IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 1 Ver. II (Jan - Feb. 2015), PP 16-23
www.iosrjournals.org

Uniformity of the Local Convergence of Chord Method for Generalized Equations

M. H. Rashid[1], A. Basak[2] and M. Z. Khaton[3]
[1,2]*Department of Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh*
[3]*Department of Mathematics, Sapahar Government Degree College, Naogaon-6560, Bangladesh*

Abstract: *Let X be a real or complex Banach space and Y be a normed linear space. Suppose that $f: X \to Y$ is a Frechet differentiable function and $F: X \rightrightarrows 2^Y$ is a set-valued mapping with closed graph. Uniform convergence of Chord method for solving generalized equation $y \in f(x) + F(x) \ldots\ldots (*)$, where $y \in Y$ a parameter, is studied in the present paper. More clearly, we obtain the uniform convergence of the sequence generated by Chord method in the sense that it is stable under small variation of perturbation parameter y provided that the set-valued mapping F is pseudo-Lipschitz at a given point (possibly at a given solution).*
Keywords: *Chord method, Generalized equation, Local convergence, pseudo-Lipschitz mapping, Set-valued mapping.*
AMS (MOS) Subject Classifications: *49J53; 47H04; 65K10.*

1. Introduction

Let X be a real or complex Banach space and Y be a normed linear space. We are concerned with the problem of finding the solution x^* of parameterized perturbed generalized equation of the form

$$y \in f(x) + F(x), \tag{1.1}$$

where $y \in Y$ is a parameter, $f: X \to Y$ is a single-valued function and $F: X \rightrightarrows 2^Y$ is set-valued mapping with closed graph.

The Generalized equation problems were introduced by S.M. Robinson [1, 2] as a general tool for describing, analyzing, and solving different problem in a unified manner. Typical examples are systems of inequalities, variational inequalities, linear and non-linear complementary problems, systems of nonlinear equations, equilibrium problems, etc.; see for example [1--3].

It is remark that when $F = \{0\}$, (1.1) is an equation. When F is positive orthand in R^n, (1.1) is a system of inequalities. When F is the normal cone to a convex and closed set in X, (1.1) represents variational inequalities.

Newton-type method can be considered to solve this generalized equation when the single-valued function involved in (1.1) is differentiable. Such an approach has been used in many contributions to this subject; see for example [4--6].

To solve (1.1), Dontchev [4] introduced a Newton type method of the form

$$y \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}),\ k = 0,1,\ldots, \tag{1.2}$$

where $\nabla f(x_k)$ is the Frechet derivative of f at the point x_k and obtained the stability of the method (1.2).

A large number of iterative methods have been presented for solving (1.1). Pietrus [5] showed the stability of this method under mild conditions. Other achievements on this topic can be found in [6, 7].

Marinov [8] associated the following method name as “Chord method” for solving the generalized equation (1.1):

$$y \in f(x_k) + A(x_{k+1} - x_k) + F(x_{k+1})\ , \tag{1.3}$$

where $A \in L(X, Y)$. It should be noted that when $A = \nabla f(x_k)$, the method (1.3) reduces to the well-known Newton-type method (1.2).

In the present paper, we are intended to present a kind of convergence of the sequence generated by the Chord method defined by (1.3) which is uniform in the sense that the attraction region does not depend on small variations of the value of the parameter y near y^* and for such values of y the method finds a solution x of (1.1) whenever F is pseudo-Lipschitz at (x^*, y^*) and the Frechet derivative of f is continuous. We will prove the uniformity of the local convergence of the Chord method defined by (1.3) in two different ways.

This work is organized as follows: In Section 2, we recall few preliminary results that will be used in the subsequent sections. In Section 3, we prove the existence and uniform convergence of the sequence generated by the Chord method defined by (1.3) for solving generalized equation (1.1). Finally in Section 4, we will give conclusion of the major results obtained in this study.

II. Notations And Preliminary Results

Throughout this work we suppose that X is a real or complex Banach space and Y is a normed linear space. All the norms are denoted by $\|\cdot\|$. The closed ball centered at x with radius $r > 0$ denoted by $B_r(x)$ and the set of linear operators denoted by $L(X, Y)$.

The following definition is taken from [7, 9].

Definition 2.1: Let $F: X \rightrightarrows 2^Y$ be a set valued mapping. Then the Domain of F is denoted by $\operatorname{dom} F$, and is define by

$$\operatorname{dom} F = \{x \in X : F(x) \neq \emptyset\}.$$

The inverse of F is denoted by F^{-1}, and is defined by

$$F^{-1}(y) = \{x \in X: y \in f(x)\},$$

and

The Graph of F is denoted by gph F, and is define by

$$\operatorname{gph} F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

The following definition of distance from a point to a set and excess are taken from [7, 9].

Definition 2.2: Let X be Banach space A be subset of X. Then the distance from a point x to a set A is denoted by dist (x, A), and is define by

$$\operatorname{dist}(x, A) = \inf\{\|x - a\| : a \in A\}.$$

Definition 2.3: Let X be Banach space and $A, B \subseteq X$. The excess e from the set A to a set B is given by

$$e(B, A) = \sup\{\|x - A\| : x \in B\}.$$

The following definition is taken from [9].

Definition 2.4: Let $F: X \rightrightarrows 2^Y$ be a set-valued mapping. Then F is said to be pseudo Lipschitz-around $(x_0, y_0) \in$ gph F with constant M if there exist positive constants $\alpha, \beta > 0$ such that

$$e\left(F(x_1) \cap B_\beta(y_0), F(x_2)\right) \leq M\|x_1 - x_2\|,$$

for every $x_1, x_2 \in B_\alpha(x_0)$. When F is single-valued, this corresponds to the usual concept of Lipschitz continuity.

The definition of Lipschitz continuity is equivalent to the definition of Aubin continuity, which is given below:

A set-valued map $\Gamma: Y \rightrightarrows 2^X$ is Aubin continuous at $(y_0, x_0) \in \operatorname{gph} \Gamma$ with positive constants α, β and M if for every $y_1, y_2 \in B_\beta(y_0)$ and for every $x_1 \in \Gamma(y_1) \cap B_\alpha(x_0)$**,** there exists an $x_2 \in \Gamma(y_2)$ such that

$$\|x_1 - x_2\| \leq M\|y_1 - y_2\|.$$

The constant M is called the modulus of Aubin continuity.

The following definition of continuity is taken from the book [10].

Definition 2.5: A map $f: \Omega \subseteq X \to Y$ is said to be continuous at $\bar{x} \in \Omega$ if for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$\|f(x) - f(\bar{x})\| < \varepsilon, \text{ for all } x \in \Omega \text{ for which } \|x - \bar{x}\| < \delta.$$

Definition 2.6: A map $f: \Omega \subseteq X \to Y$ is said to be Lipschitz continuous if there exists constant c such that

$$\|f(x) - f(y)\| \leq c\|x - y\|, \text{ for all } x \text{ and } y \text{ in the domain of } f.$$

The following definition of linear convergence is taken from the monogram [11].

Definition 2.7: Let $\{x_n\}$ be a sequence which is converges to the number $\bar{x}$. Then the sequence $\{x_n\}$ is said to be converges linearly to $\bar{x}$, if there exists a number $0 < c < 1$ such that

$$\|x_{n+1} - \bar{x}\| \leq c\|x_n - \bar{x}\|.$$

The following Lemma is a version of fixed point theorem, which is taken from [7].

Lemma 2.1: Let $\Phi: X \rightrightarrows 2^X$ be a set valued mapping and let $\eta_0 \in X$, $r > 0$ and $0 < \lambda < 1$ be such that

(a) $\operatorname{dist}\left(\eta_0, \Phi(\eta_0)\right) < r(1 - \lambda)$ and

(b) $e\left(\Phi(x_1) \cap B_r(\eta_0), \Phi(x_2)\right) \leq \lambda\|x_1 - x_2\|$, for all $x_1, x_1 \in B_r(\eta_0)$.

Then Φ has a fixed point in $B_r(\eta_0)$, that is, there exists $x \in B_r(\eta_0)$ such that $x \in \Phi(x)$. If Φ is single-valued, then x is the unique fixed point of Φ in $B_r(\eta_0)$.

III. Uniform Convergence Of Chord Method

This section is devoted to study the stability of the Chord method for solving generalized equations (1.1) involving set-valued mapping and parameters. Let $f: X \to Y$ be a single-valued function which is Frechet differentiable on an open set in X and $F: X \rightrightarrows 2^Y$ be a set-valued mapping with closed graph. Let $\bar{y} \in F(\bar{x})$ and F^{-1} be pseudo-Lipschitz around $(\bar{y}, \bar{x})$. Then through Theorem 2.1 in [12], we have that $(f + F)^{-1}$ is pseudo-Lipschitz around $(\bar{y} + f(\bar{x}), \bar{x})$. Let $x \in X$ and define the mapping $P_x: X \rightrightarrows 2^Y$ by

$$P_x(\cdot) = f(x) + A(\cdot - x) + F(\cdot).$$

Moreover, the following equivalence is obvious, for any $z \in X$ and $y \in Y$,

$$z \in P_x^{-1}(y) \Leftrightarrow y \in f(x) + A(z - x) + F(z). \tag{3.1}$$

In particular, $x^* \in P_{x^*}{}^{-1}(y^*)$ for each $(x^*, y^*) \in \text{gph}(f+F)$. Let $a > 0$, $b > 0$. Throughout the whole paper, we suppose that $B_a(x^*) \subseteq \Omega \cap \text{dom}\, F$ and that $(x^*, y^*) \in \text{gph}\, P_{x^*}$, for every$(x^*, y^*) \in \text{gph}(f+F)$. Then by Lemma 3.1 in [12], we have that the mapping $P_{x^*}^{-1}(\cdot)$ is pseudo-Lipschitz around (y^*, x^*) with constant M, that is,

$$e\left(P_{x^*}^{-1}(y_1) \cap B_a(x^*), P_{x^*}^{-1}(y_2)\right) \leq M\|y_1 - y_2\|, \text{for any } y_1, y_2 \in B_b(y^*). \tag{3.2}$$

Choose $\|A - \nabla f(x^*)\| > 0$ and set

$$\bar{r} = \min\left\{b - 2a\|A - \nabla f(x^*)\|, \frac{a(1 - M\|A - \nabla f(x^*)\|)}{4M}\right\}. \tag{3.3}$$

Then $\bar{r} > 0 \Leftrightarrow \|A - \nabla f(x^*)\| < \min\left\{\frac{b}{2a}, \frac{1}{M}\right\}$. (3.4)

The following Lemma is useful for proving the existence of a sequence generated by the method (1.3). The proof is analogous to the proof of Rashid et al. [7].

Lemma 3.1 Suppose that $P_{x^*}^{-1}(\cdot)$ is pseudo-Lipschitz around (y^*, x^*) with constant M. Let $x \in B_{\frac{a}{2}}(x^*)$ and let $A \in L(X, Y)$ be such that $M\|\nabla f(x^*) - A\| < 1$. Suppose that

$$\sup_{x \in B_{\frac{a}{2}}(x^*)} \|\nabla f(x) - \nabla f(x^*)\| \leq \|A - \nabla f(x^*)\| \leq \min\left\{\frac{b}{2a}, \frac{1}{M}\right\}.$$

Then $P_x^{-1}(\cdot)$ is pseudo-Lipschitz around (y^*, x^*) with constant $\frac{M}{1 - M\|A - \nabla f(x^*)\|}$,

that is, $e\left(P_x^{-1}(y_1) \cap B_{\frac{a}{2}}(x^*), P_x^{-1}(y_2)\right) \leq \frac{M}{1 - M\|A - \nabla f(x^*)\|}\|y_1 - y_2\|$ for any $y_1, y_2 \in B_{\bar{r}}(y^*)$.

Proof: Let $y_1, y_2 \in B_{\bar{r}}(y^*)$ and $x' \in P_x^{-1}(y_1) \cap B_{\frac{a}{2}}(x^*)$. (3.5)

We need to prove that there exists $x'' \in P_x^{-1}(y_2)$ such that

$$\|x' - x''\| \leq \frac{M}{1 - M\|A - \nabla f(x^*)\|}\|y_1 - y_2\|.$$

To finish this, we will verify that there exists a sequence $\{x_k\} \subset B_a(x^*)$ such that

$$y_2 \in f(x) + A(x_{k-1} - x) + \nabla f(x^*)(x_k - x_{k-1}) + F(x_k) \tag{3.6}$$

and $\|x_k - x_{k-1}\| \leq M\|y_1 - y_2\|(M\|A - \nabla f(x^*)\|)^{k-2}$ for each $k = 2,3,4, \ldots$ (3.7)

We prove by induction on k. Denote

$$z_i = y_i - f(x) - A(x' - x) + f(x^*) + \nabla f(x^*)(x' - x^*) \text{ for each } i = 1,2. \tag{3.8}$$

Note by (3.5) that $\|x - x'\| \leq \|x - x^*\| + \|x^* - x'\| \leq a$. Following from (3.5) and using (3.3) with the relation $\bar{r} \leq b - 2a\|A - \nabla f(x^*)\|$, we have that

$$\begin{aligned}
\|z_i - y^*\| &\leq \|y_i - y^*\| + \|f(x) - f(x^*) - \nabla f(x^*)(x - x^*)\| + \left\|\left(A - \nabla f(x^*)\right)(x - x')\right\| \\
&\leq \bar{r} + \int_0^1 \left\|\nabla f\left(x^* + t(x - x^*)\right) - \nabla f(x^*)(x - x^*)dt\right\| + \left\|\left(A - \nabla f(x^*)\right)(x - x')\right\| \\
&\leq \bar{r} + \|A - \nabla f(x^*)\|\|x - x^*\| + \|A - \nabla f(x^*)\|\|x - x'\| \\
&= \bar{r} + \|A - \nabla f(x^*)\|(\|x - x^*\| + \|x - x'\|) \\
&\leq \bar{r} + \|A - \nabla f(x^*)\|\left(\frac{a}{2} + a\right) \leq b.
\end{aligned}$$

So we get $z_i \in B_b(y^*)$ for each $i = 1,2$. Setting $x_1 = x'$. Then $x_1 \in P_x^{-1}(y_1)$ by (3.5), and it comes from (3.1) that $y_1 \in f(x) + A(x_1 - x) + F(x_1)$, which can be written as

$$y_1 + f(x^*) + \nabla f(x^*)(x_1 - x^*) \in f(x) + A(x_1 - x) + F(x_1) + f(x^*) + \nabla f(x^*)(x_1 - x^*).$$

By the definition of z_1 in (3.8), we obtain that

$$z_1 \in f(x^*) + \nabla f(x^*)(x_1 - x) + F(x_1).$$

Hence $x_1 \in P_{x^*}^{-1}(z_1)$ by (3.1), and then $x_1 \in P_{x^*}^{-1}(z_1) \cap B_{r_{x^*}}(x^*)$ by (3.5). Noting that $z_1, z_2 \in B_b(y^*)$, and from (3.2) we claim that their exists $x_2 \in P_{x^*}^{-1}(z_2)$ such that

$$\|x_2 - x_1\| \leq M\|z_1 - z_2\| = M\|y_1 - y_2\|. \tag{3.9}$$

Setting $x_1 = x'$. By the definition of z_2 in (3.8), we have that

$x_2 \in P_{x^*}^{-1}(z_2) = P_{x^*}^{-1}\left(y_2 - f(x) - A(x_1 - x) + f(x^*) + \nabla f(x^*)(x_1 - x^*)\right)$, which, together with (3.1), implies that

$$y_2 \in f(x) + A(x_1 - x) + \nabla f(x^*)(x_2 - x_1) + F(x_2). \tag{3.10}$$

Therefore (3.10) and (3.9) shows respectively that (3.6) and (3.7) are true for the constructed points x_1, x_2. We assume that $x_1, x_2, x_3, \ldots., x_n$ are constructed such that (3.6) and (3.7) are true for $k = 2,3, \ldots., n$. We need to construct x_{n+1} such that (3.6) and (3.7) are also true for $k = n + 1$. For this purpose, we can write

$z_i^n = y_2 - f(x) - A(x_{n+i-1} - x) + f(x^*) + \nabla f(x^*)(x_{n+i-1} - x^*)$, for each $i = 0,1$.

Then, by the inductive assumption, we have that

$$\begin{aligned}\|z_0^n - z_1^n\| &= \|(A - \nabla f(x^*))(x_n - x_{n-1})\| \\ &\le \|A - \nabla f(x^*)\|\|x_n - x_{n-1}\| \\ &\le \|y_1 - y_2\|(M\|A - \nabla f(x^*)\|)^{n-1}. \end{aligned} \tag{3.11}$$

Since $\|x_1 - x^*\| \le \frac{a}{2}$ and $\|y_1 - y_2\| \le 2\bar{r}$ by (3.5), it follows from (3.7) that

$$\begin{aligned}\|x_n - x^*\| &\le \sum_{k=2}^{n}\|x_k - x_{k-1}\| + \|x_1 - x^*\| \\ &\le 2M\bar{r}\sum_{k=2}^{n}(M\|A - \nabla f(x^*)\|)^{k-2} + \frac{a}{2} \\ &\le \frac{2M\bar{r}}{1 - M\|A - \nabla f(x^*)\|} + \frac{a}{2}.\end{aligned}$$

From (3.3), we see $\bar{r} \le \frac{a(1-M\|A-\nabla f(x^*)\|)}{4M}$, and so $\|x_n - x^*\| \le a.$ (3.12)

Accordingly, $\|x_n - x\| \le \|x_n - x^*\| + \|x^* - x\| \le \frac{3a}{2}.$ (3.13)

Furthermore, using (3.5) and (3.13), one has, for each $i = 0,1$, that

$$\begin{aligned}\|z_i^n - y^*\| &\le \|y_2 - y^*\| + \|f(x) - f(x^*) - \nabla f(x^*)(x - x^*)\| \\ &\quad + \|(A - \nabla f(x^*)(x - x_{n+i-1}))\| \\ &\le \bar{r} + \|A - \nabla f(x^*)\|\|x - x^*\| + \|A - \nabla f(x^*)\|\|x - x_{n+i-1}\| \\ &= \bar{r} + \|A - \nabla f(x^*)\|(\|x - x^*\| + \|(x - x_{n+i-1})\|) \\ &\le \bar{r} + \|A - \nabla f(x^*)\|\left(\frac{a}{2} + \frac{3a}{2}\right) \\ &= \bar{r} + 2\|A - \nabla f(x^*)\|a.\end{aligned}$$

By the assumption of $\bar{r}$ in (3.3), we have that $z_i^n \in B_b(y^*)$ for each $i = 0,1$. Since assumption (3.6) holds for $k = n$ we have

$$y_2 \in f(x) + A(x_{n-1} - x) + \nabla f(x^*)(x_n - x_{n-1}) + F(x_n),$$ which can be written as

$$\begin{aligned}y_2 + f(x^*) + \nabla f(x^*)(x_{n-1} - x^*) &\in f(x) + A(x_{n-1} - x) + \nabla f(x^*)(x_n - x_{n-1}) \\ &\quad + F(x_n) + f(x^*) + \nabla f(x^*)(x_{n-1} - x^*),\end{aligned}$$

that is, $z_0^n \in f(x^*) + \nabla f(x^*)(x_n - x^*) + F(x_n)$ by the assumption of z_0^n. This, jointly with (3.1) and (3.12), gives $x_n \in P_{x^*}^{-1}(z_0^n) \cap B_a(x^*)$. Utilize (3.2) again, their exists an element $x_{n+1} \in P_{x^*}^{-1}(z_1^n)$ such that

$$\|x_{n+1} - x_n\| \le M\|z_0^n - z_1^n\| \le M\|y_1 - y_2\|(M\|A - \nabla f(x^*)\|)^{n-1}, \tag{3.14}$$

where the last inequality holds by (3.11). By the assumption of z_1^n, we get

$x_{n+1} \in P_{x^*}^{-1}(z_1^n) = P_{x^*}^{-1}\big(y_2 - f(x) - A(x_n - x) + f(x^*) + \nabla f(x^*)(x_n - x^*)\big)$, which, together with (3.1), implies that

$$y_2 \in f(x) + A(x_n - x) + \nabla f(x^*)(x_{n+1} - x_n) + F(x_{n+1}).$$

This, together with (3.14), finishes the induction step, and proves the existence of sequence $\{x_k\}$ satisfying (3.6) and (3.7). Since $M\|A - f(x^*)\| < 1$, we see from (3.7) that, $\{x_k\}$ is a Cauchy sequence, and hence it is convergent. Let $x'' = \lim_{k\to\infty} x_k$. Then, taking limit in (3.6) and noting that F has closed graph, we get $y_2 \in f(x) + A(x'' - x) + F(x'')$ and so $x'' \in P_x^{-1}(y_2)$.

Furthermore,
$$\begin{aligned}\|x' - x''\| &\le \lim_{n\to\infty}\sup\sum_{k=2}^{n}\|x_k - x_{k-1}\| \\ &\le \lim_{n\to\infty}\sum_{k=2}^{n}(M\|A - \nabla f(x^*)\|)^{k-2}M\|y_1 - y_2\| \\ &= \frac{M}{1-M\|A-\nabla f(x^*)\|}\|y_1 - y_2\|.\end{aligned}$$

This completes the proof of the lemma.

3.1 Uniformity of Linear Convergence of First Kind

This section is devoted to study the uniformity of linear convergence of the Chord method defined by (1.3).

Theorem 3.1 Let x^* be a solution of (1.1) for $y = 0$. Suppose that $P_{x^*}^{-1}(\cdot)$ is pseudo-Lipschitz around $(0, x^*)$ with constants a, b and M. Let $x \in B_{\frac{a}{2}}(x^*)$ and $A \in L(X,Y)$. Suppose that ∇f is continuous on $B_{\frac{a}{2}}(x^*)$ with constant $\|A - \nabla f(x^*)\|$ such that $3M'\|A - \nabla f(x^*)\| < 1$, i.e,

$$\sup_{x\in B_{\frac{a}{2}}(x^*)}\|\nabla f(x) - \nabla f(x^*)\| \le \|A - \nabla f(x^*)\| \le \min\left\{\frac{b}{2a}, \frac{1}{M}\right\}.$$

Then there exists positive constants σ and c such that for every $y \in B_b(0)$ and for every $x_0 \in B_\sigma(x^*)$ there exists a sequence $\{x_k\}$ generated by (1.3) with initial point x_0, which is convergent to a solution x of (1.1) for y, i.e.

$$\|x_{k+1} - x\| \le c\|x_k - x\|.$$

Proof: Since $P_x^{-1}(\cdot)$ is pseudo-Lipschitz around $(0.x^*)$ with constant M. Then there exist positive constants a and b such that

$$e\left(P_{x^*}^{-1}(y_1)\cap B_a(x^*), P_{x^*}^{-1}(y_2)\right)\le M\|y_1-y_2\|, \text{for each } y_1,y_2\in B_b(0).$$

Let $x_0\in B_\sigma(x^*)$, $y\in B_b(0)$ and $\bar{r}$ be defined in (3.3). Put $\alpha=\frac{a}{2}$, $\beta=\bar{r}$ and $M^{'}=\frac{M}{1-M\|A-\nabla f(x^*)\|}$ are the constants in Lemma 3.1. Then we have that

$$e\left(P_x^{-1}(y_1)\cap B_\alpha(x^*), P_x^{-1}(y_2)\right)\le M\|y_1-y_2\|, \text{for each } y_1,y_2\in B_\beta(0).$$

Now, we have by Lemma 3.1, for $M\|A-\nabla f(x^*)\|<1$, that

$$M^{'}\|A-\nabla f(x^*)\|=\frac{M\|A-\nabla f(x^*)\|}{1-M\|A-\nabla f(x^*)\|}<1.$$

Choose $\sigma>0$ be such that

$$\sigma\le\min\left\{\alpha,\frac{\alpha-M^{'}b\left(1+3M^{'}\|A-\nabla f(x^*)\|\right)}{M^{'}(5+6M^{'}\|A-\nabla f(x^*)\|)\|A-\nabla f(x^*)\|+2},\frac{(\beta-b)-3M^{'}b\|A-\nabla f(x^*)\|}{6M^{'}\|A-\nabla f(x^*)\|^2+3\|A-\nabla f(x^*)\|},1\right\}. \tag{3.15}$$

Since $x_0\in B_\sigma(x^*)$, then $x^*\in P_{x_0}^{-1}\left(-f(x^*)+f(x_0)+A(x^*-x_0)\right)\cap B_\alpha(x^*)$ and

$$\begin{aligned}
&\|f(x^*)-f(x_0)-A(x^*-x_0)\|\\
&\le\|f(x_0)-f(x^*)-\nabla f(x^*)(x_0-x^*)\|+\|A-\nabla f(x^*)\|\|x^*-x_0\|\\
&\le\int_0^1\left\|\left(\nabla f\left(x^*+t(x_0-x^*)\right)-\nabla f(x^*)\right)(x_0-x^*)\right\|dt+\|A-\nabla f(x^*)\|\|x^*-x_0\|\\
&\le\|A-\nabla f(x^*)\|\|x_0-x^*\|+\|A-\nabla f(x^*)\|\|x_0-x^*\|\\
&=2\|A-\nabla f(x^*)\|\|x_0-x^*\|\\
&\le 2\sigma\|A-\nabla f(x^*)\|\le\beta.
\end{aligned}$$

Now from Lemma 3.1, there exists $x_1\in P_{x_0}^{-1}(y)$, i.e. $y\in f(x_0)+A(x_1-x_0)+F(x_1)$, such that

$$\begin{aligned}
\|x_1-x^*\|&\le M^{'}\|y+f(x^*)-f(x_0)-A(x^*-x_0)\|\\
&\le M^{'}(\|y\|+\|f(x_0)-f(x^*)-\nabla f(x^*)(x_0-x^*)\|)\\
&\qquad+M^{'}\|A-\nabla f(x^*)\|\|x^*-x_0\|\\
&\le M^{'}(\|y\|+\|A-\nabla f(x^*)\|\|x^*-x_0\|+\|A-\nabla f(x^*)\|\|x^*-x_0\|)\\
&\le M^{'}(\|y\|+2\sigma\|A-\nabla f(x^*)\|)\\
&\le M^{'}b+2M^{'}\sigma\|A-\nabla f(x^*)\|\le\alpha.
\end{aligned}$$

Then we have that

$$\begin{aligned}
\|x_1-x_0\|&\le\|x_1-x^*\|+\|x^*-x_0\|\\
&\le M^{'}b+2M^{'}\sigma\|A-\nabla f(x^*)\|+\sigma\\
&=M^{'}b+(2M^{'}\|A-\nabla f(x^*)\|+1)\sigma.
\end{aligned}$$

Note that $x_1\in P_{x_1}^{-1}\left(y+f(x_1)-f(x_0)-A(x_1-x_0)\right)\cap B_\alpha(x^*)$, and

$$\begin{aligned}
&\|y+f(x_1)-f(x_0)-A(x_1-x_0)\|\\
&\le\|y\|+\|f(x_1)-f(x_0)-\nabla f(x_0)(x_1-x_0)\|+(\|\nabla f(x_0)-\nabla f(x^*)\|+\|\nabla f(x^*)-A\|)\|x_1-x_0\|\\
&\le\|y\|+\|\nabla f(x^*)-A\|\|x_1-x_0\|+(\|\nabla f(x_0)-\nabla f(x^*)\|+\|\nabla f(x^*)-A\|)\|x_1-x_0\|\\
&\le\|y\|+\|\nabla f(x^*)-A\|\|x_1-x_0\|\\
&\qquad+(\|\nabla f(x^*)-A\|+\|\nabla f(x^*)-A\|)\|x_1-x_0\|\\
&\le b+3\|\nabla f(x^*)-A\|\|x_1-x_0\|\\
&\le b+3\|\nabla f(x^*)-A\|(M^{'}b+(2M^{'}\|A-\nabla f(x^*)\|+1)\sigma)\le\beta,
\end{aligned}$$

where the last inequality holds by third relation in (3.15). Then, from Lemma 3.1 there exists $x_2\in P_{x_1}^{-1}(y)$ i.e. $y\in f(x_1)+A(x_2-x_1)+F(x_2)$ such that

$$\begin{aligned}
\|x_2-x_1\|&\le M^{'}\|f(x_1)-f(x_0)-A(x_1-x_0)\|\\
&\le M^{'}(\|f(x_1)-f(x_0)-\nabla f(x_0)(x_1-x_0)\|)\\
&\qquad+M^{'}(\|\nabla f(x_0)-\nabla f(x^*)\|+\|\nabla f(x^*)-A\|)\|x_1-x_0\|\\
&\le M^{'}\|\nabla f(x^*)-A\|\|x_1-x_0\|\\
&\qquad+M^{'}(\|\nabla f(x^*)-A\|+\|\nabla f(x^*)-A\|)\|x_1-x_0\|\\
&=3M^{'}\|\nabla f(x^*)-A\|\|x_1-x_0\|.
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
\|x_2-x^*\|&\le\|x_2-x_1\|+\|x_1-x_0\|+\|x_0-x^*\|\\
&\le(1+3M^{'}\|\nabla f(x^*)-A\|)\|x_1-x_0\|+\|x_0-x^*\|\\
&\le(1+3M^{'}\|\nabla f(x^*)-A\|)(M^{'}b+(2M^{'}\|A-\nabla f(x^*)\|+1)\sigma)+\sigma\\
&\le M^{'}b(1+3M^{'}\|A-\nabla f(x^*)\|)+(M^{'}(5+6M^{'}\|A-\nabla f(x^*)\|)\|A-\nabla f(x^*)\|+2)\sigma\le\alpha.
\end{aligned}\tag{3.16}$$

Proceeding by induction, suppose that there exists integer $n\ge2$ and points $x_2,x_3,x_4,\ldots,x_n$ are obtained by the Chord method defined by (1.3), we get, $x_k\in P_{x_{k-1}}(y)$, i.e. $y\in f(x_{k-1})+A(x_k-x_{k-1})+F(x_k)$, such that $\|x_k-x_{k-1}\|\le(3M^{'}\|\nabla f(x^*)-A\|)^{k-1}\|x_1-x_0\|$, $k=2,3,4,\ldots\ldots,n$.

Then, repeating the argument in (3.16) we obtain that $x_n\in B_\alpha(x^*)$. Moreover, we have that

$$\|y+f(x_n)-f(x_{n-1})-A(x_n-x_{n-1})\|$$

$\le \|y\| + \|f(x_n) - f(x_{n-1}) - \nabla f(x_{n-1})(x_n - x_{n-1})\|$
$+(\|\nabla f(x_{n-1}) - \nabla f(x^*)\| + \|\nabla f(x^*) - A\|)\|x_n - x_{n-1}\|$
$\le b + \|f(x_n) - f(x_{n-1}) - \nabla f(x_{n-1})(x_n - x_{n-1})\| + (\|\nabla f(x^*) - A\| + \|\nabla f(x^*) - A\|)\|x_n - x_{n-1}\|$
$\le b + \|\nabla f(x^*) - A\|\|x_n - x_{n-1}\| + 2\|\nabla f(x^*) - A\|\|x_n - x_{n-1}\|$
$= b + 3\|\nabla f(x^*) - A\|\|x_n - x_{n-1}\|$
$\le b + (3\|\nabla f(x^*) - A\|)^n\|x_1 - x_0\|$
$\le b + 3\|\nabla f(x^*) - A\|\|x_1 - x_0\| \le \beta.$

Then from $x_n \in P_{x_n}^{-1}\big(y + f(x_n) - f(x_{n-1}) - A(x_n - x_{n-1})\big) \cap B_\alpha(x^*)$, and from Lemma 3.1, we conclude that there exists a chord iterate

$$x_{n+1} \in P_{x_n}^{-1}(y), \tag{3.17}$$

satisfying $\|x_{n+1} - x_n\| \le 3M'\|\nabla f(x^*) - A\|\|x_n - x_{n-1}\|.$ (3.18)
$\le (3M'\|\nabla f(x^*) - A\|)^n\|x_1 - x_0\|.$

This completes the induction step. Hence, there exists a sequence $\{x_n\}$ which is Cauchy sequence, and passing to the limit in (3.17), we obtain that $\{x_n\}$ is geometrically convergent to solution $x \in P_x^{-1}(y)$. Put $c > 3M'\|\nabla f(x^*) - A\|$. Now, if we pass to the limit in (3.18), we obtain that $\|x_{n+1} - x\| \le c\|x_n - x\|$. This completes the proof of theorem.

3.2 Uniformity of Linear Convergence of Second Kind

This section provides the uniformity of linear convergence of the Chord method defined by (1.3) in different way.

Theorem 3.2 Let x^* be a solution of (1.1) for $y = 0$. Suppose that $P_{x^*}^{-1}(\cdot)$ is pseudo-Lipschitz around $(0, x^*)$ with constants a, b and M. Let $x \in B_{\frac{a}{4}}(x^*)$ and $A \in L(X, Y)$. Suppose that ∇f is continuous on $B_{\frac{a}{4}}(x^*)$ with constant $\|A - \nabla f(x^*)\|$ such that $3M'\|A - \nabla f(x^*)\| < 1$, i.e.

$$\sup_{x \in B_{\frac{a}{4}}(x^*)} \|\nabla f(x) - \nabla f(x^*)\| \le \|A - \nabla f(x^*)\| \le \min\left\{\frac{b}{2a}, \frac{1}{M}\right\}.$$

Then there exists positive constants σ and γ such that for every $y \in B_b(0)$ and for every $x_0 \in B_\sigma(x^*)$ there exists a sequence $\{x_k\}$ generated by (1.3) with initial point x_0, which is convergent to a solution x of (1.1) for y, i.e.

$$\|x_{k+1} - x\| \le \gamma\|x_k - x\|.$$

Proof: Since $P_x^{-1}(\cdot)$ is pseudo-Lipschitz around $(0, x^*)$ with constant M. Then there exist positive constants a and b such that

$$e\left(P_{x^*}^{-1}(y_1) \cap B_a(x^*), P_{x^*}^{-1}(y_2)\right) \le M\|y_1 - y_2\|, \text{for each } y_1, y_2 \in B_b(0).$$

Let $x_0 \in B_\sigma(x^*)$, $y \in B_b(0)$ and r be defined in (4.4). Put $\alpha = \frac{a}{2}$, $\beta = \bar{r}$ and $M' = \frac{M}{1 - M\|A - \nabla f(x^*)\|}$ are the constants in Lemma 3.1. Then

$$e\left(P_x^{-1}(y_1) \cap B_{\frac{\alpha}{2}}(x^*), P_x^{-1}(y_2)\right) \le M\|y_1 - y_2\|, \text{for each } y_1, y_2 \in B_\beta(0).$$

To prove this theorem, we use Lemma 3.1. Set

$$\sigma \le \min\left\{\frac{\alpha}{2}, \frac{\alpha\left(1 - (3M'\|\nabla f(x^*) - A\|)^2\right)}{2(3M'\|\nabla f(x^*) - A\|)^2}, \frac{\beta - b}{M'(3\|\nabla f(x^*) - A\|)^2} - \frac{\alpha}{2}\right\}. \tag{3.19}$$

Then $\|x - x_0\| \le \|x - x^*\| + \|x^* - x_0\| \le \frac{\alpha}{2} + \sigma$. Note that

$$x \in P_{x_0}^{-1}\big(y - f(x) + f(x_0) + A(x - x_0)\big) \cap B_{\frac{\alpha}{2}}(x^*), \text{ and}$$

$\|y - f(x) + f(x_0) + A(x - x_0)\|$
$\le \|y\| + \|f(x) - f(x_0) - \nabla f(x_0)(x - x_0)\| + (\|\nabla f(x_0) - \nabla f(x^*)\| + \|\nabla f(x^*) - A\|)\|x - x_0\|$
$\le b + \int_0^1 \left\|\big(\nabla f(x_0 + t(x - x_0)) - \nabla f(x_0)(x - x_0)\big)\right\| dt$
$+(\|\nabla f(x_0) - \nabla f(x^*)\| + \|\nabla f(x^*) - A\|)\|x - x_0\|$
$\le b + \|\nabla f(x^*) - A\|\|x - x_0\| + (\|\nabla f(x^*) - A\| + \|\nabla f(x^*) - A\|)\|x - x_0\|$
$= b + \|\nabla f(x^*) - A\|\|x - x_0\| + 2\|\nabla f(x^*) - A\|\|x - x_0\|$
$= b + 3\|\nabla f(x^*) - A\|\|x - x_0\|$
$\le b + 3\|\nabla f(x^*) - A\|\left(\frac{\alpha}{2} + \sigma\right) \le \beta.$

Now from Lemma 3.1, there exists $x_1 \in P_{x_0}^{-1}(y)$, i.e. $y \in f(x_0) + A(x_1 - x_0) + F(x_1)$, such that $\|x - x_1\| \le M'\|f(x) - f(x_0) - A(x - x_0)\|$
$\le M'(\|f(x) - f(x_0) - \nabla f(x_0)(x - x_0)\|)$

$$
\begin{aligned}
&+M'(\|\nabla f(x_0)-\nabla f(x^*)\|+\|\nabla f(x^*)-A\|)\|x-x_0\| \\
&\le M'\|\nabla f(x^*)-A\|\|x-x_0\| \\
&\quad +M'(\|\nabla f(x_0)-\nabla f(x^*)\|+\|\nabla f(x^*)-A\|)\|x-x_0\| \\
&\le M'\|\nabla f(x^*)-A\|\|x-x_0\| \\
&\quad +M'(\|\nabla f(x^*)-A\|+\|\nabla f(x^*)-A\|)\|x-x_0\| \\
&= M'\|\nabla f(x^*)-A\|\|x-x_0\|+2M'\|\nabla f(x^*)-A\|\|x-x_0\| \\
&= 3M'\|\nabla f(x^*)-A\|\|x-x_0\|. \qquad (3.20)
\end{aligned}
$$

Hence we have that $\|x_1-x^*\| \le \|x_1-x\|+\|x-x^*\|$

$$
\begin{aligned}
&\le 3M'\|\nabla f(x^*)-A\|\|x-x_0\|+\|x-x^*\| \\
&\le 3M'\|\nabla f(x^*)-A\|\left(\tfrac{\alpha}{2}+\sigma\right)+\tfrac{\alpha}{2}\le\alpha.
\end{aligned}
$$

Furthermore, $x_1 \in P_{x_1}^{-1}\big(y-f(x)+f(x_1)+A(x-x_1)\big)\cap B_{\frac{\alpha}{2}}(x^*)$, and from (3.20), we get

$$\|x-x_1\|\le 3M'\|\nabla f(x^*)-A\|\left(\frac{\alpha}{2}+\sigma\right)\le\alpha.$$

Therefore, we have that

$$
\begin{aligned}
&\|y-f(x)+f(x_1)+A(x-x_1)\| \\
&\le \|y\|+\|f(x)-f(x_1)-\nabla f(x_1)(x-x_1)\| \\
&\quad +(\|\nabla f(x^*)-\nabla f(x_1)\|+\|\nabla f(x^*)-A\|)\|x-x_1\| \\
&\le b+\|\nabla f(x^*)-A\|\|x-x_1\| \\
&\quad +(\|\nabla f(x^*)-\nabla f(x_1)\|+\|\nabla f(x^*)-A\|)\|x-x_1\| \\
&\le b+\|\nabla f(x^*)-A\|\|x-x_1\| \\
&\quad +(\|\nabla f(x^*)-A\|+\|\nabla f(x^*)-A\|)\|x-x_1\| \\
&= b+3\|\nabla f(x^*)-A\|\|x-x_1\| \\
&\le b+(3\|\nabla f(x^*)-A\|)(3M'\|\nabla f(x^*)-A\|)\left(\tfrac{\alpha}{2}+\sigma\right) \\
&= b+M'(3\|\nabla f(x^*)-A\|)^2\left(\tfrac{\alpha}{2}+\sigma\right)\le\beta,
\end{aligned}
$$

where the last inequality holds by third relation in (3.19). It follows from Lemma 3.1 that there exists $x_2 \in P_{x_1}^{-1}(y)$ with $y\in f(x_1)+A(x_2-x_1)+F(x_2)$, such that

$$
\begin{aligned}
\|x_2-x\| &\le M'\|f(x)-f(x_1)-A(x-x_1)\| \\
&\le M'\|f(x)-f(x_1)-\nabla f(x_1)(x-x_1)\| \\
&\quad +M'(\|\nabla f(x_1)-\nabla f(x^*)\|+\|\nabla f(x^*)-A\|)\|x-x_1\| \\
&\le M'\|\nabla f(x^*)-A\|\|x-x_1\| \\
&\quad +M'(\|\nabla f(x^*)-A\|+\|\nabla f(x^*)-A\|)\|x-x_1\| \\
&= M'(\|\nabla f(x^*)-A\|+2\|\nabla f(x^*)-A\|)\|x-x_1\| \\
&= 3M'\|\nabla f(x^*)-A\|\|x-x_1\|.
\end{aligned}
$$

Moreover, we have that

$$
\begin{aligned}
\|x_2-x^*\| &\le \|x_2-x\|+\|x-x^*\| \\
&\le 3M'\|\nabla f(x^*)-A\|\|x-x_1\|+\tfrac{\alpha}{2} \\
&\le (3M'\|\nabla f(x^*)-A\|)\left(3M'\|\nabla f(x^*)-A\|\left(\tfrac{\alpha}{2}+\sigma\right)\right)+\tfrac{\alpha}{2} \\
&= (3M'\|\nabla f(x^*)-A\|)^2\left(\frac{\alpha}{2}+\sigma\right)+\frac{\alpha}{2}\le\alpha.
\end{aligned}
$$

Proceeding by induction, suppose that there exists integer $k>2$ and points $x_2,x_3,x_4,\ldots,x_k$ are obtained by the chord method (1.3), we get, $x_k\in P_{x_{k-1}}(y)$, i.e.

$$y\in f(x_{k-1})+A(x_k-x_{k-1})+F(x_k),$$

such that

$$
\begin{aligned}
\|x_k-x\| &\le M'\|f(x)-f(x_{k-1})-A(x-x_{k-1})\| \\
&\le M'\|f(x)-f(x_{k-1})-\nabla f(x_{k-1})(x-x_{k-1})\| \\
&\quad +M'(\|\nabla f(x_{k-1})-\nabla f(x^*)\|+\|\nabla f(x^*)-A\|)\|x-x_{k-1}\| \\
&\le M'\|\nabla f(x^*)-A\|\|x-x_{k-1}\|+M'(\|\nabla f(x^*)-A\|+\|\nabla f(x^*)-A\|)\|x-x_{k-1}\| \\
&= 3M'\|\nabla f(x^*)-A\|\|x-x_{k-1}\|.
\end{aligned}
$$

This implies that the theorem is true for x_k. To complete this theorem, it remains to prove that the induction is true for x_{k+1}. Moreover, we have that

$$x_k\in P_{x_k}^{-1}\big(y-f(x)+f(x_k)+A(x-x_k)\big)\cap B_{\frac{\alpha}{2}}(x^*).$$

Then, we have that

$$
\begin{aligned}
&\|y-f(x)+f(x_k)+A(x-x_k)\| \\
&\le \|y\|+\|f(x)-f(x_k)-\nabla f(x_k)(x-x_k)\|+(\|\nabla f(x^*)-\nabla f(x_k)\|+\|\nabla f(x^*)-A\|)\|x-x_k\| \\
&\le b+\|\nabla f(x^*)-A\|\|x-x_k\|
\end{aligned}
$$

$+(\|\nabla f(x^*) - \nabla f(x_k)\| + \|\nabla f(x^*) - A\|)\|x - x_k\|$
$\le b + \|\nabla f(x^*) - A\|\|x - x_k\| + (\|\nabla f(x^*) - A\| + \|\nabla f(x^*) - A\|)\|x - x_k\|$
$= b + 3\|\nabla f(x^*) - A\|\|x - x_k\|$
$\le b + 3\|\nabla f(x^*) - A\|(3M'\|\nabla f(x^*) - A\|\|x - x_{k-1}\|)$
$\le b + M'(3\|\nabla f(x^*) - A\|)^2\|x - x_{k-1}\| \le \beta.$
Now, from Lemma 3.1, there exists $x_{k+1} \in P_{x_k}^{-1}(y)$, i.e. $y \in f(x_k) + A(x_{k+1} - x_k) + F(x_{k+1})$, such that
$\|x_{k+1} - x\| \le M'\|f(x) - f(x_k) - A(x - x_k)\|$
$\le M'\|f(x) - f(x_k) - \nabla f(x_k)(x - x_k)\|$
$+M'(\|\nabla f(x_k) - \nabla f(x^*)\| + \|\nabla f(x^*) - A\|)\|x - x_k\|$
$\le M'\|\nabla f(x^*) - A\|\|x - x_k\|$
$+M'(\|\nabla f(x^*) - A\| + \|\nabla f(x^*) - A\|)\|x - x_k\|$
$\le M'(\|\nabla f(x^*) - A\| + 2\|\nabla f(x^*) - A\|)\|x - x_k\|$
$= 3M'\|\nabla f(x^*) - A\|\|x - x_k\|.$
Taking $\gamma = 3M'\|\nabla f(x^*) - A\|$. Then from the last inequality, we have that

$$\|x_{k+1} - x\| \le \gamma\|x - x_k\|.$$

This completes the proof.

IV. Concluding Remarks

In this study, we have established the uniformity of the local convergence results for the Chord method defined by (1.3) under the assumptions that $P_{x^*}^{-1}(\cdot)$ is pseudo-Lipschitz and ∇f is continuous. In particular, the uniformity of the convergence results of the Chord method defined by (1.3) are presented in two different ways when $P_{x^*}^{-1}(\cdot)$ is pseudo-Lipschitz and ∇f is continuous. These results improve and extend the corresponding one [4].

References

[1]. S.M. Robinson, Generalized equations and their solutions, Part I, Basic theory, Math. Program. Stud., 10, 1979, 128–141.
[2]. S.M. Robinson, Generalized equations and their solutions, part II: Application to nonlinear programming, Math. Program. Stud., 19, 1982, 200–221.
[3]. M.C. Ferris and J.S. Pang, Engineering and economic applications of complementarily problems, SIAM Rev., 39, 1997, 669–713.
[4]. A. L. Dontchev, Uniform convergence of the Newton method for Aubin continuous maps, Serdica Math. J., 22 , 1996, 385–398.
[5]. A. Pietrus, Does Newton's method for set-valued maps converge uniformly in mild differentiability context?, Rev. Columbiana Mat., 34, 2000, 49–56.
[6]. M.H. Geoffroy and A. Pietrus, A general iterative procedure for solving nonsmooth generalized equations, Comput. Optim. Appl., 31(1), 2005, 57–67.
[7]. M. H. Rashid, S. H. Yu, C. Li and S. Y. Wu, Convergence Analysis of the Gauss-Newton Method for Lipschitz--like Mappings, J. Optim. Theory Appl., 158(1), 2013, 216–-233.
[8]. R.T. Marinov, Convergence of the method of chords for solving generalized equations, Rendiconti del Circolo Matematico di Palermo, 58, 2009, 11–-27.
[9]. A. L. Dontchev and W. W. Hager, An inverse mapping theorem for set-valued maps, Proc. Amer. Math. Soc., 121, 1994, 481–-489.
[10]. C. L. Evans, Partial Differential Equations (American Mathematics Society, Providence, 1998).
[11]. K. A. Atkinson: An introduction to Numerical Analysis (second edition), American Mathematics Society, Providence, 1988
[12]. A. Basak, A study on Chord methods for generalized equations, Master Thesis, Rajshahi University, 2013.