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Line gracefulness of some cycle related graphs

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Abstract: *We investigate line graceful labeling for the spliting graph, total graph, shadow graph and mirror graph of cycle. Moreover we prove that the graphs obtained by duplication of each edge of cycle by a vertex and duplication of each vertex of cycle by an edge admit line graceful labeling.*

Keywords: *Graceful labeing, edge graceful labeling, line graceful labeling, total graph, spliting graph, shadow graph.*
AMS Subject Classification (2010): *05C78, 05C38, 05C76.*

I. Introduction

Labeling of discrete structures is a one of the potential areas of research due to its diversified applications. According to Prasanna et al. [1] graph labeling plays vital role in expanding the utility of the channel assignment process in communication network. The optimal linear arrangement concern to network problems in electrical engineering and placement problems in production engineering can be formalized as a graph labeling problems as stated by Yegnanaryanan and Vaidhyanathan [2]. A dynamic survey on different graph labeling schemes with an extensive bibliography can be found in Gallian [3]. The graphs considered here are finite, connected and undirected without loops and multiple edges. For standard terminology and notation we refer to West [4].

Definition 1.1: A graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s).

Definition 1.2: A function f is called graceful labeling of graph if $f: V(G) \to \{0,1,2,3,...,q\}$ is injective and the induced function $f^*: E(G) \to \{1,2,...,q\}$ defined as $f^*(e=uv) = |f(x) - f(y)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.

Most of the graph labeling techniques trace their origin with the graceful labeling which was introduced independently by Rosa [5] and Golomb [6]. A variant of graceful labeling termed as edge graceful labeling has been introduced by Lo [7].

Definition 1.3: A graph $G = (V(G), E(G))$ is said to be edge graceful if there exists a bijection $f: E(G) \to \{1,2,3,...,q\}$ such that the induced mapping $f^*: V(G) \to \{0,1,...,p-1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i) \pmod p$ is bijection.

Lo [7] have also derived a necessary condition for a graph with p vertices and q edges to be edge-graceful is $q(q+1) \equiv \frac{p(p+1)}{2} \pmod p$ and also investigate edge graceful labeling of various graph families. All trees of odd order are edge graceful as conjunctured by Lee [8]. Many graphs are proved to be edge graceful by Lee et al. [9,10,11]. The edge gracefulness in the context of cartesian product of odd cycles is studied by Wilson and Risking [12]. A variant of edge graceful labeling termed as line graceful labeling has been introduced by Gnanajothi [13] in her Ph.D. thesis.

Definition 1.4: A mapping $f: E(G) \to \{0,1,2,..., p\}$ is called line graceful of graph with p vertices, if induced function $f^*: V(G) \to \{0,1,2,..., p-1\}$ defined by $f^*(v) = \sum_{vv_i \in E(G)} f(vv_i)$ $(mod\ p)$ is bijective.

Vaidya and Kothari [14,15] have discussed line gracefulness in the context of switching of vertex operation, graphs arising from different graph operations on path while this paper is aimed to discuss line gracefulness in the context of graph operations on cycle and some standard graphs. We will give brief summary of definitions which are essential for the present investigations.

Definition 1.5: For every vertex $v \in V(G)$, the open neighbourhood set $N(v)$ is the set of vertices which are adjacent to v in G.

Definition 1.6: For a graph G the splitting graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 1.7: The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \bigcup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.8: The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G, say G' and G'' and joining each vertex u' of G' to the neighbours of corresponding vertex u' of G''.

Definition 1.9: Let G be a bipartite graph with a partite sets V_1 and V_2 and G' be the copy of G with corresponding partite sets V_1' and V_2'. The mirror graph of G is obtained from G and G' by joining each vertex of V_1 to its corresponding vertex in V_1' by an edge.

Definition 1.10:[16] Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.

Definition 1.11:[16] Duplication of a vertex v_i by a new edge $e = v_i'v_i''$ in graph G produces a new graph G' such that $N(v_i') \bigcap N(v_i'') = \{v_i\}$.

II. Main results

Proposition 2.1[13]: If the graph is line graceful then its order is not congruent to $2 \pmod 4$.
This necessary condition serves as a sufficient condition for the graph which does not admit line graceful labeling.

Theorem 2.2: $S'(C_n)$ is line graceful only for $n \equiv 0, 2 \pmod 4$.
Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(C_n)$. We note that $|V(S'(C_n))| = 2n$ and $|E(S'(C_n))| = 3n$. Define edge labeling $f : E(S'(C_n)) \to \{0, 1, \ldots, 2n\}$ as follows.
Case 1: $n \equiv 0, 2 \pmod 4$
for $1 \le i \le n-1$

$$f(v_i v_{i+1}) = \begin{cases} 0 & \text{for odd } i \\ n & \text{for even } i \end{cases}$$

$$f(u_i u_{i+1}) = i \quad \text{for } 1 \le i \le n-1$$

$$f(u_i v_{i-1}) = 0 \quad \text{for } 2 \le i \le n$$

$$f(u_1 v_n) = 0$$

$$f(v_n v_1) = f(u_n v_1) = n$$

Above defined edge labeling function satisfies the condition for line graceful labeling.
Case 2: $n \equiv 1, 3 \pmod 4$
In this case $|V(S'(C_n))| = 2n \equiv 2 \pmod 4$.
Then according to Proposition 2.1 $S'(C_n)$ is not line graceful.
In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(S'(C_n)) \to \{0, 1, \ldots, 2n-1\}$ such that $f^*(v) = \sum_{e \in E(S'(C_n))} f(e) \pmod{2n}$ for $n \equiv 0, 2 \pmod 4$. Hence $S'(C_n)$ is line graceful graph only for $n \equiv 0, 2 \pmod 4$.

Example 2.3: $S'(C_n)$ and its line graceful labeling is shown in Fig.1

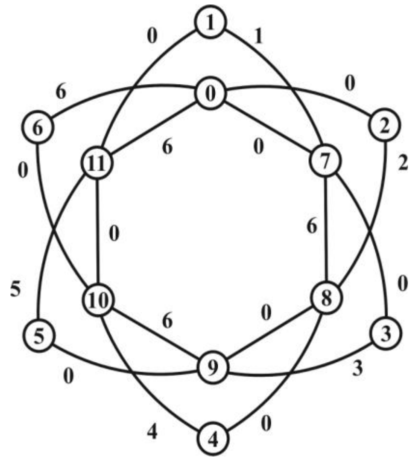

Fig. 1

Theorem 2.4: $T(C_n)$ is line graceful only for $n \equiv 0 \pmod 2$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n. Then $V(T(C_n)) = V(C_n) \bigcup E(C_n)$ and $E(T(C_n)) = E(C_n) \bigcup \{v_i e_i; 1 \le i \le n, v_i e_{i-1}; 2 \le i \le n, v_1 e_n, e_i e_{i+1}; 1 \le i \le n-1, e_n e_1\}$. We note that $|V(T(C_n))| = 2n$ and $|\mathrm{E}(\mathrm{C}_n)| = 4\mathrm{n}$. Define edge labeling $f : E(T(C_n)) \to \{0, 1, \ldots, 2n\}$ as follows.

Case 1: $n \equiv 0 \pmod 2$

for $1 \le i \le n-1$

$$f(v_i v_{i+1}) = \begin{cases} 0 & \text{for odd } i \\ n & \text{for even } i \end{cases}$$

$$\begin{aligned} f(u_i u_{i+1}) &= 0 \quad \text{for } 1 \le i \le n-1 \\ f(u_n u_1) &= 0 \\ f(u_i v_{i+1}) &= i \quad \text{for } 1 \le i \le n-1 \\ f(u_i v_i) &= 0 \quad \text{for } 1 \le i \le n \\ f(v_n v_1) &= f(u_n v_1) = n \end{aligned}$$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Case 2: $n \equiv 1 \pmod 2$

In this case $|V(T(C_n))| = 2n \equiv 2 \pmod 4$.

Then according to Proposition 2.1 $T(C_n)$ is not line graceful.

In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(T(C_n)) \to \{0, 1, \ldots, 2n-1\}$ such that $f^*(v) = \sum_{e \in E(T(C_n))} f(e) \pmod{2n}$ for $n \equiv 0 \pmod 2$. Hence $T(C_n)$ is line graceful graph only for $n \equiv 0 \pmod 2$.

Example 2.5: $T(C_n)$ and its line graceful labeling is shown in Fig. 2

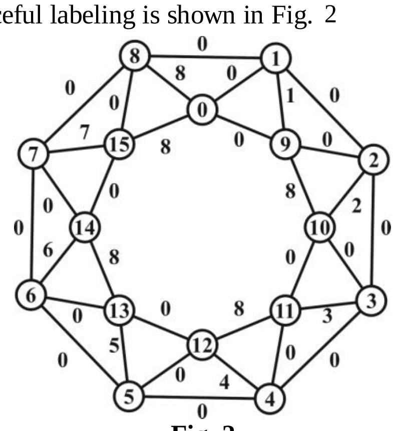

Fig. 2

Theorem 2.6: $D_2(C_n)$ is line graceful only for $n \equiv 0 \pmod 2$.

Proof: Consider two copies of cycle C_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of first copy of cycle C_n and $u_1, u_2, \ldots, u_n$ be the vertices of second copy of cycle C_n. Hence $V(D_2(C_n)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(D_2(C_n)) = E(C_n) \bigcup \{u_i u_{i+1}; 1 \le i \le n-1, u_n u_1, u_i v_{i+1}; 1 \le i \le n-1, u_n v_1, u_i v_{i-1}; 2 \le i \le n, u_1 v_n\}$. We note that $|V(D_2(C_n))| = 2n$ and $|E(D_2(C_n))| = 4n$. Define edge labeling $f : E(D_2(C_n)) \to \{0,1,\ldots,2n\}$ as follows.

Case 1: $n \equiv 0 \pmod 2$

for $1 \le i \le n-1$

$$f(v_i v_{i+1}) = \begin{cases} 1 & \text{for odd i} \\ 0 & \text{for even i} \end{cases}$$

$$f(u_i u_{i+1}) = 0 \quad \text{for } 1 \le i \le n-1$$

$$f(u_i v_{i+1}) = 2i \quad \text{for } 1 \le i \le n-1$$

$$f(u_i v_{i-1}) = 0 \quad \text{for } 2 \le i \le n$$

$$f(v_n v_1) = f(u_n u_1) = f(u_n v_1) = f(u_1 v_1) = 0$$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Case 2: $n \equiv 1 \pmod 2$

In this case $|V(D_2(C_n))| = 2n \equiv 2 \pmod 4$.

Then according to Proposition 2.1 $D_2(C_n)$ is not line graceful.

In view of above defined edge labeling function will induce the bijective vertex labeling function $f^* : V(D_2(C_n)) \to \{0,1,\ldots,2n-1\}$ such that $f^*(v) = \sum_{e \in E(D_2(C_n))} f(e) \pmod{2n}$ for $n \equiv 0 \pmod 2$. Hence $D_2(C_n)$ is line graceful graph only for $n \equiv 0 \pmod 2$.

Example 2.7: $D_2(C_n)$ and its line graceful labeling is shown in Fig. 3.

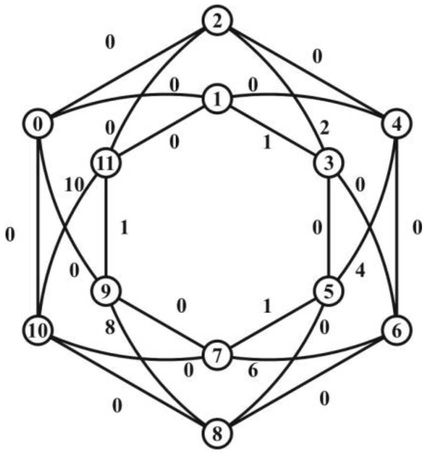

Fig. 3

Theorem 2.8: Mirror graph of even cycle C_n is line graceful.

Proof: Consider two copies of cycle C_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of first copy of cycle C_n and $u_1, u_2, \ldots, u_n$ be the vertices of second copy of cycle C_n, were n is even. Let $V_1 = \{v_1, v_3, \ldots, v_{n-1}\}$, $V_2 = \{v_2, v_4, \ldots, v_n\}$ be the partite sets of first copy of C_n and $U_1 = \{u_1, u_3, \ldots, u_{n-1}\}$, $U_2 = \{u_2, u_4, \ldots, u_n\}$ be the partite sets of second copy of C_n. The mirror graph G of cycle C_n is obtained by joining each vertex of V_1 to its corresponding vertex in U_1. Hence $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(G) = E(C_n) \bigcup \{u_i u_{i+1}; 1 \le i \le n-1, u_n u_1, u_i v_i; 1 \le i \le n-1 \text{for odd} i\}$.

We note that $|V(G)| = 2n$ and $|E(G)| = 2n + \frac{n}{2}$.

Define edge labeling $f : E(G) \to \{0,1,\ldots,2n\}$ as follows.

for $1 \le i \le n-1$

$$f(v_i v_{i+1}) = \begin{cases} n+i & \text{for odd i} \\ 0 & \text{for even i} \end{cases}$$

$f(v_n v_1) = f(u_n u_1) = 0$

for $1 \le i \le n-1$

$$f(u_i u_{i+1}) = \begin{cases} i & \text{for odd i} \\ 0 & \text{for even i} \end{cases}$$

$f(u_i v_i) = 2n-1$ for odd i

Above defined edge labeling function satisfies the condition for line graceful labeling.

In view of above defined edge labeling function will induce the bijective vertex labeling function $f^*: V(G) \to \{0,1,\ldots,2n-1\}$ such that $f^*(v) = \sum_{e \in E(G)} f(e) \pmod{2n}$ for $n \equiv 0 \pmod 2$. Hence mirror graph of even cycle is line graceful.

Example 2.9: Mirror graph of cycle and its line graceful labeling is shown in Fig. 4

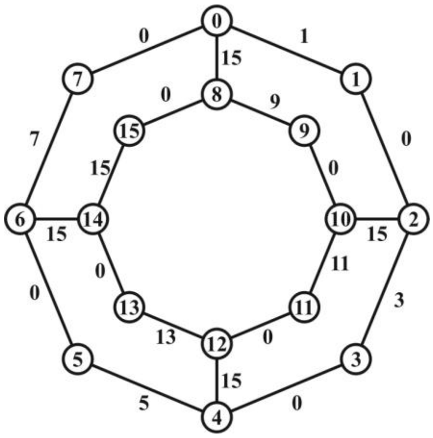

Fig. 4

Theorem 2.10: The graph obtained by duplication of each vertex of C_n by an edge admits line graceful labeling except $n \equiv 2 \pmod 4$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and G be the graph obtained by duplication of each vertex v_i of cycle C_n by an edge $u_i u_{i+1} (1 \le i \le n)$.

Then $V(G) = V(C_n) \bigcup \{u_1, u_2, \ldots, u_{2n}\}$ and $E(G) = E(C_n) \bigcup \{u_{2i-1}v_i, u_{2i}v_i, u_i u_{i+1}; 1 \le i \le 2n-1 \text{for odd} i\}$. We note that $|V(G)| = 3n$ and $|E(G)| = 4n$.

Define edge labeling $f: E(G) \to \{0,1,\ldots,2n\}$ as follows.

Case 1: $n \equiv 0,1 \pmod 4$

$f(v_i v_{i+1}) = 3\left\lfloor \frac{i}{2} \right\rfloor$ for $1 \le i \le n-1$

$f(v_n v_1) = 0$

$f(u_{2i-1} v_i) = 1$ for $1 \le i \le n-1$

$f(u_{2n-1} v_n) = \left\lfloor \frac{3n - f(v_{n-1} v_n)}{2} \right\rfloor$

$f(u_{2i} v_i) = 2$ for $1 \le i \le n-1$

$f(u_{2n} v_n) = \left\lceil \frac{3n - f(v_{n-1} v_n)}{2} \right\rceil$

$f(u_i u_{i+1}) = 3(i-1)$ for $1 \le i \le 2n-3,\ i \equiv 1 \pmod 2$

$f(u_{2n-1} u_{2n}) = 3n - 1 - f(u_{2n} v_n)$

Case 2: $n \equiv 3 \pmod 4$

$f(v_1 v_2) = 0$

for $1 \le i \le n-1$

$$f(v_i v_{i+1}) = \begin{cases} 3n - \frac{3i}{2} & \text{for even i} \\ f(v_{i-1}v_i) & \text{for odd i} \end{cases}$$

$f(u_{2i-1}v_i) = 3i - 2$ for $1 \le i \le n-1$

$f(u_{2n-1}v_n) = \left\lfloor \frac{3n - f(v_{n-1}v_n)}{2} \right\rfloor$

$f(u_{2i}v_i) = 3i - 1$ for $1 \le i \le n-1$

$f(u_{2n}v_n) = \left\lceil \frac{3n - f(v_{n-1}v_n)}{2} \right\rceil$

$f(u_{2i-1}u_{2i}) = 0$ for $1 \le i \le n-1, i \equiv 1 (\text{mod } 2)$

$f(u_{2n-1}u_{2n}) = 3n - 2 - f(u_{2n-1}v_n)$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Case 3: $n \equiv 2\ (\text{mod} 4)$

In this case $|V(G)| = 3n \equiv 2\ (\text{mod } 4)$.

Then according to Proposition 2.1 G is not line graceful.

In view of above defined edge labeling function will induce the bijective vertex labeling function $f^*: V(G) \to \{0,1,\dots,3n-1\}$ such that $f^*(v) = \sum_{e \in E(G)} f(e)\ (\text{mod } 3n)$ for $n \equiv 0,1,3\ (\text{mod } 4)$. Hence we proved that G is line graceful graph except for $n \equiv 2\ (\text{mod } 4)$.

Example 2.11: Graph obtained by duplication of vertex of C_n by an edge and its line graceful labeling is shown in Fig. 5.

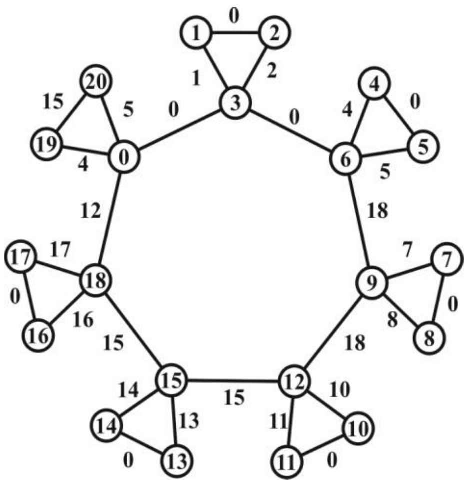

Fig. 5

Theorem 2.12: The graph obtained by duplication of an edge of C_n by a vertex admits line graceful labeling only for $n \equiv 0\ (\text{mod } 2)$.

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n and G be the graph obtained by duplication of an edge $v_i v_{i+1}$ by a vertex $u_i (1 \le i \le n)$. Then $V(G) = V(C_n) \cup \{u_1, u_2, \dots u_n\}$ and $E(G) = E(C_n) \cup \{u_i v_i; 1 \le i \le n, u_i v_{i+1}; 1 \le i \le n-1, u_n v_1\}$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n$. Define edge labeling $f: E(G) \to \{0,1,\dots,2n-1\}$ as follows.

Case 1: $n \equiv 0\ (\text{mod } 2)$

for $1 \le i \le n-1$

$$f(v_i v_{i+1}) = \begin{cases} 0 & \text{for odd i} \\ n & \text{for even i} \end{cases}$$

$f(u_i v_{i+1}) = i$ for $1 \le i \le n-1$

$f(u_i v_i) = 0$ for $1 \le i \le n$

$f(v_n v_1) = f(u_n v_1) = n$

Above defined edge labeling function satisfies the condition for line graceful labeling.

Case 2: $n \equiv 1 \pmod 2$

In this case $|V(G)| = 2n \equiv 2 \pmod 4$.

Then according to Proposition 2.1 G is not line graceful.

Thus above defined edge labeling function will induce a bijective vertex labeling function $f^*: V(G) \to \{0,1,\ldots,2n-1\}$ such that $f^*(v) = \sum_{e \in E(G)} f(e) \pmod{2n}$ for $n \equiv 0 \pmod 2$. Hence, G is line graceful graph only for $n \equiv 0 \pmod 2$.

Example 2.13: The graph obtained by duplication of an edge of C_n by a vertex and its line graceful labeling is shown in Fig. 6.

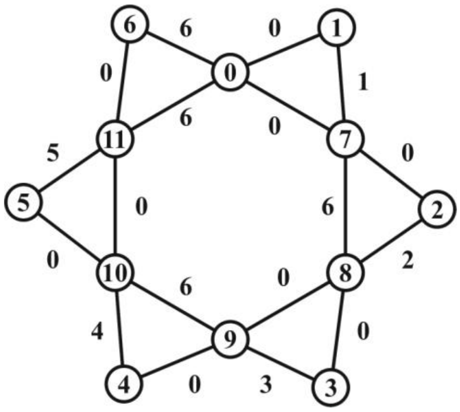

Fig. 6

III. Conclusion

Edge gracefulness and line gracefulness of a graph are two independent concepts. A graph may possess one or both of these or neither as mentioned below.

1. C_{2n+1} is edge graceful as well as line graceful.
2. P_n is neither edge graceful nor line graceful for $n \equiv 2 \pmod 4$.
3. C_{4n} is not edge graceful but line graceful.
4. Triangular snake T_n is edge graceful only for $n = 3$ while it is line graceful for all n.

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