed the flow of generalized Oldroyd-B fluid with fractional derivatives within an oscillating rectangular duct. Both cases of cosine and sine oscillations of the duct have been analyzed and the solutions have been determined by means of discrete Lplace and double finite Fourier sine transforms. The solutions corresponding to the generalized Maxwell fluid and the ordinary Oldroyd-B fluid have been determined as particular cases. Finally, the effects of various parameters on the velocity distribution characteristics are revealed by graphical illustrations.

Figures are sketched to show the profiles of the velocity field of generalized Oldroyd-B fluid with fractional derivatives in the case of cosine oscillations of the duct, Eq.(30) (Panel a), and the case of sine oscillations of the duct, Eq.(31) (Panel b).

Figs. 1 and 2 prepared to show the variations of the non- integer fractional parameters α and β, respectively, as well as a comparison between the velocity in the case of cosine oscillation of the duct (Panel a) and the velocity in the case of sine oscillation of the duct (Panel b) for fixed values of other parameters. It is clearly seen that the smaller the values of α, the more rapidly the velocity decays for both cases. However, one sees a opposite trend for the variation of β.

Figs. 3 and 4 provide the graphically illustrations for effects of relaxation and retardation parameters λ_1 and λ_r on the velocity field. The velocity is decreasing with the increased the λ_1 and λ_r for both cases, cosine and sine oscillations.

Fig. 5 demonstrates the influence of frequency of oscillation ω on the velocity profile for two cases cosine and sine oscillations. The velocity is increasing with the increase of the values of ω.

Fig. 6 represents the variation of velocity profiles for two cases cosine and sine oscillations for different value of y. It is seen that the amplitude of oscillation decreases as y increases. Fig. 7 represents the variation of velocity profile for different values of t. It is seen that effect of t on transient velocity is opposite to that of y. Comparison shows that the velocity profile in the case of cosine oscillation are greater in magnitude when compared to those of the case of sine oscillation.

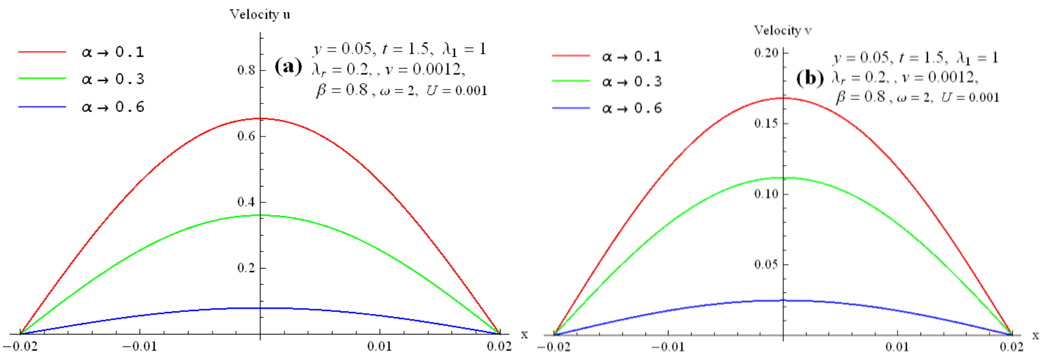

Fig. 1. The velocity for different value of α when keeping other parameters fixed a) cosine oscillation b) sine oscillation

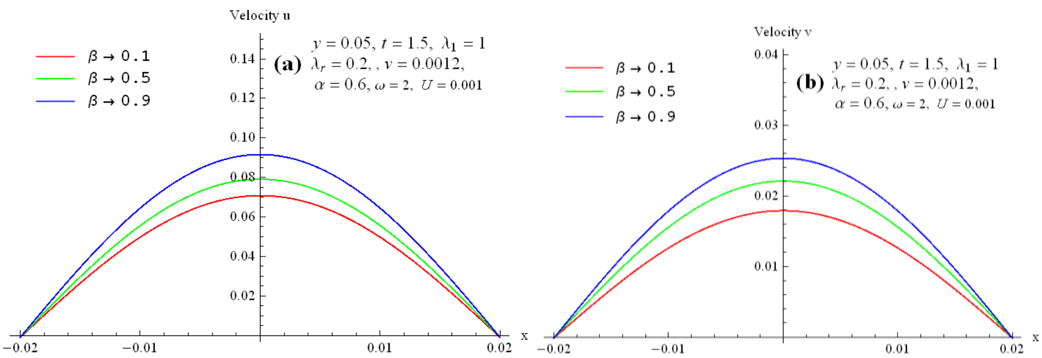

Fig. 2. The velocity for different value of β when keeping other parameters fixed a) cosine oscillation b) sine oscillation

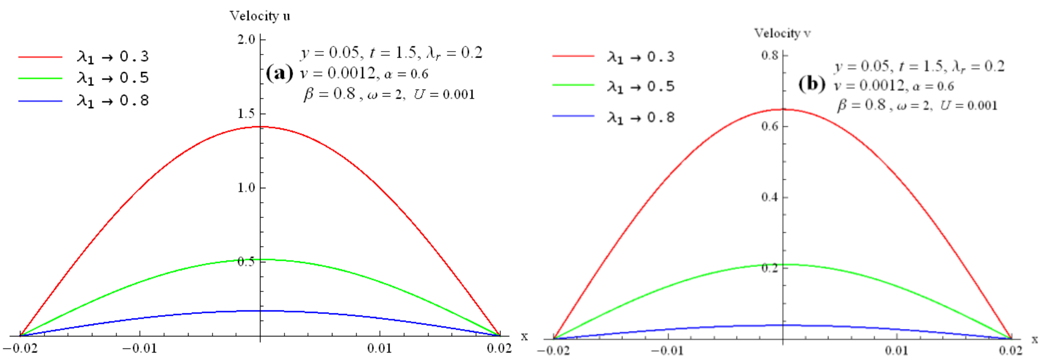

Fig. 3. The velocity for different value of λ_1 when keeping other parameters fixed a) cosine oscillation b) sine oscillation

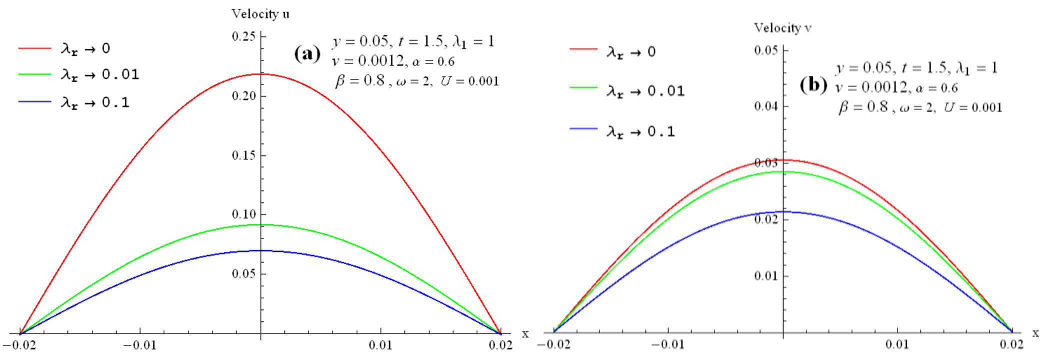

Fig. 4. The velocity for different value of λ_r when keeping other parameters fixed a) cosine oscillation b) sine oscillation

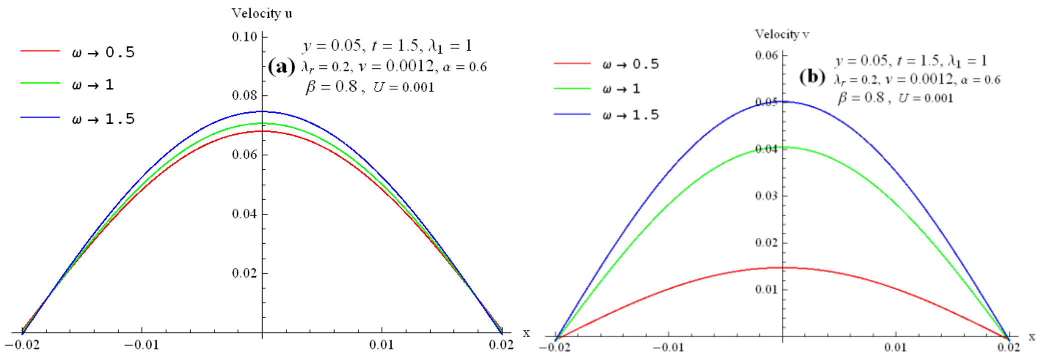

Fig. 5. The velocity for different value of ω when keeping other parameters fixed a) cosine oscillation b) sine oscillation

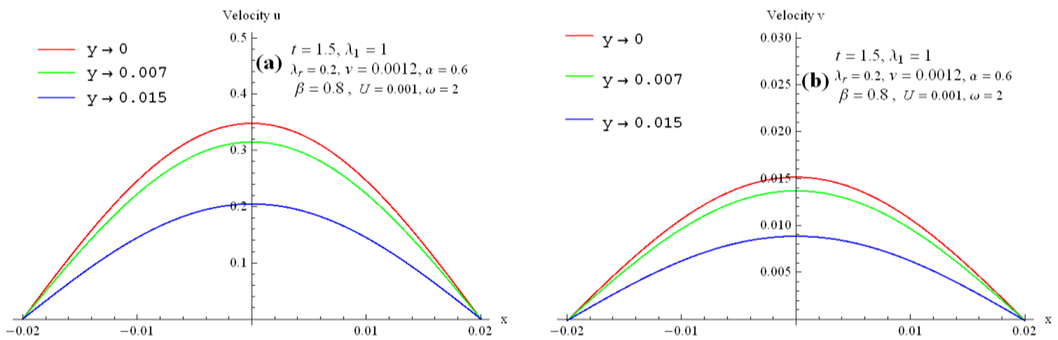

Fig. 6. The velocity for different value of y when keeping other parameters fixed a) cosine oscillation b) sine oscillation

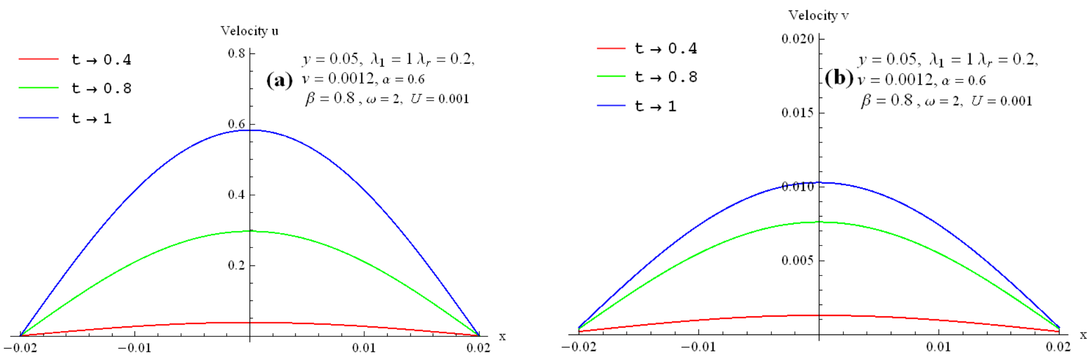

Fig. 7. The velocity for different value of t when keeping other parameters fixed a) cosine oscillation b) sine oscillation

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