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S-Mono form Modules

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Abstract: *In this paper, we introduce the concept S-monoform modules as a generalization of monoform modules. We study this class of modules, also we give several properties of these module and other related modules.*
Key Words: *Monoform module, small monoform module, S-monoform module, small submodule, small prime module, uniform module, S-uniform module, quasi-Dedekind module, hollow module.*

I. Introduction

Let R be a commutative ring with unity and let M be a unitary R-module, M is called a monoform module if for each nonzero submodule N of M and for each $f \in Hom(N,M)$, $f \neq 0$ implies ker f = (0), [15]. Equivalently M is monoform if and only if every nonzero submodule N of M is rational, that is $N \leq K \leq M$, $Hom(\frac{N}{K}, M) = 0$, [14].

The concept small monoform appeared in [5] where an R-module M is called small monoform if for each $0 \neq N \leq M$ and for each nonzero $f \in Hom(N,M)$, implies ker $f \ll N$. Also this class of modules studied in [8]. In this paper we introduce another generalization of monoform. M is called S-monoform module if for each nonzero small submodule N of M and for each nonzero $f \in Hom(N,M)$, implies ker f = 0. And a proper submodule N of M is called small (denoted by $N \ll M$) if $N + K \neq M$ for any proper submodule, [9].

We give the basic properties of S-monoform module and their relationships with small monoform, monoform module and other related modules.

II. S-Mono form Modules-Basic Results

In this section, as a generalization of monoform modules, S-monoform modules are introduced. Basic properties of S-monoform modules are given (see theorem (1.4)).

Definition (1.1):

Let M be an R-module. M is called **S-monoform** if for each $N \ll M$, $N \neq (0)$ and $f \in Hom(N,M)$ implies ker f = (0).

A ring R is called **S- monoform** if it is S-monoform R-module.

Remarks and Examples (1.2):

(1) It is clear that Z_4 as Z-module is S-monoform.

(2) Z_8 as Z-module is not S-monoform, since there exists $f : <\bar{2}> \longrightarrow Z_8$, such that $f(\bar{x}) = 2\bar{x}$, for each $\bar{x} \in <\bar{2}>$ and hence ker f = $\{\bar{0}, \bar{4}\} \neq (\bar{0}) \neq (0)$. Also notice that Z_8 is small monoform. Thus small monoform does not imply S-monoform.

(3) Clearly every monoform module is S-monoform, but the converse is not true. For example: Z_4 as Z-module is S-monoform, but not monoform.

(4) If M is semisimple, then M is S-monoform.

Proof:

As M is semisimple, (0) is the only small submodule of M. Hence the result follows directly. In particular each of Z_6, Z_{10}, $Z_2 \oplus Z_2$ as Z-module is S-monoform.

(5) The epimorphic image of S-monoform modules not necessarily S-monoform, for example: the Z-module Z is S-monoform. But $\pi: Z \longrightarrow Z/8Z \cong Z_8$, where π is the natural epimorphism. However Z_8 as Z-module is not S-monoform (see remarks and examples (1.2)(2)).

(6) Every nonzero submodule of S-monoform module is S-monoform module.

Proof:

Let M be an S-monoform R-module and let $(0) \neq N \leq M$, for any $(0) \neq U \ll N$, let $f : U \longrightarrow N$, $f \neq (0)$. Consider the diagram

$$U \xrightarrow{f} N \xrightarrow{i} M,\ i \circ f \neq (0)$$

Where i is the inclusion mapping N to M. But $U \ll N$, implies $U \ll M$. Hence $\ker (i \circ f) = (0)$, since M is S-monoform. But $\ker f \subseteq \ker (i \circ f)$, so $\ker f = (0)$. Thus N is S-monoform.

Note (1.3):

The direct sum of S-monoform modules is not necessarily S-monoform module. Now, consider the following example:

Let $M = Z_4 \oplus Z_4$ as Z-module, let $N = <\bar{2}> \oplus <\bar{2}> \ll M$ and let $f : N \longrightarrow M$ defined by $f(\bar{x}, \bar{y}) = (\bar{x}, 2\bar{y})$, for each $(\bar{x}, \bar{y}) \in N$, $\ker f = \{(\bar{0},\bar{0}),(\bar{0},\bar{2})\} \neq \{(\bar{0},\bar{0})\}$, then M is not S-monoform, but Z_4 as Z-module is S-monoform (see remarks and examples (1.2) (1)).

Recall that an R-module M is called fully stable if for each $N \leq M$, N is stable; that is for each $f \in Hom(N,M)$, $f(N) \subseteq N$, see [1]. Equivalently M is fully stable if and only if $\underset{M}{ann}(\underset{R}{ann}(x)) = (x)$, for each $x \in M$, see [1,corollary 3.5,p.22].

Theorem (1.4):

Let $M = M_1 \oplus M_2$, $M_1, M_2 \leq M$ such that for each $f : N_1 \oplus N_2 \longrightarrow M$, $f \neq 0$ implies $f(N_1) \neq (0)$, $f(N_2) \neq 0$ (i.e. $f|N_1 \neq 0$, $f|N_2 \neq 0$ and M is fully stable, then M_1 and M_2 are S-monoform if and only if M is S-monoform.

Proof:

$(\Leftarrow)$ It is clear by remarks and examples (1.2)(6).

$(\Rightarrow)$ Let $(0) \neq N \ll M$ and let $f \in Hom(N,M)$, $f \neq 0$. Since M is fully stable, $N = N_1 \oplus N_2$, where $N_1 = N \cap M_1$, $N_2 = N \cap M_2$. Moreover, $f(N) \subseteq N$. As $N \ll M$, we get $N_1 \ll M_1$ and $N_2 \ll M_2$ by [2, proposition 5.20]. Let $g_1 = f/N_1 : N_1 \longrightarrow M$, $g_2 = f/N_2 : N_2 \longrightarrow M$. Again, since M is fully stable $g(N_1) \subseteq N_1$, $g(N_2) \subseteq N_2$. Thus $g_1: N_1 \longrightarrow M_1$, $g_2: N_2 \longrightarrow M_2$ and so $\ker g_1 \oplus \ker g_2 = <0> \oplus <0> = <0>$. Now, let $n \in \ker f \leq N$, then $n = n_1 + n_2$ for some $n_1 \in N_1$, $n_2 \in N_2$ and $f(n) = 0$. Thus $0 = f(n) = f(n_1) + f(n_2) = g(n_1) + g(n_2)$. Hence $g(n_1) = -g(n_2) \in N_1 \cap N_2 = (0)$, it follows that $g(n_1) = g(n_2) = 0$; that is $n_1 \in \ker g_1 = (0)$, $n_2 \in \ker g_2 = (0)$. Therefore $n_1 + n_2 = 0$ and hence $\ker f = (0)$.

Note (1.5):

The condition M is fully stable in theorem (1.4) cannot be dropped, since the module M (in note (1.3)) is not fully stable, since for $W = <\bar{2}> \oplus <\bar{0}>$, there exists $f : W \longrightarrow M$ defined by $f(\bar{x}, \bar{0}) = (\bar{0}, \bar{x})$, for each $(\bar{x}, \bar{0}) \in W$, then $f(W) = <\bar{0}> + <\bar{2}> \not\subseteq W$.

2 S-Monoform Modules and S-Uniform Modules

It is known that monoform (small monoform) module implies uniform (see [3,theorem (2.3)] where an R-module M is called uniform if every nonzero submodule N of M is essential (large), and a submodule N of M, $N \neq (0)$ is called essential (denoted by $N \underset{e}{\leq} M$) if $N \cap W \neq (0)$ for each $W \neq (0)$, see [6]. However this is not true for S-monoform (see remarks and examples (1.2)(4)). However we introduce the concept of S-uniform and we see that there are some connections between S-monoform module and S-uniform module (see theorems (2.5,2.13), propositions (2.2,2.14) and corollary (2.7).

Definition (2.1):

Let M be an R-module. M is called **S-uniform** if every nonzero small submodule of M is essential in M.

It is clear that every uniform module is S-uniform, but the converse is not true as the following example shows:

Z_6 as Z-module is S-uniform, since Z_6 has no nonzero small submodule. However Z_6 is not uniform.

Let M be an R-module, put Z(M) = {m ∈ M: $\underset{R}{\text{ann}}(M) \underset{e}{\leq} R$} , Z(M) is called a singular submodule of M. M is called singular if Z(M) = M and M is called nonsingular if Z(M) = (0), see [6].

Proposition (2.2):
Let M be a nonsingular R-module, if M is S-uniform. Then M is S-monoform.

Proof:
Let (0) ≠ N ≪ M and let f ∈ Hom(N,M), f ≠ (0). To prove ker f = (0). By 1st Fundamental theorem N/ker f ≅ f (N). But f (N) ⊆ M and M is nonsingular, hence f (N) is nonsingular by [6,proposition 1.22,p.32]. Thus N/ker f is nonsingular. But ker f ⊆ N and N ≪ M, so ker f ≪ M. As M is S-uniform we have ker f $\underset{e}{\leq}$ M. Hence ker f $\underset{e}{\leq}$ N. Also, since N ≤ M, so N is nonsingular, hence N/ker f is singular by [6,proposition 1.21,p.32]. Thus N/ker f singular and nonsingular. It follows that N/ker f = (0); that is N = ker f and so that f = 0 which is a contradiction. Thus ker f = (0).

Remark (2.3):
The converse of proposition (2.2), is not true in general. For example:
Consider Z_{12} as Z-module; $<\overline{6}>$ is the only nonzero small submodule of Z_{12}, let f : $<\overline{6}> \longrightarrow Z_{12}$, f ≠ (0), then f is the inclusion mapping. Thus ker f = (0). Hence Z_{12} is S-monoform. However Z_{12} is not S-uniform, since $<\overline{6}> \ll Z_{12}$. But $<\overline{6}> \underset{e}{\not\leq} Z_{12}$, since $<\overline{6}> \cap <\overline{4}> = (0)$.

Corollary (2.4):
Let M be a nonsingular R-module, if M is small monoform. Then M is S-monoform.

Proof:
Since M is small monofrm, then M is uniform by [8,proposition 1.6]. Hence M is S-uniform and so by proposition (2.2), M is S-monoform.

Recall that M is an R-module, then M is monoform if and only if M is uniform prime, see [13,theorem 2.3]. We prove the following:

Theorem (2.5):
If M is S-uniform and semiprime R-module, then M is S-monoform.

Proof:
Let (0) ≠ N ≪ M and f ∈ Hom(N,M) such that f ≠ 0. To prove ker f = (0). Suppose ker f ≠ (0). As ker f ⊆ N ≪ M, ker f ≪ M and since M is S-uniform, then ker f $\underset{e}{\leq}$ M. Since f ≠ 0, then there exists x ∈ N such that f (x) ≠ 0. Hence (ker f) ∩ <x> ≠ (0); that is there exists r ∈ R, r ≠ 0 such that 0 ≠ r x ∈ ker f. Thus f(rx)= r f (x) = 0

On the other hand N ≪ M, rx ∈ M implies <r x> ≪ M. But M is S-uniform, so that <rx> $\underset{e}{\leq}$ M and hence <rx> ∩ < f (x)> ≠ (0). Then there exists r_1 ∈ R, r_1 ≠ 0 such that r_1rx ≠ 0 and r_1rx ∈ < f (x)>. This implies r_1rx = c f (x), for some 0 ≠ c ∈ R. It is follows that r_1r^2x = crf (x) = 0. Thus $(r_1r)^2x$ = 0. Hence r_1rx = 0, since M is semiprime, which is a contradiction. Thus ker f = (0) and hence M is S-monoform.

Remark (2.6):
The converse of theorem (2.5), is not true. For example:
The Z-module Z_4 is S-monoform. But Z_4 is not semiprime since $2^2 \cdot \overline{1} = \overline{0}$, but $2 \cdot \overline{1} \neq \overline{0}$.

Corollary (2.7):
If M is an S-uniform and prime R-module, then M is S-monoform.

Proof:
Since every prime module is semiprime, the result follows directly.

Recall that an R-module M is called small prime if $\underset{R}{\text{ann}}\, M = \underset{R}{\text{ann}}\, N$ for each N ≪ M, see [10].

To prove the next two corollaries, we need the folowng lemma:

Lemma (2.8):

Let M be a small prime. Then for each $x \neq 0$ with $(x) \ll M$ and for each $f \in Hom((x),M)$ with $f \neq 0$, then $\ker f = (0)$.

Proof:

Let $x \neq 0$ and let $(x) \ll M$, let $0 \neq f \in Hom((x),M)$ and let $rx \in \ker f$, then $f(rx) = 0$. This implies $r f(x) = 0$. But M is small prime; that is (0) is a small prime submodule. Hence either $f(x) = 0$ or $r \in ((0) \underset{R}{:} M) = annM$. As $f(x) \neq 0$, we get $r \in annM$. Thus $rx = 0$, which implies $\ker f = (0)$.

Corollary (2.9):

Let M be an R-module such that every submodule of M is cyclic and small. If M is small prime, then M is S-monform.

Proof:

It is follows by lemma (2.8).

Recall that an R-module M is a hollow module if $M \neq (0)$ and every proper submodule of M is small in M, see [4].

Corollary (2.10):

Let M be a small prime such that every submodule is hollow and Noetherian R-module. Then M is S-monoform.

Proof:

Let $N \leq M$ and $N \neq (0)$. Since M is Noetherian, then N is a finitely generated submodule of M. But M is hollow, so that N is cyclic submodule. Hence $N = (x)$, for some $x \in M$, $x \neq (0)$. Thus the result is obtained by lemma (2.8).

An R-module M is called quasi-Dedekind if every nonzero R-submodule N of M is quasi-invertable; that is $Hom(M/N,M) = 0$. A ring R is quasi-Dedekind if it is quasi-Dedekind R-module see [11,definition 1.1,p.24]. Equivalently M is quasi-Dedekind module if and only if for each nonzero $f \in End(M)$, f is monomorphism see [11, theorem 1.5,p.26].

The following proposition shows that S-monoform implies monoform under the class hollow quasi-Dedekind module.

Proposition (2.11):

Let M be a hollow module and quasi-Dedekind R-module. If M is S-monoform, then M is monoform.

Proof:

Let $(0) \neq N \leq M$ and let $f \in Hom(N,M)$ with $f \neq 0$. If $N \neq M$. Since M is hollow, then $N \ll M$. But M is S-monoform by hypothesis, implies $\ker f = (0)$. If $M = N$, then $\ker f = (0)$, since M is quasi-Dedekind. Thus M is monoform.

Note (2.12):

The condition M is quasi-Dedekind in proposition (2.11) is necessarily. For example: Z_4 as Z-module is S-monoform and hollow. Also it is not quasi-Dedekind and it is not monoform.

Under the class of fully stable modules, we have the following result:

Theorem (2.13):

Let M be a fully stable R-module. If M is a small prime and S-uniform, then M is S-monoform.

Proof:

Let $(0) \neq N \ll M$ and let $f \in Hom(N,M)$ with $f \neq 0$. To prove $\ker f = (0)$, suppose $\ker f \neq (0)$. Since $\ker f \leq N \ll M$, then $\ker f \ll M$. But M is S-uniform, so $\ker f \underset{e}{\leq} M$. Hence $\langle x \rangle \cap \ker f \neq (0)$, for any $x \in N$, $x \neq 0$. This means there exists $r \neq 0$, $0 \neq rx \in \ker f$ which implies $0 = f(rx) = r f(x)$. But $f \in Hom(N,M)$ and N is stable, so $f(N) \subseteq N$, hence $\langle f(x) \rangle \subseteq N$. But $N \ll M$, so $\langle f(x) \rangle \ll M$. As M is small prime and $r f(x) = 0$, we get that either $f(x) = 0$ or $r \in annM$. But $x \notin \ker f$. Thus $r \in annM$, so $rx = 0$ which is a contradiction. Therefore $\ker f = (0)$.

Recall that a submodule N of an R-module M is called rational in M if $\underset{R}{Hom}(X/N,M) = 0$ for any $N \leq X \leq M$, see [3].

It is known that every rational submodule is essential [3]. Also it is known that: M is monoform if and only if Hom(X/N,M) = 0 for each $N \leq M$ and for each $N \leq X \leq M$, see [14].

We have the following result:

Proposition (2.14):

Let M be an R-module. If for each $N \ll M$, Hom(X/N,M)= 0, $N \leq X \leq M$ (i.e. for each $N \ll M$, N is rational, then M is S-monoform and S-uniform.

Proof:

Let $(0) \neq W \ll M$ and let $f \in$ Hom(W,M) with $f \neq 0$. If ker f = (0), then nothing to prove. If ker $f \neq (0)$, then W/ker f $\cong$ f (W) $\subseteq$ M. Hence there exists an isomorphism g such that g: W/ker f $\longrightarrow$ f (W). Consider the diagram

$$W / \ker f \xrightarrow{g} f(W) \xrightarrow{i} M$$

Where i is the inclusion mapping. Thus $i \circ g \in$ Hom(W/ker f,M) and $i \circ g \neq 0$. On the other hand ker f $\leq W \ll M$, so ker $f \ll M$. But Hom(W/ker f,M) $\neq$ (0), so we get a contradiction with the hypothesis. Thus ker f = (0). Also for each $N \ll M$, then $N \underset{e}{\leq} M$, since N is rational submodule. Therefore M is S-uniform.

Recall that an R-module M is called multiplication if for each submodule N of M, N = IM for some ideal I of R. Equivalently M is multiplication R-module if for each submodule N of M, $N = (N \underset{R}{:} M)M$, where $(N \underset{R}{:} M) = \{r \in R : rM \subseteq N\}$, see [12].

Under the class of multiplication module we have the following result:

Proposition (2.15):

Let M be a multiplication R-module with $\underset{R}{\text{ann}}$ (M) is a prime ideal of R. Consider the following:

(1) For each $N \ll M$, N is rational submodule.
(2) For each $N \ll M$, N is essential (i.e. M is S-uniform).
(3) M is S-monoform.

Then (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3)

Proof:

(1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Let $(0) \neq N \ll M$. Suppose there exists $N \leq X \leq M$ such that Hom(X/N,M) $\neq$ (0), then there exists $f \in$ Hom(X/N,M), $f \neq 0$. So there exists $x + N \in X/N$, $x \notin N$ such that $f(x + N) = m \neq 0$. Since $N \underset{e}{\leq} M$, there exists $r \neq 0$ such that $0 \neq rx \in N$. It follows that $rm = r f(x + N) = f(rx + N) = 0$, then rm = 0. Since M is multiplication Rm = <m> = IM, for some ideal $I \in R$. Hence rIM = (0). Thus $rI \subseteq \underset{R}{\text{ann}}$ M. But $\underset{R}{\text{ann}}$ M is prime ideal, so either $r \in \underset{R}{\text{ann}}$ M or $I \subseteq \underset{R}{\text{ann}}$ M. If $r \in \underset{R}{\text{ann}}$ M, then rM = (0) which implies rx = 0, which is a contradiction. Thus $I \subseteq \underset{R}{\text{ann}}$ M, hence <m> = IM = (0), which is a contradiction. Thus Hom(X/N,M) = (0), for each $N \leq X \leq M$ (i.e. N is rational submodule).

(1) $\Rightarrow$ (3) It follows by proposition (2.14).

Recall that an R-module M is called comonoform module if for every N < M, Hom(M,N/L) = (0), for all $L \leq N$, see [7].

Proposition (2.16):

M is comonoform and S-monoform quasi-Dedekind R-module, then M is monoform.

Proof:

Since M is comonoform, then M is hollow by [7, lemma 17]. Byproposition (2.11), M is monoform.

Now we introduce the following:

Definition (2.17):

An –R-module M is called small polyform if for each N ≪ M, N ≠ (0), f ∈ Hom(N,M), ker f $\not\leq_e$ N.

The following result explains some connection between S-monoform module and small polyform module.

Proposition (2.18):

If M is S-monoform, then M is small polyform.

Proof:

Let (0) ≠ N ≪ M and f ∈ Hom(N,M), f ≠ 0. Since M is S-monoform, ker f = (0) $\not\leq_e$ N. Thus M is small polyform.

Proposition (2.19):

If M is small polyform and S-uniform, then M is S-monoform.

Proof:

Let (0) ≠ N ≪ M and f ∈ Hom(N,M) with f ≠ 0. To prove ker f = (0). Suppose ker f ≠ (0). It is clear that ker f ≤ N ≪ M, hence ker f ≪ M. On the other hand, M is S-uniform implies ker f $\leq_e$ M. But this contradits the hypothesis, M is small polyform. Thus ker f = (0) and so that M is S-monoform.

Corollary (2.20):

If M is S-uniform, then M is small polyform if and only if M is S-monoform.

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