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Adomian Decomposition Method for Certain Space-Time Fractional Partial Differential Equations

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Abstract: *The fractionalspace and time, as ageneralization of telegraph, diffusion and the waveequations, are considered. The (direct case and the modified cases) of Adomiandecompositionmethod are adopted to treat the certain space-time fractional partial differentialequations, in thispaper, four distinguished cases willbemodified and applied to solvedifferent certain space-timefractionalorder (homogeneous or inhomogeneous; linear or nonlinear), the steps of solutions willbegiven, sevendifferentexampleswillbesolved to show the powerful of thismethod to solvedifferentkinds of problems, finally figures and tables of resultswillbegiven by using programs of matlab.*

Keywords: *Adomian decomposition Method, Fractional Partial Differential Equations.*

I. Introduction

George Adomian (1980s), [1, 2, 3], introduced a powerful method for solving linear and nonlinear partial functional equations. Since then, this method is known as an Adomian decomposition method (ADM). Adomian method, [4,5,6], relatively new approaches to provide an analytical and approximation solution, moreover they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations, without linearization or discretization. Application of this method is extended for fractional differential equations. The convergence, of Adomian decomposition method, was been discussed similarity, with other analytical and numerical solutions of initial value problems for differential equations, are solved by many authors. N. Bellomo and D.Sarafyan [7], On Adomian's decomposition methods and some comparisons with Picard's iterative scheme, identifies carefully all substantial differences between the two methods. M. A. Golberg, in his work [8] "notes, on the decomposition method for operator equation" has illustrated that Adomian's decomposition method is equivalent to the classical method of successive approximations (Picard iterations) for linear operator equations. A. M. Wazwaz [9], A comparison between Adomian decomposition method and Taylor series method in the series solutions, it been show compare the performance of the ADM and the Taylor series method applied to the solution of linear and nonlinear ODE. J.Y. Edwards, J. A. Roberts,and N. J. Ford [10], A comparison of Adomian's decomposition method and Runge-Kutta methods for approximation solution of some model equations, the relationship between existence/uniqueness of solutions of the model equations. D.B. dhaigude and Gunvant a.birajdar [11], numerical solution of system of time FPDE by discrete Adomian decomposition method. Mehdi Safari, Mohammad Danesh [12], Application of Adomian's Decomposition Method for the Analytical Solution of Space Fractional Diffusion Equation by Adomian's decomposition method. guo - chengwuyong-guoshikai- tengwu [13], Adomian decomposition method and non- analytical solutions of fractional differential equations. Using Adomian decomposition method to approximate solution of fractional differential equations.The iteration procedure is based on a fractional Taylor series. jinfacheng and yu-mingchu[14], use ADM for Solution of the general form of the linearFDE with constant coefficients, A.M.A. EL-Sayed, S.H.Behiry, W.E. Raslan [15], Adomian's decomposition method for solving an intermediate fractional advection dispersion equation, the Caputo sense and Adomian's decomposition method was been used for solving the process between advection and dispersion via fractional derivative.

our working in this paper, are showing the powerful of ADM and modified ADM in applying for fractional order PDE, with the Caputo's Fractional derivatives and R-L fractional Integral, andgiving the short steps for solving certain S-TFPDE, Our discussion will be given, through solving of seven different examples of our considering the fractional telegraph equation as a model, the generalized telegraph equation to fractional order has the form,

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = a\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} + b\frac{\partial^{\sigma} u(x,t)}{\partial t^{\sigma}} + cu(x,t) + f(x,t) \qquad (1)$$

Where $0< x< L$; $t > 0$; $0 <\sigma \le 1<(\alpha \& \beta) \le 2$; a, b and c are constants, f is given function, Subject to initial and boundary conditions respectively:

I.C $u(0, t) = f_1(t)$, $u_x(0,t) = f_2(t)$,

B.C $u(x, 0) = g_1(x)$, $u_t(x,0) = g_2(x)$,
One can see easy these cases, if (f = 0), then equation will be liner homogeneous S-TFPDE,if
(a = c = 0), then equation yields the S-TFPDE heat equation, If (b = c = 0), then equation will be the S-TFPDE wave equation.

II. Definitions And Theorems[1-15]

Fractional integral and derivatives with its properties will be given:

2.1 Riemann-Liouville Fractional integral (R-LFI)of order β > 0, given by the formula:

$J_t^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s)ds$ where$J_t^0 = I$ (2)

2.2 Caputo fractional derivative (CFD) of order β,$n-1<\beta\leq n \in \mathbb{N}$, given by the form:

${}_0^C D_t^\beta = J_t^{n-\beta} D_t^n f(t)$:

$${}_0^C D_t^\beta f(t) = \begin{cases} \left[\frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)ds\right] n-1<\beta<n \\ \frac{d^n}{dt^n} f(t) \beta = n \end{cases} \quad (3)$$

2.3 Properties of R-LFI and CFD:

- let $\alpha \geq 0$, $\beta \geq 0$ then: $J_t^\beta J_t^\alpha = J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}$
- let p > -1 and α > 0 then ,$J_t^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)} t^{(p+\alpha)}$, $D_t^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{(p-\alpha)}$

  if β→(n-1) then , ${}_0^C D_t^\beta f(t) \to J_t^1 D_t^n f(t) = D_t^{n-1} f(t) - D_t^{n-1} f(0)$
- $\begin{cases} {}_0^C D_t^\beta J_t^\beta f(t) = f(t) \\ J_t^\beta {}_0^C D_t^\beta f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!} \end{cases}$, $t > 0$

Note1: why the using of Caputo's fractional derivative is better than other formula, for PDE? The reason for adopting the Caputo definition to solve differential equations (both integer and fractional order), we need to specify additional conditions in order to produce a unique solution. For the case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are therefore familiar to us. In contrast, for Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point x = 0, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details of the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types.

Note2: The fractional derivatives of some functions:

One can use thelinearity and the fractional derivatives of power function to find fractional derivatives ofexpansion of some functions as the following:
If f(x) is developed in the form:

$f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$ Then for α ϵ R,the fractional derivatives of order α of f(x) given by:

- $f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{(i-\alpha)}}{\Gamma(i-\alpha+1)}$ (4)
- if $f(x) = e^{ax}$ then $(e^{ax})^{(\alpha)} = a^\alpha e^{ax}$, α ϵ R (5)

if $f(x) = \cos(x) = \sum_{k=-\infty}^{+\infty} a_k \frac{x^k}{k!}$, where $a_{2k+1} = 0$; $a_{2k} = (-1)^k$

and$f(x) = \sin(x) = \sum_{k=-\infty}^{+\infty} b_k \frac{x^k}{k!}$,where $b_{2k+1} = (-1)^k$; $b_{2k} = 0$, then

- $\sin^{(\alpha)}(x) = \sin\left(x + \frac{\alpha\pi}{2}\right)$ (6a)
- $\cos^{(\alpha)}(x) = \cos\left(x + \frac{\alpha\pi}{2}\right)$ (6b)
- $(e^x \sin(x))^{(\alpha)} = 2^{\alpha/2} e^x \sin\left(x + \frac{\alpha\pi}{4}\right)$ (7)
- $(e^x \cos(x))^{(\alpha)} = 2^{\alpha/2} e^x \cos\left(x + \frac{\alpha\pi}{4}\right)$ (8)

we can see easy these three functional derivatives be true if the sum define from -∞ to +∞, and these derivatives are not true if the sum define from 0 to -∞.

$\lim_{x\to 0}(\sin x)^{(\frac{1}{2})} = 0$; while $\lim_{x\to 0}\left(\sin(x+\frac{\pi}{4})\right) = \frac{\sqrt{2}}{2}$

$\lim_{x\to 0}(\cos x)^{(\frac{1}{2})} = \pm\infty$; while $\lim_{x\to 0}\left(\cos(x+\frac{\pi}{4})\right) = \frac{\sqrt{2}}{2}$

$\lim_{x\to 0}(e^x)^{(\frac{1}{2})} = \pm\infty$; while $\lim_{x\to 0}(e^x) = 1$

2.4 The Adomian decomposition method:

This method proved to be powerful, effective, and it can easy handle so kinds of differential equations. (integer and fractional order), this method attacks the problems in a direct way and thea straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion, moreover it consists ofdecomposing the unknown function u(x, t) of an y equation into a sum of an infinite number of components defined by decomposition series, where the components are to be determined in recursive manner. In this paper, Four different cases that used in applied of this method will be discussed, and used to solve S-TFPDE's.

Case1: (direct way)

In this casethe general steps of Adomian decomposition for linear S-TFPDE will be given in the following:
Consider the boundary partial differential equation which has the form:

$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{\partial^\beta}{\partial x^\beta}u(x,t) + f(x,t); where\ 0 < x < l; t \geq 0\ ; 0 < \ \alpha,\beta \leq 2$ (9)

Subject to boundary and initial conditions respectively:

B.C $u(0,t) = F_1(t)$; $u(l,t) = F_2(t)$;

$I.C\ u(x,0) = g_1(x)$; $u^{'}(x,0) = g_2(x)$(10)

Step1: write the FPDE eq(9) by using the operators form as

$D_t^\alpha u(x,t) = D_x^\beta u(x,t) + f(x,t)$; Where$D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$;$D_x^\beta = \frac{\partial^\beta}{\partial t^\beta}$.

Step2: take the invers operator (J_t^α in eq(2)) to both side of equation in step1:

$J_t^\alpha u(x,t) = J_t^\alpha D_x^\beta u(x,t) + J_t^\alpha f(x,t)$; then use the property in section (2.3) to write

$u(x,t) = \sum_{j=0}^{m-1}\frac{\partial^j}{\partial t^j}u(x,0^+)\frac{t^j}{j!} + J_t^\alpha f(x,t)] + J_t^\alpha D_x^\beta u(x,t)$, Where m-1< α ≤ m

Step3: set the solution u(x,t) into decomposition finite series in step1 as:

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t) = (u_0 + u_1 + \cdots) = \sum_{j=0}^{m-1}\frac{\partial^j}{\partial t^j}u(x,0^+)\frac{t^j}{j!} + J_t^\alpha f(x,t) + J_t^\alpha\sum_{n=0}^{\infty}D_x^\beta u_n(x,t)$$

Step4: write the recursive relation equations as:

$u_0(x,t) = [\sum_{j=0}^{m-1}\frac{\partial^j}{\partial t^j}u(x,0^+)\frac{t^j}{j!} + J_t^\alpha f(x,t)]$ (11)

$u_{k+1}(x,t) = J_t^\alpha D_x^\beta u_k(x,t)$; $k \geq 0$ (12)

Step 5: use the recursive scheme eq's(11-12) to determine the successive components of the approximate solutions u_k(x,t), to find k^{th} solution, $u_k(x,t) = \sum_{j=0}^{k}u_j(x,t)$.

use the determined component approximation solutions in first equation in step3to obtain solution in series form.

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t) = \lim_{k\to\infty}\sum_{j=0}^{k}u_j(x,t)$$

so that we can find the approximate solution u_k(x,t)and the exact solution u(x,t) can be obtained in many equations if such a closed form solution exact.(see example 1,2 and 3).

Case2(using Adomian polynomials A_k)

In this case, the steps of ADM which are using the helping of $\mathbf{A_k}$ to find the approximation solution and exact solutions for nonlinear S-TFPDE which has the general form:

$D_t^{\alpha} u(x,t) = D_x^{\beta} u(x,t) + G(u) + f(x,t)$. Where G (u) is the nonlinear term.
Adomian polynomials (A_k) which were be defined by Adomian himself as the following:
$A_n\ (u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[\ G\ (\sum_{i=0}^{\infty} u_i \lambda_i)]_{\lambda=0}$ (13)
he used this polynomial to define the nonlinear term as:
G (u)= $\sum_{n=0}^{\infty} A_n\ (u_0, u_1, u_2, \dots, u_n)$.
The easy way to calculate these polynomials is:
Assume the inverse operator $\mathbf{J}^{\alpha}$exists, now if we set :$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and
The nonlinear S-TFPDE will be yield:

$$u(x,t) = \sum_{j=0}^{m-1} \frac{\partial^j}{\partial t^j} u(x,0^+) \frac{t^j}{j!} + J_t^{\alpha} f(x,t)$$

$$+J_t^{\alpha}[D_x^{\beta} u(x,t) + G\ (u)]$$

And the recursive relation equations for this equation are:

$u_0(x,t) = [\sum_{j=0}^{m-1} \frac{\partial^j}{\partial t^j} u(x,0^+) \frac{t^j}{j!} + J_t^{\alpha} f(x,t)]$ (14)

$u_{k+1} = J_t^{\alpha}[D_x^{\beta} u(x,t) + G\ (u)]$; G (u)= $\sum_{n=0}^{\infty} A_n$ (15)

So we need to calculate Adomian polynomials to find approximate solutions and exact solutions, the following steps, which be given below, will be shown the easy way to calculate Adomian polynomials and series-solutions:

Step1: from general equation get the nonlinear term G(u) and choose the integer number k, The number of Adomian polynomials calculate.

step2: put the zeros solution as: $\boldsymbol{A_0 = G\ (u_0)}$**, and** calculate k Adomian's polynomials $\boldsymbol{A_k}$ where:
$k = 0,1, \dots, n-1$,n the number of approximate solutions you want to calculate, where:
$A_k\ (u_0, \dots, u_k)$= $A_k\ (u_0 + u_1\lambda, \dots, u_k + (k+1)u_{k+1}\ \lambda)$
by using:
$A_k : u_i \longrightarrow u_i + (i+1)u_{i+1}\ \lambda$for i = 0,1,2,…, k
take the first order derivative of A_k withrespect λ, and put λ = 0 so from step2 yield:
$\frac{d}{d\lambda} A_k|_{\lambda=0} = (k+1)A_{k+1}$we can get the n polynomials

Step3: calculate the approximate or the exact solution by using the recursive relations equations in: eq's (13-14)**.** (see example 4).

Case3:(the noise term):one from the powerful of ADM is the finding of noise term, which is helping us to find the exact solutions of inhomogeneous differential equations in second or third steps.

Noise Term:A useful summary about noise term phenomenon can be drawn as follows:
1. The noise terms are the identical terms with opposite signs that may appear in the components u_0 and u_1.
2. The noise terms appear only for specific types of inhomogeneous equations whereas noise terms do not appear
3. for homogeneous equations.
4. Noise terms may appear if the exact solution is part of the zeros component u_0.
5. Verification that the remaining non-canceled terms satisfy the equation is necessary and essential.(See example 5and 6).

Case4: (using both the boundary and initial conditions):

In this case the boundary condition will be usedboundary and initial conditions to find the first start approximation solution u_0.
Consider the linear homogeneous S-TFPDE heat equation:
$D_t^{\alpha} u(x,t) = hD_x^{\beta} u(x,t)$where 0 < x< L ; t ≥ 0,0< α ≤ 1 < β ≤ 2 **,**
subject to boundary and initial conditions
B.C u(0,t) = f_0 (t) ; u(L,t)= f_L (t) ;
I.C u(x,0) = g(x).

If we solve this equation by using the general steps in case1for t and for x respectively as the following: recursive relation equations for t given as:

$$\begin{cases} u_0 = g(x) \\ u_{k+1} = hJ_t^{\alpha} D_x^{\beta} u_k \end{cases} \quad (16a)$$

Take the approximation of the first derivative as $u_t'(0,t) = \frac{f_L - f_0}{L-0}$, and

Solve equation for x, then the recursive relation equations will be given as:

$$\begin{cases} u_0 = f_0 + \frac{x}{L}[f_L - f_0] \\ u_{k+1} = \frac{1}{h} J_x^{\beta} D_t^{\alpha} u_k \end{cases} \quad (16b)$$

Now the new recursive relation equations by adding half of equations (16a and 16b), then the new recursive relation equations will be given as:

$$\begin{cases} u_0 = \frac{1}{2}[g(x) + f_0 + \frac{x}{L}[f_L - f_0]] \\ u_{k+1} = \frac{1}{2}[hJ_t^{\alpha} D_x^{\beta} + \frac{1}{h} J_x^{\beta} D_t^{\alpha}] u_k \end{cases} \quad (17)$$

At the end we can find the numerical solution by the same way in Adomian decomposition (see example 7).

III. Convergence Of Adomian Decomposition Method

Since our problems (ODE's, PDE's fractional or not), in different sciences, can be written as the general form:

u = f + N(u), where N(u) is the nonlinear operator term. Since the Adomian decomposition method and (DJM) method, is placed the solution as infinite series.

$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$, and the nonlinear term in infinite composition of known functions, which can be calculated by easy way, as:

N(u)= $\sum_{n=0}^{\infty} A_n$ Where A_n are the Adomian polynomials given by :

$A_n\ (u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[\ N\ (\sum_{i=0}^{\infty} u_i \lambda_i)]_{\lambda=0}$.we can see the same way in (DJM),

$N(u) = \sum_{i=0}^{\infty} G_i$ where $G_n = N(\sum_{i=0}^{n} u_i) - N(\sum_{i=0}^{n-1} u_i), n = 1,2,$ ...and $G_0 = N(u_0)$ where

N(u) = N(u_0)+ [N(u_0+u_1) – N(u_0)]+ [N($u_0 + u_1 + u_2$) –N($u_0 + u_1$)]+ …

The recursive relation equations given as:

$u_0 = f,\ \ u_n = G_{n-1}$, n=1,2,…

two facts, that is used to prove of convergence of Adomian method, the equivalent between Adomian and (DJM) methods, and the similarity between the infinite series of nonlinear term and the Taylor series expanded round the solution. See [15].

IV. Material And Method

considerthe general linear FPDE telegraph equation (1),

$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = a\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} + b\frac{\partial^{\sigma} u(x,t)}{\partial t^{\sigma}} + cu(x,t) + f(x,t)$, Where 0< x< L; t > 0; 0 <σ ≤ 1< α& β ≤ 2;

a, b and c are constants, f is given function, Subject to boundary conditions respectively:

B.C u(x, 0) = g_1(x), u_t(x,0) = g_2(x)

I.C u(0, t) = f_1(t) , u_x(0,t) = f_2(t),

if f = 0 (see example 1), if a = c = 0,(see example 3), If b = c = 0, (see example 2).

Now the steps in not1will be used to find the general solution of this equation by using Adomian method.

Step1: write this equation by the operators:

($\frac{\partial^{\beta}}{\partial x^{\beta}} = D_x^{\beta}$; $\frac{\partial^{\alpha}}{\partial t^{\alpha}} = D_t^{\alpha}$; $\frac{\partial^{\sigma}}{\partial t^{\sigma}} = D_t^{\sigma}$; $J_x^{\beta}\ is\ invers\ of D_x^{\beta}$

$D_x^{\beta}\ u = aD_t^{\alpha}\ u + b\ D_t^{\sigma}\ u + c\ u + f(x,t)$, Subject to same conditions,

Step2: take the invers J_x^{β} and using properties of Caputo fractional derivatives and the initial and boundary condition, the equations has the form:

u =[f_1(t) +x f_2(t)+J_x^{β} f(x,t)]+ $J_x^{\beta}[aD_t^{\alpha} u + bD_t^{\sigma} u + \ c\ u]$ (18)

By using decomposition

$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \ (u_0, u_1, u_2, \dots)$

Step3: now the recursive equations given by:

$$\begin{cases} u_0 = \ [f1(t) + x\ f2(t)] + J_x^{\beta} f(x,t) \\ u_{k+1} = J_x^{\beta}[aD_t^{\alpha} u_k + b\ D_t^{\sigma} u_k + c\ u_k\ ] k\ \geq\ 0 \end{cases} \quad (19)$$

V. Numerical Andexamples

Example 1: consider linear-homogeneous S-TFPDE from stander telegraph equation (1) where (a = b =1 and c= -1), and f(x,t)=0,

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial^\sigma u(x,t)}{\partial t^\sigma} - u(x,t)$$

Subject by these initial conditions:
I.C $u(0,t) = e^{-t}$; $u_x(0,t) = e^{-t}$
We can write the recursive equations (19) as:
$u_0(x,t) = [e^{-t} + x\, e^{-t}] = e^{-t} + xe^{-t}$
$u_0(x,t) = (1 + x)e^{-t}$
$u_k = J_x^\beta [\,(D_{tt}^\alpha + D_t^\sigma - 1)u_k\,(x,t)\,]$;k ≥ 0
From equation (4) we get:
$[\,(D_{tt}^\alpha + D_t^\sigma - 1)u_0(x,t) = (1 + x)e^{-t}$then by proprieties of Caputo's fractional derivatives and the fractional derivatives in note2, let a^k=$(-1)^k$

$$u_1 = J_x^\beta [\,(D_{tt}^\alpha + D_t^\sigma - 1)u_0\,(x,t)\,]$$

$= J_x^\beta (1 + x)e^{-t}$
$= [\frac{x^{(\beta)}}{\Gamma(\beta+1)} + \frac{x^{(\beta+1)}}{\Gamma(\beta+2)}](a^\alpha + a^\sigma + 1)e^{-t}$
$u_2 = [\frac{x^{(2\beta)}}{\Gamma(2\beta+1)} + \frac{x^{(2\beta+1)}}{\Gamma(2\beta+2)}](\,a^\alpha + a^\sigma + 1)^2 e^{-t}$

$$\vdots$$

$$u_k = [\frac{x^{(k\beta)}}{\Gamma(k\beta+1)} + \frac{x^{(k\beta+1)}}{\Gamma(k\beta+2)}](\,a^\alpha + a^\sigma + 1)^k e^{-t}$$

This is approximate solution $u_k(x,t)$, since the solution in this method given as:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = \lim_{k\to\infty} \sum_{j=0}^{k} u_j(x,t)$$

$= e^{-t} \sum_{n=0}^{\infty} \frac{x^{(k\beta)}}{\Gamma(k\beta+1)} + \frac{x^{(k\beta+1)}}{\Gamma(k\beta+2)}](\,a^\alpha + a^\sigma + 1)^k$,
If β = α =2 and $\sigma = 1$ then the series yield:
$u(x,t)$=$e^{x^2}(\,e^{-t})$, then this closed with exact solution.

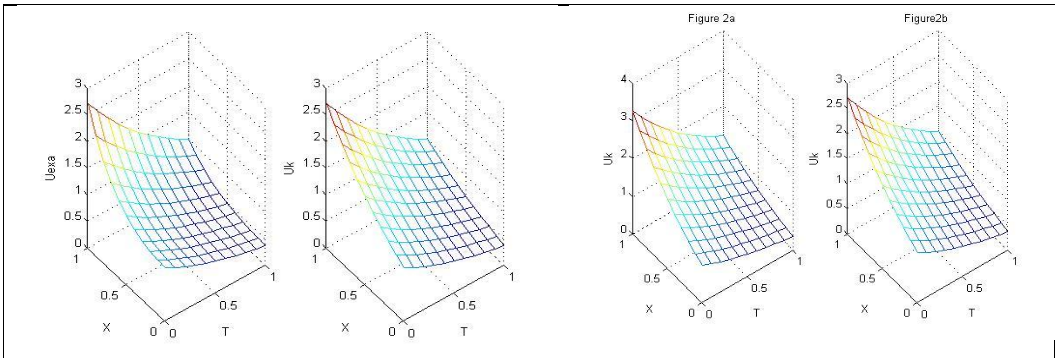

Figure1	Figure2

Figure1 shows the approximate solution u_k where k = 7, at (β = α =2 and $\sigma = 1$).
Figure2,in 2a shows the approximate solution u_k where k=7; at (β = 1.5; α =2 and $\sigma = 1$); in 2b shows the appr-oximate solution u_k where k=7; at (β = 2, α =1.5 and $\sigma = 1$).

Example 2: consider the telegraph equation (16) where (b=c=0 and f(x,t)=0) we get the stander fractional wave equation given by the form:
$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a\frac{\partial^\beta u(x,t)}{\partial x^\beta}$ Where 1 < β, α ≤ 2; 0 < x < L; t > 0 subject by initial conditions:
I.C u(x, 0) = $f_1(x)$ = cos(x), $u_t(x,0)$ = $f_2(x)$ = cos(x).
by using the general steps in case1 to solve this equation: then the recursive relation equations are been given by:
$u_0(x,t) = [f_1(x) + t\, f_2(x)]$;
$u_{k+1}(x,t) = J_t^\alpha [\,D_x^\beta u_k\,(x,t)\,]$Where J_t^α invers of D_t^α.

Let $0 < x < 1$; $t > 0$; a=1

I.C $f_1(x) = \cos(x)$; $f_2(x) = \cos(x)$ for $t \geq 0$.

Solution: by using recursive relation equations in case1we get:

$$u_0(x,t) = \sum_{j=0}^{2-1} \frac{\partial^j}{\partial t^j} u(x,0^+) \frac{t^j}{j!} = (1+t)cos(x)$$

$$u_1(x,t) = J_t^\alpha [\, D_x^\beta u_0\,(x,t)\,]$$

$$u_1 = J_t^\alpha (1+t)[cos\left(x+\frac{\beta\pi}{2}\right)]$$

$u_1 = [\frac{t^{(\alpha)}}{\Gamma(\alpha+1)} + \frac{t^{(\alpha+1)}}{\Gamma(\alpha+2)}]\,[cos\left(x+\frac{\beta\pi}{2}\right)]$ by the same way we get:

$$u_2(x,t) = [\frac{t^{(2\alpha)}}{\Gamma(2\alpha+1)} + \frac{t^{(2\alpha+1)}}{\Gamma(2\alpha+2)}]\,[cos\left(x+\frac{2\beta\pi}{2}\right)]$$

⋮

$$u_k(x,t) = [\frac{t^{(k\alpha)}}{\Gamma(k\alpha+1)} + \frac{t^{(k\alpha+1)}}{\Gamma(k\alpha+2)}]\,[cos\left(x+\frac{k\beta\pi}{2}\right)]$$

or we can write:

$$u_k(x,t) = [\frac{t^{(k\alpha)}}{\Gamma(k\alpha+1)} + \frac{t^{(k\alpha+1)}}{\Gamma(k\alpha+2)}]\,[cos\left(x+\frac{k\beta\pi}{2}\right)]$$

Then the u_k approximate solution will be given as:

$u_k(x,t) = \sum_{j=0}^{k} u_j(x,t)$; then Since:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \lim_{k\to\infty} \sum_{j=0}^{k} u_j(x,t)$$

$$u = \sum_{k=0}^{\infty} [\frac{t^{(k\alpha)}}{\Gamma(k\alpha+1)} + \frac{t^{(k\alpha+1)}}{\Gamma(k\alpha+2)}]\,[cos\left(x+\frac{k\beta\pi}{2}\right)]$$

If α=β=2 then the series become:

$$u = \sum_{k=0}^{\infty} [\frac{(t)^{(2k)}}{\Gamma(2k+1)} + \frac{(t)^{(2k+1)}}{\Gamma(2k+2)}]\,[cos(x+k\pi)]$$

$$u = cos(x) \sum_{k=0}^{\infty} (-1)^k [\frac{(t)^{(2k)}}{\Gamma(2k+1)} + \frac{(t)^{(2k+1)}}{\Gamma(2k+2)}]$$

$u = cos(x)\,[cos(t) + sin(t)]$.

This solution closed with exact solution at β = α=2.

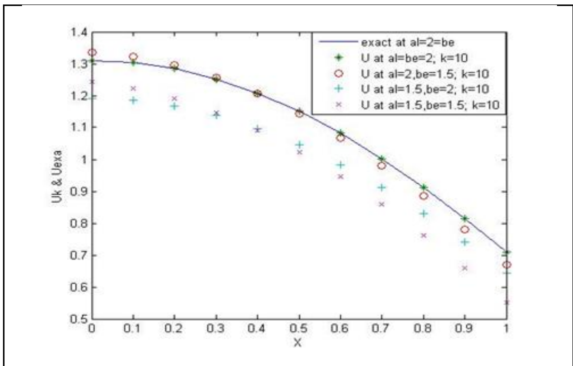

Figure3

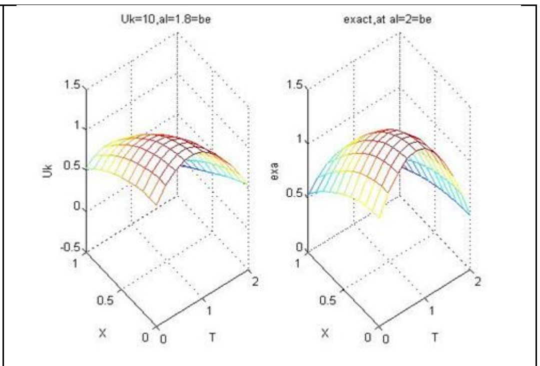

Figure4

Figure3 shows the exact and the approximate solutions at fixed t=0.4 and for x=(0:0.1:1), $U_{k=10}$, at β = α=2, and he approximate U_k at different values of (β and α).

Figure4 shows the exact solution at β = α=2, and approximate solution at k=10 and at β = α=1.8.

Example 3: if (a=c=0 and f(x,t) = 0)then the S-TFPDE Telegraph equation yield the general linear-homogeneous S-TFPDE heat equation will be given by

$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = b\,\frac{\partial^\beta u(x,t)}{\partial x^\beta}$where $0 < \alpha \leq 1 < \beta \leq 2$, and $0 < x < 1$; $t > 0$, subject by initial conditions, I.C u(x,0) $=f_1(x)=\sin(x)$.

Solution: using the general steps in case1and property in section (2.3) and Note2. Then the recursive relation

$$u_0(x,t) = f1(x) = sin(x).$$

$u_{k+1}(x,t) = J_t^{\alpha}[\, D_{xx}^{\beta} u_k\ (x,t)\ ]$. Then:

$u_0(x,t) = sin(x)$.

$u_1(x,t) = J_t^{\alpha}[\, D_{xx}^{\beta} u_0\ (x,t)\ ] = \frac{t^{(\alpha)}}{\Gamma(\alpha+1)}[sin\left(x+\frac{\beta\pi}{2}\right)]$

By the same way we have the approximate solution $u_k(x,t)$ k=0, 1, 2, 3,…

$u_k(x,t) = \frac{t^{(k\alpha)}}{\Gamma(k\alpha+1)}[sin\left(x+\frac{k\beta\pi}{2}\right)]$, then the solution is:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \lim_{k\to\infty} \sum_{j=0}^{k} u_j(x,t)$$

At α=1 and β=2 the series become:

$u(x,t) = sin(x)\ lim_{k\to\infty} \sum_{j=0}^{k} (-1)^k \frac{t^k}{\Gamma(k+1)}$, then series become the exact solution $u(x,t) = e^{-t}\, sin(x)$.

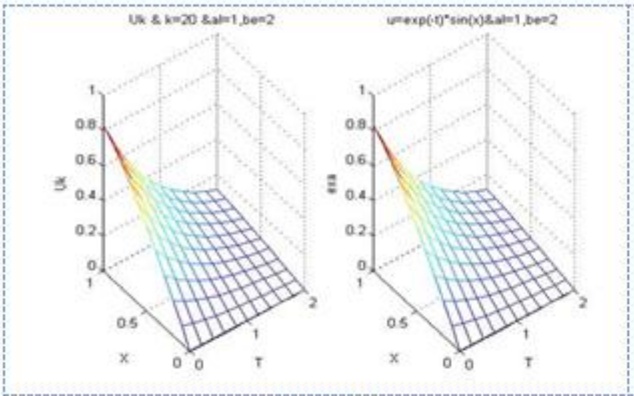

Figure5a

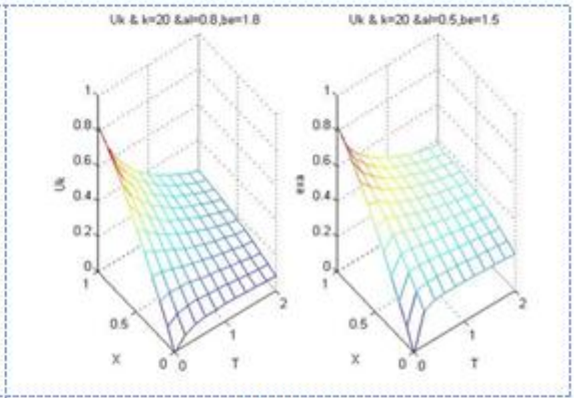

Figure5b

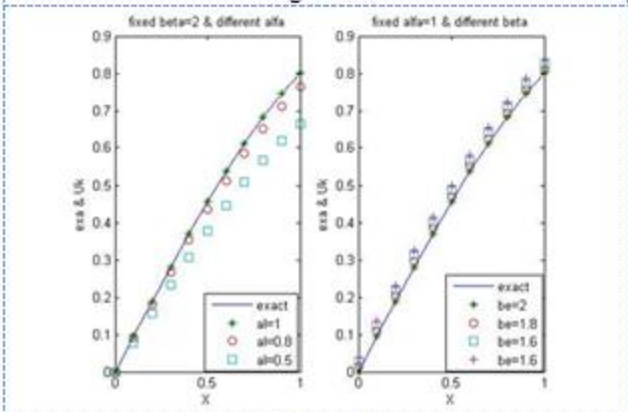

Figure6a

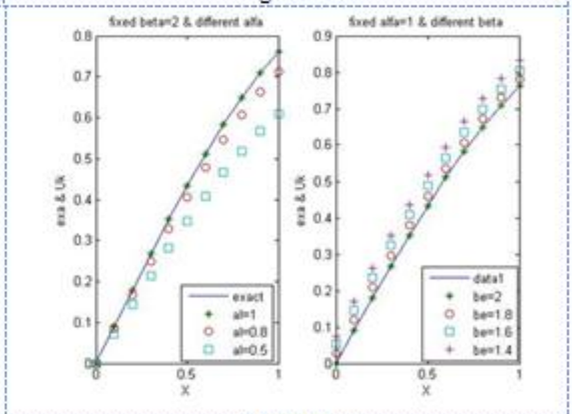

Figure6b

t	x	u			
0.05	0	0	0.0000	0.0341	0.0000
	0.1000	0.0950	0.0950	0.1303	0.0875
	0.2000	0.1890	0.1890	0.2251	0.1741
	0.3000	0.2811	0.2811	0.3177	0.2589
	0.4000	0.3704	0.3704	0.4071	0.3412
	0.5000	0.4560	0.4560	0.4924	0.4201
	0.6000	0.5371	0.5371	0.5728	0.4947
	0.7000	0.6128	0.6128	0.6475	0.5644
	0.8000	0.6824	0.6824	0.7158	0.6285
	0.9000	0.7451	0.7451	0.7769	0.6863
	1.0000	0.8004	0.8004	0.8302	0.7373

Table 1

Figure5a-b show the exact solution and the approximate solutions at different values of β and α, we can see that in the titles of sub-figures.

Figure6(a and b) show the carves of exact and approximate $U_{k=10}$ solutions at different values of α with fixed β at left figures and the carves at different values of β with fixed α at right figures and all at k=10, x=0:0.1:1 and (t=0.05 and 0.1) respectively.

Table1 shows the comparison between exact and approximate solutions at different values of β, α and at x with fixed t=0.5, at all these work we calculate the approximate solutions at k=10.

Example 4: consider the nonlinear S-TFPDE which has the form:

$D_t^{\alpha}u(x,t) = D_x^{\beta}u(x,t) + 6u(1-u)$ where

$0< \alpha \leq 1< \beta \leq 2$, and I.C $u(x,0) = \frac{1}{(1+e^x)^2}$

In this case of zero solution, the using of Taylor series is good way with using of Adomian polynomials to calculate the solutions.

Solution: since the Taylor series for I.C given by: $\frac{1}{(1+e^x)^2} = \sum_{n=0}^{\infty} u_n(x,0) = \frac{1}{4} - \frac{x}{4} + \frac{x^2}{16} + \frac{x^2}{48} - \frac{x^2}{96} + \cdots$; then the nonlinear term is $F(u)=u^2 = \sum_{n=0}^{\infty} A_n$. Now by using the steps in case2 and the information in note2 and section (2.3).we can write the recursive relation equation for this problem will be given in one equation only as:

$u_{k+1}(x,t) = u_{k+1}(x,0) + J_t^{\alpha}[D_x^{\beta}u_k(x,t) + 6u_k(x,t) \ - 6A_n$

Where $u_0(x,0) = \frac{1}{4}; u_1(x,0) = \frac{-x}{4}; u_2(x,0) = \frac{x^2}{16}; \cdots$

And $A_0 = u_0^2; A_0 = u_0^2; A_1 = 2u_1u_0; A_0 = 2u_2u_0 + u_1^2$

Then we can write the solutions in the following:

$u_0 = \frac{1}{4}; \ u_1 = -\frac{x}{4} + \frac{9}{8}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$

$$u_2 = \frac{x^2}{16} - \frac{3x}{4}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{27}{8}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_2 = \frac{x^2}{16} - \frac{3x}{4}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{27}{8}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$u_3 = \frac{x^3}{48} - \frac{3x^2}{16}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{2-\beta}}{8\Gamma(3-\beta)}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{9x}{16}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{27}{16}\frac{t^{-3\alpha}}{\Gamma(3\alpha+1)},$

$$u_4 = -\frac{x^4}{96} + \frac{x^3}{4}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{3-\beta}}{8\Gamma(4-\beta)}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{117x^2}{64}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{3x^{2-\beta}}{8\Gamma(3-\beta)}$$

we can write solution by series; $u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots$

atα=1 and β =2 this solution agreement with the exact solution in [14]

Example5: solve the linear-inhomogeneous S-TFPDE

${}_a^C D_t^{0.5}u(x,t) = {}_a^c D_x^{1.5}u(x,t) + g(x,t); 0 < x < 1, t \geq 0$

Where $g(x,t) = \frac{1}{\Gamma(1.5)}x^2 j_t^{0.5}e^{-t} - \frac{\Gamma(3)}{\Gamma(1.5)}x^{0.5}e^{-t}$; subject to conditions :

B.C u(0,t) =0; u(1 ,t) $=e^{-t}t \ \geq 0$

I C u(x ,0) = x^2 ; 0 < x<1

Solution: we solve the equation for general Real number α ;β , since from recursive relation

$u_0(x,t) = F(x,t)$where F(x ,t) = u(x,0) + $J_t^{\alpha}g(x,t)$

$u_{r+1}(x,t) = J_t^{\alpha}\,{}_a^c D_x^{\beta}u_r(x,t)$for r ≥ 0 . we have

$u_0(x,t) \ = F(x,t)$= u(x,0) + $J_t^{\alpha}g(x,t)$

$$u_0(x,t) \ = x^2 + j_t^{\alpha}[x^2 D_t^{\alpha}e^{-t} - e^{-t}\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{2-\beta}]$$

$$u_0(x,t) \ = x^2 + x^2(e^{-t}-1) - j_t^{\alpha}e^{-t}\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{2-\beta}$$

$$u_0(x,t) \ = x^2e^{-t} - j_t^{\alpha}e^{-t}\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{2-\beta}$$

$u_1(x,t) = J_t^{\alpha}\,{}_a^R D_x^{\beta}u_0(x,t)$; so

$$u_1(x,t) = j_t^{\alpha}e^{-t}\frac{\Gamma(3)x^{2-\beta}}{\Gamma(3-\beta)} - j_t^{2\alpha}e^{-t}\frac{\Gamma(3)x^{2-2\beta}}{\Gamma(3-2\beta)}$$

Now from u_1 and u_0 we can see the noise term is [$\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{2-\beta}j_t^{\alpha}e^{-t}$]

If we deleted it and the other term of zeros u_0 satisfy the equation and all conditions, so that the exact solution is $u_0(x,t) = x^2 e^{-t}$.

Example6: solve the linear-inhomogeneous S-TFPDE;
${}_a^C D_t^{0.5} u(x,t) = {}_a^c D_x^{1.5} u(x,t) + g(x,t)$;Where $0 < x < \pi , t \geq 0$; and
g(x,t) = $- cos(x)\, {}_a^c D_t^{0.5}(e^{-t}) - {}_a^c D_x^{1.5}(cos(x)) + e^{-t}\, {}_a^c D_x^{1.5}(cos(x))$, Subject by conditions:
B.C u(0 ,t) $= 1 - e^{-t}$;(π ,t) = $e^{-t} - 1$ $t \geq 0$
I .C u(x ,0) = 0 0≤ x ≤ π

Solution:
$u_0(x,t) = F(x,t)$ = u(x,0) + $J_t^{0.5} g(x,t)$
$= - cos(x)\,(e^{-t} - e^0) - 0 + J_t^{0.5} e^{-t} [{}_a^c D_x^{1.5}(cos(x))]$
$= cos(x)\,(1 - e^{-t}) + J_t^{0.5} e^{-t}\ [\ {}_a^c D_x^{1.5}(cos(x))\ ]$
$u_1(x,t) = -J_t^{0.5} e^{-t}\, {}_a^c D_x^{1.5}(cos(x)) + J_t^1 e^{-t}\, {}_a^c D_x^3(cos(x))]$
From u_1 and u_0 if we delete the noise term in u_0, $J_t^{0.5} e^{-t}\, {}_a^R D_x^{1.5}(cos(x))$, then the other terms satisfy the FPDE and all conditions, so that the exact solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = (1 - e^{-t}) cos(x)$$

Example 7: Solvethe S-TFPDE which has the form,
$D_t^{\alpha} u(x,t) = D_x^{\beta} u(x,t)$, where 0< x< 1; t ≥ 0, 0< α ≤ 1< β ≤ 2, subject to conditions
B.C u(0,t) = f_0 =at$^{\alpha}$; u(1,t) =f_1 = at$^{\alpha}$ +1
I.C u(x,0) = g(x)= x$^{\beta}$ where $a = \Gamma(\beta+1)/\Gamma(\alpha+1)$

Solution: by using steps in case4 we can write the recursive relation equations as:

$$u_0 = \frac{1}{2}[x^{\beta} + at^{\alpha} + x]$$

$$u_1 = \frac{1}{2}[hJ_t^{\alpha} D_x^{\beta} + \frac{1}{h} \mathrm{J}_x^{\beta} \mathrm{D}_t^{\alpha}](\frac{1}{2}[\mathrm{x}^{\beta} + \mathrm{x} + \mathrm{at}^{\alpha}])$$

$\mathrm{u}_1 = \left(\frac{1}{2}\right)^2 (\mathrm{h}\frac{\Gamma(\beta+1)\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)} + \frac{\mathrm{a}\Gamma(\alpha+1)}{\Gamma(\beta+1)} \mathrm{x}^{\beta})$
$\mathrm{u}_2 = \left(\frac{1}{2}\right)^3 (\ \mathrm{at}^{\alpha} + \mathrm{x}^{\beta}\mathrm{by})$
thenby the same way one can get the kth approximation solution as
$\mathrm{u}_k = \left(\frac{1}{2}\right)^{k+1} (\mathrm{x}^{\beta} + \mathrm{at}^{\alpha})$, then the solution is
$\mathrm{u(x,t)} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} (\mathrm{x}^{\beta} + \mathrm{at}^{\alpha})$ =$(\mathrm{x}^{\beta} + \mathrm{at}^{\alpha})$, this is exact solution for S-TFPDE, and it is closed with the exact solution$(\mathrm{x}^2 + 2\mathrm{t})$, for the integer order PDE $[\mathrm{D_t}\ \mathrm{u(x,t)} = \mathrm{D}_x^2 \mathrm{u(x,t)}\ ]$,

Figure7-a shows the approximate and exact solutions at(β =2 and α=1) and the approximate solution at different values of (α=0.8, 0.5), with fixed (β =2),
Figure7-b shows the approximate solution at different values of (β =1.8, 1.6,1.4), with fixed (α =2).all that at step t=0.01 and k=30.

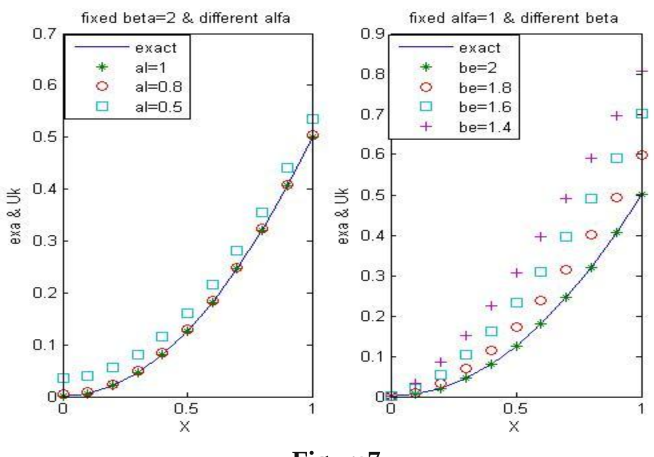

Figure7

VI. Conclusion

The telegraph equation is a general model equation, which admits the behavior of the heat equations as well as the wave equation, many authors have studied the heat equation with fractional order derivatives in space or time also the wave equation has been studied with fractional derivatives order in space or time.

We considered the telegraph equation with fractional derivatives in both space and time. We considered four cases at case one, we considered linear and homogeneous telegraph with fractional derivatives in both space and time. We preformed three examples for different values of fractional order as shown in figure 1,2,3,4,5,(6-a) and (6-b).also we presented more numerical data in label1 for fixed value of t=0.05, and different values of x,α and β.

From this study one can see easy the easy way and the good method(ADM &MADM), one can use the steps to find the numerical solutions (sometime the exact solution), for different kind of S-TFPDE's

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