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$i(G)$-Graph - $G(i)$ Of Some Special Graphs

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Abstract: *The $i(G)$-graph is defined as a graph whose vertex set correspond 1 to 1 with the $i(G)$-sets of G. Two $i(G)$- sets say S_1 and S_2 are adjacent in $i(G)$ if there exists a vertex $v \in S_1$, and a vertex $w \in S_2$ such that v is adjacent to w and $S_1 = S_2 - \{w\} \cup \{v\}$ or equivalently $S_2 = S_1 - \{v\} \cup \{w\}$. In this paper we obtain $i(G)$-graph of some special graphs.*

I. Introduction

By a graph we mean a finite, undirected, connected graph without loops and multiple edges. For graph theoretical terms we refer Harary [12] and for terms related to domination we refer Haynes et al. [14].

A set $S \subseteq V$ is said to be a dominating set in G if every vertex in $V - S$ is adjacent to some vertex in S. The domination number of G is the minimum cardinality taken over all dominating sets of G and is denoted by $\gamma(G)$. A subset S of the vertex set in a graph G is said to be independent if no two vertices in S are adjacent in G. The maximum number of vertices in an independent set of G is called the independence number of G and is denoted by $\beta_0(G)$. Any vertex which is adjacent to a pendent vertex is called a support. A vertex whose degree is not equal to one is called a non-pendent vertex and a vertex whose degree is $p-1$ is called a universal vertex. Let u and v be (not necessarily distinct) vertices of a graph G. A $u-v$ walk of G is a finite, alternating sequence $u = u_0, e_1, e_2, \ldots, e_n, u_n = v$ of vertices and edges beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}, u_i, i = 1,2,3,\ldots,n$. The number n is called the length of the walk. A walk in which all the vertices are distinct is called a path. A closed walk $(u_0, u_1, u_2, \ldots, u_n)$ in which $u_0, u_1, u_2, \ldots, u_n$ are distinct is called a cycle. A path on p vertices is denoted by P_p and a cycle on p vertices is denoted by C_p.

Gerd H.Frickle et. al [11] introduced γ-graph.The γ-graph of a graph G denoted by $G(\gamma) = (V(\gamma), E(\gamma))$ is the graph whose vertex set corresponds $1-\text{to}-1$ with the γ-sets. Two γ-sets say S_1 and S_2 are adjacent in $E(\gamma)$ if there exist a vertex $v \in S_1$ and a vertex $w \in S_2$ such that v is adjacent to w and $S_1 = S_2 - \{w\} \cup \{u\}$ or equivalently $S_2 = S_1 - \{u\} \cup \{w\}$. Elizabeth et.al [10] proved that all graphs of order $n \le 5$ have connected γ-graphs and determined all graphs G on six vertices for which $G(\gamma)$ is connected. We impose an additional condition namely independency on γ-sets and study $i(G)$-graphs denoted by $G(i)$. The $i(G)$-graph is defined as a graph whose vertex set correspond 1 to 1 with the $i(G)$-sets of G. Two $i(G)$- sets say S_1 and S_2 are adjacent in $i(G)$ if there exists a vertex $v \in S_1$, and a vertex $w \in S_2$ such that v is adjacent to w and $S_1 = S_2 - \{w\} \cup \{v\}$ or equivalently $S_2 = S_1 - \{v\} \cup \{w\}$. In this paper we obtain $i(G)$-graph of some special graphs.

II. Main Results

Definition 2.1 A set $S \subset V$ is said to the independent if no two vertices in S are adjacent. The minimum cardinality of a maximal independent dominating set is called the independent domination number and is denoted by $i(G)$. A maximal independent dominating set is called a $i(G)-$ set.

Definition 2.2 Consider the family of all independent dominating sets of a graph G and define the graph

$G(i)=(V(i),E(i))$ to be the graph whose vertices $V(i)$ correspond 1-1 with independent dominating sets of G and two sets S_1 and S_2 are adjacent ib $G(i)$ if there exists a vertex $v \in S_1$, and $w \in S_2$ such that (i) v is adjacent to w and (ii) $S_1 = S_2 - \{w\} \cup \{v\}$ and $S_2 = S_1 - \{v\} \cup \{w\}$.

Proposition 2.3 If a graph G has a unique $i(G)$-set then $G(i) \cong K_1$ and conversely.

Corollary 2.4 $K_{1,n}(i) = K_1$.

Proof. Since the central vertex of $K_{1,n}$ is the only $i(G)$-set, $K_{1,n}(i) = K_1$.

Proposition 2.5 $\overline{K_n}(i) \cong K_1$, whereas $K_n(i) \cong K_n$.

Proof. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of K_n. Each singleton set $S_i = \{v_i\}, i = 1,2,3,\ldots,n$ is an element of $V(i)$ and each pair $(S_i, S_j), (1 \le i, j \le n)$ form an edge in $K_n(i)$. Hence $K_n(i) \cong K_n$. Since the set of all vertices of $\overline{K_n}$ is the only independent dominating set of $\overline{K_n}, \overline{K_n}(i) \cong K_1$.

Proposition 2.6 For $1 \le m \le n$,

$$K_{m,n}(i) \cong \begin{cases} K_2 & \text{ifm = n = 1} \\ \overline{K_2} & \text{ifm = nandm} \ge 2 \\ K_1 & \text{ifm < n} \end{cases}$$

Proof. Let $S_1 = \{u_1, u_2, u_3, \ldots, u_m\}$ and $S_2 = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of $K_{m,n}$.
If $m = n = 1$, $\{u_1\}$ and $\{v_1\}$ are the $i(G)$ sets and clearly $K_{m,n}(i) = K_2$.
If $m = n$ and $m \ge 2$, S_1 and S_2 are the only two independent dominating sets of $K_{m,n}$ and they are non-adjacent vertices of $K_{m,n}(i)$. Hence $K_{m,n}(i) = \overline{K_2}$ for all values of m. If $m < n, S_1$ is the only $i(G)$-set and so $K_{m,n}(i) \cong K_1$.

Proposition 2.7 $C_{3k+2}(i) \cong C_{3k+2}$.

Proof. **Case(i).** $k = 1$
Let the cycle be $(v_1, v_2, v_3, v_4, v_5, v_1)$.
$S_1 = \{v_1, v_3\}, S_2 = \{v_1, v_4\}, S_3 = \{v_2, v_4\}, S_4 = \{v_2, v_5\}, S_5 = \{v_3, v_5\}$ are the 5 $i(G)$-sets of C_5 and $C_5(i)$ is the cycle $(S_1, S_2, S_3, S_4, S_5, S_1)$.

Case(ii). $k = 2$
Let the cycle be $(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1)$.
$S_1 = \{v_1, v_4, v_7\}, S_2 = \{v_1, v_4, v_6\}, S_3 = \{v_1, v_3, v_6\}, S_4 = \{v_2, v_5, v_8\}, S_5 = \{v_2, v_5, v_7\}, S_6 = \{v_2, v_4, v_7\}, S_7 = \{v_3, v_6, v_8\}$ are the 8 $i(G)$-sets of C_8 and $C_8(i)$ is the cycle $(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_1)$.

Case(iii). Let the vertices of the cycle be $(v_1, v_2, v_3, \ldots, v_{3k+2}, v_1)$
We know that $i(C_{3k+2}) = k+1$. $S_1 = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3k-2}, v_{3k+1}\}$ and $S_2 = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3k-2}, v_{3k}\}$ are two $i(G)$-sets of C_{3k+2}.
Now finding the first vertex of S_1 and changing the other vertices of S_1 we get $S_3 = \{v_1, v_3, v_6, v_9, \ldots, v_{3k-3}, v_{3k}\}$. Now fixing the first two vertices of S_1 and changing the other vertices of S_1 we get $S_4 = \{v_1, v_4, v_6, v_9, \ldots, v_{3k-3}, v_{3k}\}$. Proceeding like this, fixing the first $k-1$ vertices and changing the k^{th} vertex alone we get $S_{k+1} = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3k-5}, v_{3k-3}, v_{3k}\}$. Now consider the two

$i(G)$-sets $S_{k+2} = \{v_2, v_5, v_8, \ldots, v_{3k-1}, v_{3k+2}\}$ and $S_{k+3} = \{v_2, v_5, v_8, \ldots, v_{3k-4}, v_{3k-1}, v_{3k+1}\}$.
As before, fixing the first vertex and changing from the $2^{nd}, 3^{rd}$ vertices upto k^{th} of S_{k+3} we get

$S_{k+4} = \{v_2, v_4, v_7, v_{10}, \ldots, v_{3k-5}, v_{3k-2}, v_{3k+1}\}$,

$S_{k+5} = \{v_2, v_4, v_7, v_{10}, \ldots, v_{3k-5}, v_{3k-2}, v_{3k+1}\}$,

$\vdots$

$S_{2k+2} = \{v_2, v_5, v_8, v_{11}, \ldots, v_{3k-4}, v_{3k-2}, v_{3k+1}\}$.

Now consider $S_{2k+3} = \{v_3, v_6, v_9, \ldots, v_{3k}, v_{3k+2}\}$. As before, fixing the first vertex and changing from the $2^{nd}, 3^{rd}, 4^{th}$ vertices of S_{2k+3} we get

$S_{2k+4} = \{v_3, v_5, v_8, v_{11}, \ldots, v_{3k-1}, v_{3k+2}\}$

$S_{2k+5} = \{v_3, v_5, v_8, v_{11}, \ldots, v_{3k-1}, v_{3k+2}\}$

$\vdots$

$S_{3k+2} = \{v_3, v_6, v_9, v_{12}, \ldots, v_{3k-3}, v_{3k-1}, v_{3k+2}\}$.

Now $S_1, S_2, S_3, \ldots, S_{3k+2}$ are $i(G)$-sets of C_{3k+2}. Here S_1 is adjacent S_2 and S_{k+4}. $S_2, S_3, S_4, \ldots S_k$ are adjacent to preceeding and succeeding vertices. S_{k+1} is adjacent to S_2 and S_k. S_{k+2} is adjacent to S_{k+3} and S_{2k+4}. S_{k+3} is adjacent to S_{k+2} and S_{2k+2}. S_{k+4} is adjacent to S_{k+5} and S_1. $S_{k+5}, S_{k+6}, S_{k+7}, \ldots, S_{2k+1}$ are adjacent to the preceeding and suceeding vertices. S_{2k+2} is adjacent to S_{2k+1} and S_{k+3}. S_{2k+3} is adjacent to S_3 and S_{3k+2}. S_{2k+4} is adjacent to S_{k+2} and S_{2k+5}. $S_{2k+5}, S_{2k+6}, S_{2k+7}, \ldots, S_{3k+1}$ are adjacent to the preceeding and suceeding vertices. S_{3k+2} is adjacent to S_{3k+1} and S_{2k+3}.

Thus we get a cycle $(S_1, S_2, S_{k+1}, S_k, S_{k_1}, \ldots S_3, S_{2k+3}, S_{3k+2}, S_{3k-1}, S_{3k}, \ldots S_{2k+4}, S_{k+2}, S_{k+3}, S_{2k+2}, S_{2k+1}, S_{2k} \ldots S_{k+4}, S_1)$ which is isomorphic to C_{3k+2}.

Proposition 2.8 For $k \geq 2, C_{3k}(i) \cong \overline{K_3}$

Proof. Since each C_{3k} for $k \geq 3$ has 3 disjoint $i(G)$-sets, $C_{3k}(i) \cong \overline{K_3}$.

Proposition 2.9 $P_{3k}(i) \cong \overline{K_1}$

Proof. Since paths P_{3k} of order $3k$ have a unique $i(G)$-set, $P_{3k}(i) \cong \overline{K_1}$.

Proposition 2.10 $P_{3k+2}(i) \cong P_{k+2}$

Proof. Let $v_1, v_2, v_3, \ldots, v_{3k+2}$ be the vertices of P_{3k+2}. We have $i(P_{3k+2}) = k+1$. $S_1 = \{v_2, v_5, v_8, \ldots, v_{3k+2}\}, S_2 = \{v_2, v_5, v_8, \ldots, v_{3k-4}, v_{3k-1}, v_{3k+1}\}$ are two $i(G)$-sets of P_{3k+2}. Now fixing the first vertex and varying from the $2^{nd}, 3^{rd}, 4^{th}, \ldots k^{th}$ vertices we get the following $i(G)$-sets.

$S_3 = \{v_2, v_4, v_7, v_{10}, \ldots, v_{3k+1}\}$

$S_4 = \{v_2, v_4, v_7, v_{10}, \ldots, v_{3k+1}\}$

$\vdots$

$S_{k+1} = \{v_2, v_5, v_8, v_{11}, \ldots, v_{3k-1}, v_{3k+1}\}$

Also $S_{k+2} = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3k+1}\}$ is an $i(G)$-set of P_{3k+2}.

Thus there are $k+2$ $i(G)$-sets of P_{3k+2}. It is obvious that S_1 is adjacent to S_2 alone and S_{k+2} is adjacent to S_3 alone. S_{k+1} is adjcent to S_2 and S_k. $S_3, S_4, S_5, \ldots, S_k$ are adjacent to the preceeding and succeeding vertices. Thus we get a path of length P_{k+2}. Hence $P_{3k+2}(i) \cong P_{k+2}$.

Definition 2.11 Grid graph is the cartesian product of 2 paths.
The cartesian product of 2 paths P_m and P_n is denoted by $P_m \mathrm{W} P_n$ or $P_m \times P_n$.

Proposition 2.12 For $k \geq 2, (P_2 \mathrm{W} P_{2k+1})(i) \cong \overline{K_2}$.

Proof. $P_2 \mathrm{W} P_{2k+1}(i)$ for $k \geq 2$ has only two disjoint $i(G)$-sets. Therefore $(P_2 \mathrm{W} P_{2k+1})(i) \cong \overline{K_2}$.
The structure of $i(G)$-graphs of paths and cycles of order $3k+1$ can be determined. Assume that the vertices in each of these graphs have been labelled $1,2,3,\ldots,3k+1$. For $G = P_{3k+1}$ or $G = C_{3k+1}, S = \{1,4,7,\ldots,3k+1\}$ is a $i(G)$-set of size $k+1$. In each case, 1 and $3k+1$ have one external private neighbour while the other numbers of S have two non adjacent external private neighbours. So $S_1 - \{1\} \cup \{2\}$ and $S - \{3k+1\} \cup \{3k\}$ are $i(G)$-sets. Further if $S^{'}$ is an $i(G)$-set for $G = P_{3k+1}$ or $G = C_{3k+1}$ and vertex i has exactly one external private neighbour, $j = i+1$ or $j = k-1$, then $S^{'} = \{i\} \cup \{j\}$ is an $i(G)$-set. Let us refer to the process of changing from a $i(G)$-set $S^{'}$ to the γ-set $S^{'} - \{i\} \cup \{j\}$ as a swap. We see that each swap defines an edge in $G(i)$.

Definition 2.13 We define a step grid $SG(k)$ to be the induced subgraph of the $k \times k$ grid graph $P_k \mathrm{W} P_k$ that is defined as follows:
$SG(k) = (V(K), E(K))$ where $V(K) = \{(i,j) : 1 \leq i, j \leq k, i+j \leq k+2\}$ and $E(K) = \{(i,j),(i^{'},j^{'}) : i = i, j^{'} = j+1, i^{'} = i+1, j = j\}$.

Theorem 2.14 If $G = P_{3k+1}$ or $G = C_{3k+1}$ then $G(i)$ is connected.
Proof. Each independent dominating set X of P_{3k+1} is some number of swaps of sets of type 1 $(X - \{i\} \cup \{i+1\})$ or sets of type 2 $(X - \{i\} \cup \{i-1\})$ from S. Alternatively we can perform swaps from S to X. Thus each vertex in $P_{3k+1}(i)$ can be associated with an ordered pair (i,j) where i is the number of swaps of type 2 needed to convert S to X. Thus vertex 1 and $3k+1$ in P_{3k+1} can be swapped with at most one external private neighbour. However each vertex can be swapped at most once in either direction. Thus the conditions on the ordered pair (i,j) are $1 \leq i \leq k, 1 \leq j \leq k, i+j = 2$. If $q = i+1$ and $r = j+1$, we have $1 \leq q \leq k+1, 1 \leq r \leq k+1$ and $q + r \leq (k+1)+2$.
Thus every $i(G)$-set of $G = P_{3k+1}$ or $G = C_{3k+1}$ is some number of swaps from the $i(G)$-set $S = \{1,4,7,\ldots,3k+1\}$. Hence $G(i)$ is connected for these graphs.

Theorem 2.15 $P_{3k+1}(i)$ is isomorphic to a step grid of order k with 2 pendent edges where the pendent vertices correspond to the $i(G)$-sets $\{v_1, v_3, v_6, \ldots, v_k\}$ and $\{v_2, v_5, v_8, \ldots, v_{3k-1}, v_{3k+1}\}$.
Proof. We know that $i(P_{3k+1}) = k+1$. Consider the $i(G)$-set $S_1 = \{v_1, v_4, v_7, \ldots, v_{3k+1}\}$ of P_{3k+1}. Fixing the first vertex of S_1 and changing from the $2^{nd}, 3^{rd}, 4^{th}, \ldots, k^{th}$ vertex of S_1 we get,

$S_2 = \{v_1, v_3, v_6, v_9, \ldots, v_{3k}\}$
$S_3 = \{v_1, v_4, v_6, v_9, \ldots, v_{3k}\}$
$S_4 = \{v_1, v_4, v_7, v_9, \ldots, v_{3k}\}$
$\vdots$
$S_{k+1} = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3k}\}$.

Now consider $S_{k+2} = \{v_2, v_5, v_8, \ldots, v_{3k-1}, v_{3k+1}\}$. Now fixing the first and last vertices of S_{k+2} and changing the k^{th} vertex $(k-1)^{th}$ vertex alone, k^{th} vertex $(k-1)^{th}$ vertex (2 vertices),

$\ldots,2^{nd},3^{rd},\ldots,(k-1)^{th},k^{th}$ vertices, $(k-1)$ vertices, we get $(k-1)$ $i(G)$-sets. They are
$S_{k+3}=\{v_2,v_5,v_8,\ldots,v_{3k-5},v_{3k-2},v_{3k+1}\}$
$S_{k+4}=\{v_2,v_5,v_8,\ldots,v_{3k-4},v_{3k-2},v_{3k+1}\}$
$\vdots$
$S_{2k+1}=\{v_2,v_4,v_7,\ldots,v_{3k-5},v_{3k-2},v_{3k+1}\}$
Now fixing the first vertex of S_{k+2} and changing the remaining vertices including the last vertex as before we get $kC_2 i(G)$-sets. Let us denote these kC_2 $i(G)$ sets by (3). Thus the total number of $i(G)$-sets of $P_{3k+1}=2k+1+kC_2=2k+1+\dfrac{k(k-1)}{2}=\dfrac{k^2+3k+2}{2}$. Of these $\dfrac{k^2+3k+2}{2}$ $i(G)$ sets, S_1 gets deg 2, S_2 gets deg 1 and the remaining $(k-1)$ vertices of (1) get deg 3. S_{k+2} gets deg 1,remaining $(k-1)$ vertices of (2) get deg 3. Of the kC_2 vertices of $(3),(k-1)C_2$ get deg 4,remaining $[kC_2-(k-1)C_2=k-1]$ vertices get deg 2.

Thus these $\dfrac{k^2+3k+2}{2}$ vertices are connected in $P_{3k+1}(i)$ and they form the step grid of order k with 2 pendent vertices $\{v_1,v_3,v_6,v_9,\ldots,v_{3k}\}$ and $\{v_2,v_5,v_8,\ldots,v_{3k-1},v_{3k+1}\}$.

Theorem 2.16 For any triangle free graph G, $G(i)$ is triangle free.

Proof. Suppose $G(i)$ contains a traingle of 3 vertices corresponding to $i(G)$-sets S_1,S_2 and S_3. Since (S_1,S_2) corresponds to an edge in $G(i)$, $S_2=S_1-\{x\}\cup\{y\}$ for some $x,y\in V(G)$ such that $(x,y)\in E(G)$. Further since (S_2,S_3) corresponds to an edge in $G(i)$, $S_3=S_2-\{c\}\cup\{d\}$ for some $c,d\in V(G)$ such that $(c,d)\in E(G)$. However $S_3=S_2-\{c\}\cup\{d\}=S_1-\{x,c\}\cup\{y,d\}$. But since (S_2,S_3) corresponds to an edge in $G(i)$, $S_3=S_2-\{a\}\cup\{b\}$ for some $a,b\in V(G)$ such that $(a,b)\in E(G)$. Since S_3 is not two swap away from S_1, it must be the case $x=a,c=y$ and $b=d$. But this implies that $(x,y),(x,b)$ and (y,b) are edges in $E(G)$, a contradiction since G is traingle free. Thus for any traingle free graph G, there is no K_3 induced subgraph in $G(i)$.
Corollary 2.17 For any tree T, $T(i)$ is traingle free.

Theorem 2.18 For any tree T, $T(n)$ is C_n-free for any odd $n\geq 3$.

Proof. Suppose $T(i)$ contains a cycle C of $k\geq 3$ vertices where k is odd. Let x be the vertex in C and let S be the $i(G)$-set corresponding to the vertex x. Let y and z be the two vertices on C of distance $m=\dfrac{k-1}{2}$ swaps away from x with corresponding $i(G)$-sets S_1 and S_2. That is there is a path P_1 corresponding to a series of vertex swaps say x_1 for y_1, x_2 for $y_2,\ldots x_m$ for y_m so that $S_1=S-X\cup Y$ where $X=\{x_1,x_2,x_3,\ldots,x_m\}$ and $Y=\{y_1,y_2,y_3,\ldots,y_m\}$. Likewise there is a path P_2 corresponding to a series of vertex swaps say w_1 for z_1,w_2 for $z_2,\ldots,w_m$ for z_m so that $S_2=S-W\cup Z$ where $W=\{w_1,w_2,w_3,\ldots,w_m\}$ and $Z=\{z_1,z_2,z_3,\ldots,z_m\}$. However since $(y,z)\in E(T(i)),S_2=S_1-\{a\}\cup\{b\}$ for some $a,b\in V(T)$. Thus this must be the case that the set $X=W-\{w_j\}\cup\{x_j\}$ and $Y=Z-\{z_j\}\cup\{y_j\}$. This implies that $S_2=S_1-\{y_j\}\cup\{x_j\}$ and

$(x_j, y_j) \in E(T(i))$ for $1 \le j \le m$. Since x_j was swapped for y_j and x_k was swapped for y_k in P_1, we also know that $(x_j, y_j) \in E(T(i))$ and $(x_k, y_k) \in E(T(i))$. Now both x_j and y_j are in S_2. So there exists a swap x_l for y_i in P_2 such that $(x_l, y_i) \in E(T(i))$. However in path P_1, x_l was swapped for y_l and thus $(x_l, y_l) \in E(T(i))$. Similarly $y_l \in S_2$, so there exists some x_s so that in path P_2, x_s was swapped for y_l. We can continue to find the alternating path P_1 and P_2 swaps. But since m is finite , we reach a vertex y_q which swapped with x_j in P_2, thus creating a cycle in T and contradicting the fact that T is cycle-free. Hence $T(i)$ is free of odd cycle.

Theorem 2.19 Every tree T is the i-graph of some graph.

Proof. Let us prove the theorem by induction on the order n of a tree T. The trees $T = K_1$ and $T = K_2$ are the i-graphs of K_1 and K_2 respectively.
Let us assume that the theorem is true for all trees T of order at most n and let T' be a tree of order $n+1$. Let v be a leaf of T' with support u. $T' - v$ is a tree of order n. By induction we know that the tree $T' - v$ is the i-graph of some graph say G. Let $i(G) = k$ and $S_u = \{u_1, u_2, u_3, \ldots, u_k\}$ be the $i(G)$- set of G corresponding to the vertex u in $T' - v$.
Construct a new graph G' by attaching k leaves to the vertices in S_u say $S_u' = u_1', u_2', u_3', \ldots, u_k'$. Now add a new vertex x and join it to each of the vertices in S_u'. Finally attach a leaf y adjacent to x. Then every $i(G)$- set of the new graph G must either be of the form $S \cup \{x\}$ for any $i(G)$- set S in G or the one new $i(G)$- set $S_u \cup \{y\}$.
$S_u \cup \{x\}$ is adjacent to $S_u \cup \{y\}$ in the graph i-graph of G'. Also the vertex corresponding to the $i(G)$- set $S_u \cup \{y\}$ is adjacent only to the vertex corresponding to the $i(G)$- set $S_u \cup \{x\}$ and the $i(G)$- set $S_u \cup \{y\}$ corresponding to the vertex v in T'. Thus the i-graph of the graph G'' is isomorphic to the tree T'.

i-graph sequence: From a given graph we can construct the i-graph repeatedly that is $G \xrightarrow{i} G(i) \xrightarrow{i} G(i)(i)$ etc. We can also see that often the sequence ends with K_1. We can list some examples of the phenomenon.

(1). $K_{1,n} \xrightarrow{i} K_1$

(2). $C_{3k} \xrightarrow{i} \overline{K_3} \xrightarrow{i} K_1$

(3). $\overline{K_n} \xrightarrow{i} K_1$

(4). $P_4 \xrightarrow{i} P_3 \xrightarrow{i} K_i$

(5). $P_2 \mathrm{W} P_3 \xrightarrow{i} \overline{K_3} \xrightarrow{i} K_1$ the sequence can be infinite.

(6). $P_2 \mathrm{W} P_6 \xrightarrow{i} P_3 \cup P_4 \xrightarrow{i} K_1$ the sequence can be infinite.

(7). $P_2 \mathrm{W} P_{2k+1} \xrightarrow{i} \overline{K_2} \xrightarrow{i} K_1$ the sequence can be infinite.

Although all the i-graph sequences terminated after a small number of steps, for some graph the sequence can be infinite.
For example

1. $K_n \xrightarrow{i} K_n \xrightarrow{i} K_n \xrightarrow{i} \dots$
2. $C_{3k+2} \xrightarrow{i} C_{3K+2} \xrightarrow{i} C_{3k+2} \xrightarrow{i} \dots$
3. $P_3 W P_3 \xrightarrow{i} C_8 \xrightarrow{i} C_8 \xrightarrow{i} \dots$

Definition 2.20 Let us define a new class of graph as follows. These graphs are combinations of cycles and complete graphs. Consider C_k, the cycle on k vertices $(x_1, x_2, x_3, \dots, x_k)$. If k is odd, we replace each edge $(x_i, x_{(i+1)} (\text{modk})) \in E(C_k)$, $i \le i \le k$ with a complete graph of size n. That is we add vertices $a_1, a_2, a_3, \dots, a_{n-2}$ and all possible edges corresponding to these vertices and x_i and x_{i+1}. This is repeated for each of the original edges in C_k. If k is even, we replace one vertex x_1 with a complete graph K_n and add edges from X_k and K_2 to each vertex in the added K_n. Then for each of the edge $(x_i, x_{i+1} (\text{modk})$, $2 \le i \le k-1$ we make the same replacement as we did when k is odd. We call the graph formed in this manner as $K_n \circ C_k$. The graph $K_4 \circ C_3$ is given in fig 5.3.

(-3,-3.3483582)(6.4844894,4.3276935)
59.69716(1.9881554,-0.7504699)[linewidth=0.034,dimen=outer](3.3379886,2.2471151)(-0.042011347,0.46711504)
119.15412(8.365909,-2.2677374)[linewidth=0.034,dimen=outer](6.538801,2.2125077)(3.1588013,0.43250778)
179.7155(6.4535127,-2.8186145)[linewidth=0.034,dimen=outer](4.920255,-0.5112962)(1.5402552,-2.2912962)
[dotsize=0.12,dotangle=-2.0515552](1.5710514,-0.51132834)
[dotsize=0.12,dotangle=-2.0515552](4.8718,-0.5495155)
[dotsize=0.12,dotangle=-2.0515552](1.7256418,3.2455456)
[dotsize=0.12,dotangle=-2.0515552](0.062109467,0.38326332)
[dotsize=0.12,dotangle=-2.0515552](4.7872605,3.2359374)
[dotsize=0.12,dotangle=-2.0515552](6.4023414,0.27622235)
[dotsize=0.12,dotangle=-2.0515552](4.8716197,-2.2305865)
[dotsize=0.12,dotangle=-2.0515552](1.5687231,-2.252361)
[dotsize=0.12,dotangle=-2.0515552](3.255287,2.3702252)
[linewidth=0.042cm](1.7456291,3.2448297)(1.5524962,-0.470638)
[linewidth=0.042cm](0.082096644,0.38254735)(3.2345839,2.350954)
[linewidth=0.042cm](1.5701551,-2.2123866)(4.851097,-0.5687867)
[linewidth=0.042cm](1.5710514,-0.51132834)(4.8330774,-2.1891801)
[linewidth=0.042cm](3.27599,2.3894963)(3.275274,2.3695092)
[linewidth=0.042cm](3.275274,2.3695092)(6.364515,0.33761585)
[linewidth=0.042cm](4.7879763,3.2559245)(4.891787,-0.55023146)
-2.0515552(0.113585256,0.10894949)(3.0801435,-3.097705)Fig 1

Proposition 2.21 $(K_n \circ C_k)(i) \cong kK_{n-2}$.

Proof. We only prove the case when n is odd since the graph $(K_n \circ C_k)$ consists of kK_n subgraphs arranged along an odd cycle C_k. We choose vertices that will dominate the vertices in each K_n subgraph. This is minimally accomplished by choosing the vertices that are on the inner cycle. Each of these two vertices dominate two adjacent K_n subgraphs . Let $v_1, v_2, v_3, \dots, v_k$ be the vertices of the inner cycle and $a_{i1}, a_{i2}, a_{i3}, \dots a_{in_2}$ be the vertices of K_n drawn on the edge $v_i v_j$ of the cycle C_k . Then $S_1 = \{v_1,, v_3, v_5, \dots, v_{k-2}, a_{k-1,1}\}$ $S_2 = \{v_2, v_4, v_6, \dots, v_{k-1}, a_{k-1}\}$ and $S_3 = \{v_3, v_5, v_7, \dots, v_k, a_{11}\}$ are three $i(G)$ sets of $K_n \circ C_k$ with cardinality= $\frac{k+1}{2}$. Since there are 2 vertices v_{k-2} and v_1 of S, the vertices of K_n drawn on the edge $v_{k-1} v_k$ is not determined by the first $\frac{k-1}{2}$ vertices of S_1. Hence any one of the vertices of that K_n except v_{k-1} and v_k should be an element of S_1. Hence a_{k-1} and v_k dominates that K_n. There are $n-2$ choices for the last vertex of S_1. Now, varying the last vertex of S_1, these $n-2$ $i(G)$ sets including S_1 and these $n-2$ $i(G)$ sets are adjacent with each other and they form a K_{n-2}. Now fixing the first vertex of S_1 and changing from the 2^{nd} vertex, we get the $i(G)$ set $S_4=\{v_1,v_4,v_6,\ldots,v_{k-1},a_{2,1}\}$. Now changing from the $3^{rd},4^{th},5^{th},\ldots,\frac{k-1}{2}^{th}$ vertices we get $\frac{k-1}{2}$ number of K_{n-2} graph . Thus with v_1 as the first vertex we get $\frac{k-1}{2}$ number of K_{n-2} graphs. Similarly using S_2 we get $\frac{k-1}{2}$ number of K_{n-2} graph. Now changing the last vertex of S_3 by allowing all $n-2$ choices for it we get a K_{n-2} graph. Thus the total number of K_{n-2} graph= $\frac{k-1}{2}+\frac{k-2}{2}+1=k$. Therefore $(K_n \circ C_k)(i)=kK_{n-2}$.

To find the number of independent dominating sets of the comb:

Let $u_1,u_2,u_3,\ldots,u_n$ be the supports and $v_1,v_2,v_3,\ldots,v_n$ be the corresponding pendent vertices of the comb Cb_n. In each $i(G)$ -set, let us arrange the pendents and supports individually in the ascending order of suffixes. $\{v_1,v_2,v_3,\ldots,v_n\}$ is the only $i(G)$-set with n pendent vertices. Hence the $i(G)$-set with no support is 1. The sets $\{v_2,v_3,v_4,\ldots,v_n,u_1\},\{v_1,b_3,v_4,v_5,\ldots,v_n,u_2\},\{v_1,v_2,v_4,v_5,v_6,\ldots,v_n,u_2\},\ldots,\{v_1,b_2,v_3,\ldots,v_{n-1},u_n\}$ are the n $i(G)$- sets with only one support.

To find the number of independent dominating sets with 2 supports:

The $i(G)$ - sets with u_1 as first support are $\{v_2,v_4,v_5,v_6,\ldots,v_n,u_1,u_2\},\{v_2,v_3,v_5,v_6,v_7,\ldots,v_n,u_1,u_4\},\{v_2,v_3,v_4,\ldots,v_n,u_1,u_5\}\ldots,\{v_2,v_3,v_4,\ldots,v_{n-1},u_1,u_n\}$. Thus we get $n-2$ $i(G)$- sets with u_1 as first support.

The $i(G)$ - sets with u_2 as first support are $\{v_1,v_3,v_5,v_6,\ldots,v_n,u_2,u_4\},\{v_1,v_3,v_4,v_6,v_7,\ldots,v_n,u_2,u_5\},\ldots,\{v_1,v_3,v_4,v_5,\ldots,v_{n-2},v_{n-1},u_2,u_n\}$. Thus there are $n-3$ $i(G)$ - sets with u_2 as first support. Proceeding like this we see that $\{v_1,v_2,v_3,\ldots,v_{n-3},v_{n-1},u_{n-2},u_n\}$ is the only $i(G)$- set with u_{n-2} as first support.

Hence the total number of $i(G)$ - sets with 2 supports are $(n-2)+(n-3)+(n-4)+\ldots+3+2+1=\frac{(n-1)(n-2)}{2}$.

Here we see that Cb_3 is the smallest comb having $i(G)$-sets with 2 supports.

To find the number of independent dominating sets with 3 supports:

Cb_5 is the smallest comb containing $i(G)$-sets with 3 supports and $\{v_2,v_4,u_1,u_3,u_5\}$ is the only $i(G)$-set with 3 supports. For sake of brevity we use the following notation. We denote supports only. For example let us denote $\{v_2,v_4,u_1,u_3,u_5\}$ by $\{u_1,u_3,u_5\}$.

For the comb Cb_n, the $i(G)$ -set with first support u_1 and second support u_3 are $\{u_1,u_3,u_5\}$, $\{u_1,u_3,u_6\}$, $\{u_1,u_3,u_7\}$,...,$\{u_1,u_3,u_n\}$ i.e, here we fix the first 2 supports and vary the third support. Thus we get $n-4$ $i(G)$-sets. Now fixing u_1 and u_4 as the first 2 supports and varying the third support we get the

$i(G)$ -sets $\{u_1,u_4,u_6\}$, $\{u_1,u_4,u_7\}$, $\{u_1,u_4,u_8\}$,..., $\{u_1,u_4,u_n\}$. Thus we get $n-5$ $i(G)$ -sets. Proceeding like this we get $\{u_1,u_{n-2},u_n\}$ is the only $i(G)$-set with u_1 as the first support and u_{n-2} as the second support. Hence the number of $i(G)$ -sets with u_1 as the first support is $=(n-4)+(n-5)+(n-6)+\ldots+2+1=\frac{(n-4)(n-3)}{2}$.

Now fixing u_2 and u_4 as the first 2 supports and varying the third support we get $n-5$ $i(G)$-sets. Similarly by fixing u_2 and u_5 as the first 2 supports and varying the third support we get $n-6$ $i(G)$-sets. Proceeding like this, by fixing u_2 and u_{n-2} as the first 2 supports we get only one $i(G)$-set. Hence the number of $i(G)$ -sets with first support u_2 is $(n-5)+(n-4)+(n-3)+\ldots+2+1=\frac{(n-5)(n-4)}{2}$.

Continuing in a similar way , by fixing u_{n-4} and u_{n-2} as the first two supports we get only one $i(G)$-set. Thus the total number of $i(G)$-sets with 3 supports is

$$\begin{aligned} &= \frac{(n-4)(n-3)}{2}+\frac{(n-5)(n-4)}{2}+\frac{(n-6)(n-5)}{2}+\ldots+\frac{2\times 1}{2} \\ &= \frac{1}{2}\sum_{k=5}^{n}(k-4)(k-3) \end{aligned} \quad (1)$$

Hence the number of $i(G)$-sets of $Cb_5,Cb_6,Cb_7,Cb_8,Cb_9,\ldots$ are $1,4,10,20,35,\ldots$.

To find the number of independent dominating sets with 4 supports.

Cb_7 is the smallest comb having $i(G)$-set with 4 supports.

For the comb, the number of $i(G)$-sets with u_1,u_3,u_5 as first 3 supports $=n-7$ ⁝.

The number of $i(G)$-sets with u_1,u_3,u_{n-2} as first 3 supports $=1$.

Hence the number of $i(G)$-sets with u_1 and u_3 as first two supports $=\frac{(n-5)(n-6)}{2}$.

Similarly number of $i(G)$-set with u_1 and u_4 as the first 2 supports $=\frac{(n-6)(n-7)}{2}$

Number of $i(G)$-sets with u_1,u_{n-5} as first 2 supports $=3$.

Number of $i(G)$-sets with u_1,u_{n-4} as first 2 supports $=1$.

Therefore number of $i(G)$-sets with u_1 as first support.

$$\begin{aligned} &= \frac{(n-6)(n-5)}{2}+\frac{(n-7)(n-6)}{2}+\frac{(n-8)(n-7)}{2}+\ldots+ \\ &6+3+1 \\ &= \frac{1}{2}\sum_{k=7}^{9}(k-6)(k-5) \end{aligned} \quad (2)$$

Similarly number of $i(G)$-sets with first support $u_2=\frac{1}{2}\sum_{k=7}^{n-1}(k-6)(k-5)$.

Number of $i(G)$-sets with first support $u_3=\frac{1}{2}\sum_{k=7}^{n-2}(k-6)(k-5)$.

Proceeding like this we get the number of $i(G)$-sets with first support $n-7$ is 3 and the number of $i(G)$-sets with first support $n-6$ is 1. Hence the total number of $i(G)$-sets with 4 supports is $\frac{1}{2}\sum_{k=7}^{n}(k-6)(k-5)+\frac{1}{2}\sum_{k=7}^{n-1}(k-6)(k-5)+\frac{1}{2}\sum_{k=7}^{n-2}(k-6)(k-5)+\ldots+6+3+1$. Hence the number of $i(G)$-sets of $Cb_7, Cb_8, Cb_9, \ldots$ are $1,5,15,35,\ldots$.

To find the smallest comb with only one $i(G)$-set with 5 supports:

Cb_9 is the smallest comb with only one $i(G)$-set with 5 supports. For the comb Cb_7, the number of $i(G)$-sets with u_1,u_3,u_5,u_7 as first 4 supports is $n-8$. The number of $i(G)$-sets with u_1,u_3,u_5,u_8 as first 4 supports is $n-9$.

The number of $i(G)$-sets with u_1,u_3,u_5,u_7 as first 4 supports is 1. Thus the the number of $i(G)$-sets with first 3 vertices $u_1,u_3,u_5; u_1,u_3,u_6; u_1,u_3,u_7 \ldots, u_1,u_3,u_{n-4}$ are $\frac{(n-8)(n-7)}{2}, \frac{(n-9)(n-8)}{2}, \frac{(n-10)(n-9)}{2}, \ldots, 1$

Therefore number of $i(G)$-sets with u_1,u_3 as first 2 supports= $\frac{1}{2}\sum_{k=9}^{n}(k-8)(k-7)$.

Similarly number of $i(G)$-sets with u_1,u_3 as first 2 supports= $\frac{1}{2}\sum_{k=9}^{n-1}(k-8)(k-7)$

⋮

Number of $i(G)$-sets with u_1,u_{n-6} as first 2 supports is 1.

Number of $i(G)$-sets with first support $u_1=\frac{1}{2}\sum_{k=9}^{n}(k-8)(k-7)+\frac{1}{2}\sum_{k=9}^{n-1}(k-8)(k-7)+\frac{1}{2}\sum_{k=9}^{n-2}(k-8)(k-7)+\ldots+10+4+1.$

Similarly number of $i(G)$-sets with u_2 as first support $=\frac{1}{2}\sum_{k=9}^{n-1}(k-8)(k-7)+\frac{1}{2}\sum_{k=9}^{n-2}(k-8)(k-7)+\ldots+10+4+1$

⋮

Number of $i(G)$-sets with u_2 as first support is 1.

Therefore total number of $i(G)$-sets with 5 supports is $\frac{1}{2}\sum_{k=9}^{n}(k-8)(k-7)+\frac{2}{2}\sum_{k=9}^{n-1}(k-8)(k-7)+\frac{3}{2}\sum_{k=9}^{n-2}(k-8)(k-7)+\frac{4}{2}\sum_{k=9}^{n-2}(k-8)(k-7)+\ldots+(n-10)6+(n-9)4+(n-8)1.$

Thus the total number of $i(G)$-sets of $Cb_9, Cb_{10}, Cb_{11}, Cb_{12}, \ldots$ with 5 supports are $1,6,21,56,\ldots$

By a similar method we can find the number of $i(G)$-sets of the comb with more number of supports.

Note 2.22 Consider the sequence $1,4,10,20,35,56,84,120,165,\ldots \quad \ldots(1)$

This is the sequence of number of $i(G)$-sets of the comb with 3 supports. Let $t_1=1, t_2=4, t_3=10, t_4=20, \ldots$. The partial sums of the sequence are

$S_1=t_1=1$

$S_2=t_1+t_2=1+4=5$

$S_3=t_1+t_2+t_3=1+4+10=15$

$S_4=t_1+t_2+t_3+t_4=1+4+10+20=35$

⋮

Thus the sequence of partial sums of the sequence (1) is $1,5,15,35,\ldots$ ….........(2)

The terms of this sequence represents the number of $i(G)$-sets with 4 supports of the comb.
The sequence of partial sums of the sequence (2) are $1,6,21,56,\ldots\ \ldots(3)$.
The terms of this sequence represent the number of $i(G)$-sets with 5 supports of the comb.
The sequence of partial sums of the sequence (3) are $1,7,28,84,210,\ldots$. The terms of the sequence represent the number of $i(G)$-sets with 6 supports of the comb.
Thus if the number of $i(G)$-sets with n supports is known, the number of $i(G)$-sets with $n+1$ supports can be found out.

Note 2.23

1. Let us denote the partial sums of the sequence of number of $i(G)$-sets with k supports by $S_{k_{11}}, S_{k_{12}}, S_{k_{13}},\ldots$

Then $S_{3,1}=1, S_{3,2}=4, S_{3,3}=10, S_{3,4}=20,\ldots$
$S_{4,1}=1, S_{4,2}=5, S_{4,3}=15, S_{4,4}=35,\ldots$
$S_{5,1}=1, S_{5,2}=6, S_{5,3}=21, S_{3,4}=56,\ldots$ and so on.

2. Number of $i(G)$-sets of the comb Cb_n with 2 supports=number of $i(G)$-sets of the comb Cb_{n-1} with 2 supports $+(n-2)=\frac{1}{2}(n-2)(n-3)+(n-2)$.
3. Number of $i(G)$-sets of the comb Cb_n with 3 supports $=S_{3,n-5}+\frac{1}{2}(n-3)(n-4)$.
4. Number of $i(G)$-sets of the comb Cb_n with 4 supports $=S_{3,n-6}+S_{4,n-7}$.
5. Number of $i(G)$-sets of the comb Cb_n with 5 supports $=S_{4,n-8}+S_{5,n-9}$.
6. Number of $i(G)$-sets of the comb Cb_n with 6 supports $=S_{5,n-10}+S_{6,n-11}$. and so on.

Theorem 2.24 Let us denote the graph $Cb_n(i)$ by G_n. Then order of G_n =order of G_{n-1} +order of G_{n-2}.

Proof. We know that Cb_n has $2n$ vertices and $i(Cb_n)=n$. Also the maximum number of supports in an $i(G)$-set of $Cb_n=\left\lceil \frac{n}{2} \right\rceil$.

Let $u_1,u_2,u_3,\ldots,u_n$ be the supports and $v_1,v_2,v_3,\ldots,v_n$ be the pendent vertices of the comb Cb_n. Then $o(G_n)=$ Number of $i(G)$ sets with n pendents+ Number of $i(G)$-sets with $n-1$ pendents+ Number of $i(G)$ -sets with $n-2$ pendents $+\ldots+$ Number of $i(G)$ -sets with $\left\lceil \frac{n}{2} \right\rceil$ pendents

$$=1+n+\frac{1}{2}(n-1)(n-2)+S_{3,n-4}+S_{4,n-6}+S_{5,n-8}+\ldots.$$

$$
\begin{aligned}
i.e)o(G_n) &= 1+n+\frac{1}{2}(n-1)(n-2)+S_{3,n-4}+S_{4,n-6}+S_{5,n-8}+\dots \\
&= 1+[(n-1)+1]+[\frac{1}{2}(n-2)(n-3)+(n-2)]+ \\
&[S_{3,n-5}+\frac{1}{2}(n-2)(n-3)]+(S_{3,n-6}+S_{4,n-7})+ \\
&(S_{4,n-8}+S_{5,n-9})+(S_{5,n-10}+S_{6,n-11}+\dots+ \qquad (3) \\
&= 1+[(n-1)+\frac{1}{2}(n-2)(n-3)+(S_{3,n-6}+S_{4,n-7})+ \\
&(S_{4,n-8}+S_{5,n-9})+(S_{5,n-10}+S_{6,n-11})+\dots+ \\
&= 1+[(n-1)+\frac{1}{2}(n-2)(n-3)+(S_{3,n-5}+S_{4,n-7})+ \\
&S_{5,n-9}+\dots]+[1+(n-2)+\frac{1}{2}(n-3)(n-4)+S_{3,n-6}+ \\
&S_{4,n-8}+S_{5,n-10}+\dots \\
&= O(G_{n-1})+O(G_{n-2})
\end{aligned}
$$

Example 2.25 When $n=1$, $Cb_1 \cong K_2$ and

$|Cb_1(i)| = 2 = 1+1$

$|Cb_2(i)| = 3 = 1+2$

$|Cb_3(i)| = 5 = 1+3+1 = 1+(2+1)+1 = (1+2)+(1+1)$

$|Cb_4(i)| = 8 = 1+4+3 = 1+(3+1)+(1+2) = (1+3+1)+(1+2)$

$|Cb_5(i)| = 13 = 1+5+6+1 = 1+(4+1)+(3+3)+1 = (1+4+3)+(1+3+1)$

$|Cb_6(i)| = 21 = 1+6+10+4 = 1+(5+1)+(6+4)+(1+3) = (1+5+6+1)+(1+4+3)$ and so on.

Theorem 2.26 For any complete graph H, there exists a graph $G® H$ such that $G(i) \cong H$.

Proof. Let H be a complete graph with vertices $v_1, v_2, v_3, \dots, v_n$. By construction, let us prove that there exists a graph $G® H$ such that $G(i) \cong H$. To form G, we add a star $K_{1,s}$ of order $s+1$ with vertices $p_1, p_2, p_3 \dots p_{s+1}$, center p_2 and $s \geq 3$ and add an edge joining any one of the leaf of $K_{1,s}$ to a vertex v_i of $H, 1 \leq i \leq n$. Since no vertex of G is adjacent to any other vertex, $i(G) \geq 2$. Obviously $X_i = \{p_2, v_i\}, 1 \leq i \leq n$ is an $i(G)$-set for G ,since each v_i dominates all the other vertices of $K_{1,s}$. Since p_2 is the only vertex of $K_{1,s}$ which dominates all the vertices of $K_{1,s}$ there are no other $i(G)$-sets for G. Hence $X_i, 1 \leq i \leq n$ are the only $i(G)$-sets for G.

Each $i(G)$-set differs by only one vertex as p_2 appears in every $i(G)$-set of G. Hence $G(i) \cong H$. The following figure shows the construction of the graph G with $H \cong K_5$ so that $G(i) \cong H$.

(0,-2.328125)(13.48917,3.328125) [dotsize=0.12](0.08,1.0755764) [dotsize=0.12](2.86,1.0755764) [dotsize=0.12](2.86,-1.6644236) [dotsize=0.12](0.06,-1.6644236) [dotsize=0.12](4.66,-0.2644236)
[linewidth=0.042cm](2.88,1.0755764)(4.66,-0.24442361)
[linewidth=0.042cm](2.88,-1.6244236)(4.64,-0.2644236)
[linewidth=0.042cm](0.08,1.1155764)(4.64,-0.24442361)
[linewidth=0.042cm](0.06,-1.6244236)(4.66,-0.2644236)
[linewidth=0.042cm](2.82,1.0355763)(0.08,-1.6044236)
[linewidth=0.042cm](0.08,1.0755764)(2.92,-1.7044237)
[linewidth=0.042cm](4.7,-0.2644236)(7.36,-0.28442362) [dotsize=0.12](7.4,-0.28442362)
[linewidth=0.042cm](7.42,-0.28442362)(10.3,-0.28442362) [dotsize=0.12](10.34,-0.3044236)
[linewidth=0.042cm](10.34,-0.24442361)(9.32,1.7555764)
[linewidth=0.042cm](10.34,-0.2644236)(10.16,1.8355764)
[linewidth=0.042cm](10.36,-0.2644236)(11.04,1.7355764)
[linewidth=0.042cm](10.34,-0.2644236)(11.94,1.3355764)
[linewidth=0.042cm](10.38,-0.24442361)(12.44,0.6155764)

[linewidth=0.042cm](10.38,-0.2644236)(12.48,-0.28442362)
[linewidth=0.042cm](10.4,-0.28442362)(12.34,-1.1044236)
[linewidth=0.042,dimen=outer](12.18,-1.4044236)0.02 [linewidth=0.042,dimen=outer](11.96,-1.5244236)0.02
[linewidth=0.042,dimen=outer](11.64,-1.6644236)0.02 [linewidth=0.042,dimen=outer](11.36,-1.7244236)0.02
[linewidth=0.042,dimen=outer](11.0,-1.7844236)0.02 [linewidth=0.042,dimen=outer](10.68,-1.7844236)0.02
[linewidth=0.042,dimen=outer](10.36,-1.7444236)0.02 [dotsize=0.12](10.1,-1.6844236)
[dotsize=0.12](12.32,-1.1244236) [dotsize=0.12](12.46,-0.2644236) [dotsize=0.12](12.44,0.5955764)
[dotsize=0.12](11.94,1.3355764) [dotsize=0.12](11.06,1.7355764) [dotsize=0.12](10.16,1.8355764)
[dotsize=0.12](9.32,1.7755764) [linewidth=0.04cm](10.365889,-0.2503125)(10.085889,-1.6903125)
(7.3387012,-0.5003125) p_1 39.2(1.9594711,-6.852193)(10.589769,-0.6648356) p_2 (9.138701,2.0196874)
p_3 (10.0987015,2.1396875) p_4 (11.178701,2.0796876) p_5 (12.398702,1.4796875) p_6
(12.978702,0.6596875) p_7 (12.998701,-0.3003125) p_8 (12.818703,-1.2403125) p_9
(10.168701,-2.1003125) p_{s+1} [linewidth=0.032,dimen=outer](2.84,1.1355762)(0.02,-1.6844236)

Corollary 2.27 Every complete graph H of order n is the i-graph G of order $n+m$ where $m \geq 3$.

Definition 2.28 A graph obtained by attaching a pendent edge to each vertex of the n-cycle is called a crown. Let us denote it by G_n and $G_n = C_n \text{e} K_1$. Hence a crown G_n has $2n$ vertices.

Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the cycle(supports) and $v_1, v_2, v_3, \ldots, v_n$ be the corresponding pendent vertices. It is obvious that $i(G_n) = n$. In the $i(G)$-set of G_n, let us arrange the pendent vertices and supports in the increasing order of the suffixes. Note that maximum number of supports in any $i(G)$-set of $G_n = \left\lfloor \frac{n}{2} \right\rfloor$.

To find the number of independent dominating sets of a crown:

As $\{v_1, v_2, v_3, \ldots, v_n\}$ is the only $i(G)$-set with n pendent vertices, the number of $i(G)$- set with no support is 1.

The sets $\{v_2, v_3, \ldots, v_n, u_1\}, \{v_1, v_3, \ldots, v_n, u_2\}, \ldots \{v_1, v_2, v_3, \ldots, v_{n-1}, u_n\}$ are the n $i(G)$-sets with only one support. Hence the number of $i(G)$-sets with one support is n.

The $i(G)$ -sets containing 2 supports with u_1 as the first support are $\{v_2, v_4, v_5, v_6, \ldots, u_1, u_3\}, \{v_2, v_3, v_5, v_6, v_7, \ldots, v_n, u_1, u_4\}, \ldots, \{v_2, v_3, v_4, \ldots, v_n, u_1, u_{n-1}\}$. Thus we get $n-3$ $i(G)$-sets with u_1 as the first support.

The $i(G)$ -sets with u_2 as the first support are $\{v_1, v_3, v_4, v_5, \ldots, v_n, u_2, u_4\}, \{v_1, v_3, v_4, v_6, v_7, \ldots, v_n, u_2, v_5\}, \ldots \{v_1, v_3, v_4, v_5, \ldots, v_{n-2}, u_2, u_n\}$. Thus there are $n-3$ $i(G)$-sets with u_2 as the first support.

The $i(G)$ -sets with u_3 as the first support are $\{v_1, v_2, v_4, v_5, v_6, \ldots, v_n, u_3, u_5\}, \{v_1, v_2, v_4, v_5, v_6, \ldots, v_n, u_3, u_6\}, \ldots, \{v_1, v_2, v_4, v_5, v_6, \ldots, v_{n-1}, u_3, u_n\}$. Thus there are $n-4$ $i(G)$-sets with u_3 as the first support. Hence the number of $i(G)$-sets with 2 supports is

$$
\begin{aligned}
&= (n-3) + [(n-3) + (n-4) + (n-5) + \ldots + 1] \\
&= n - 3 + \frac{(n-3)(n-2)}{2} \qquad (4) \\
&= \frac{2n - 6 + n^2 - 5n + 6}{2} = \frac{n^2 - 3n}{2} \text{..........(1)}
\end{aligned}
$$

G_4 is the smallest crown with $i(G)$-sets containing 2 supports. Hence substituting $n-4,5,6,\ldots$ in (1) we get the sequence $2,5,9,14,20,27,35,44,54,65,77,90,\ldots$.....(2)

ie)the terms in the sequence (2) represent the number of $i(G)$-sets of the crown $G_4, G_5, G_6, \ldots$. Let

$t_1 = 2, t_2 = 5, t_3 = 9, t_4 = 14, t_5 = 20, t_6 = 27, t_7 = 35, t_8 = 44, t_9 = 54, t_{10} = 65, t_{11} = 77, t_{12} = 90, \ldots$

Consider the sequence of partial sums of (2).

$S_1 = 2, S_2 = 7, S_3 = 16, S_4 = 30, S_5 = 50, S_6 = 77, S_7 = 112, S_8 = 156, S_9 = 210, S_{10} = 275, \ldots$

ie)the sequence of partial sums of (2) is $2,7,16,30,50,77,112,156, \ldots..(3)$

The terms of (3) represent the number of $i(G)$-sets of the crown $G_6, G_7, G_8, \ldots$ with 3 supports . The sequence of partial sums of (3) is $2,9,25,55,105,182, \ldots....(4)$

The terms of this sequence represent the number of $i(G)$-sets of the crown $G_8, G_9, G_{10}, \ldots$ with 4 supports. In a similar manner the number of $i(G)$-sets of the crown with more number of supports can be found out.

Note 2.29

1. We denote the partial sums of the sequence of number of $i(G)$ -sets with k -supports by $S_{k,1}, S_{k,2}, S_{k,3}, \ldots.$

Then

$S_{2,1} = 2, S_{2,2} = 5, S_{2,3} = 9, S_{2,4} = 14, S_{2,5} = 20, S_{2,6} = 27, S_{2,7} = 35, \ldots S_{3,1} = 2, S_{3,2} = 7, S_{3,3} = 16, S_{3,4} = 30, S_{3,5} = 50, S_{3,6} = 77, S_{3,7} = 112, S_{3,8} = 156, \ldots S_{4,1} = 2, S_{4,2} = 9, S_{4,3} = 25, S_{4,4} = 55, S_{4,5} = 105, S_{4,6} = 182, S_{4,7} = 294, \ldots$

2. Number of $i(G)$-sets of the crown G_n with 2 supports $= S_{2,n-4} + n - 2$.
3. Number of $i(G)$-sets of the crown G_n with 3 supports $= S_{3,n-6} + S_{2,n-5}$.
4. Number of $i(G)$-sets of the crown G_n with 4 supports $= S_{4,n-8} + S_{3,n-7}$.
5. Number of $i(G)$-sets of the crown G_n with 5 supports $= S_{5,n-10} + S_{4,n-9}$. and so on.

Theorem 2.30 Let G_n be a crown of order n. Then $G_n(i) =$ order of $G_{n-1}(i) +$ order of $G_{n-2}(i)$

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the cycle and $v_1, v_2, v_3, \ldots, v_n$ be the corresponding pendent vertices. $O(G_n(i))$ = Number of $i(G)$ sets with n pendents + Number of $i(G)$-sets with $n-1$ pendents + Number of $i(G)$-sets with $n-2$ pendents + ... + Number of $i(G)$-sets with $\left\lfloor \frac{n}{2} \right\rfloor$ pendents. Therefore

$$\begin{aligned}
O(G_n(i)) &= 1 + n + S_{2,n-3} + S_{3,n-5} + S_{4,n-7} + S_{5,n-9} + \ldots + S_{3,n-(n-1)} \\
&= 1 + [(n-1)+1] + [S_{2,n-4} + (n-2)] + (S_{3,n-6} + S_{2,n-5}) \\
&\quad + (S_{4,n-8} + S_{3,n-7}) + (S_{5,n-10} + S_{4,n-9} + \ldots + \\
&= 1 + [(n-1) + (S_{2,n-4} + S_{3,n-6}) + (S_{4,n-8}] + \ldots 1 + \\
&\quad [(n-2) + (S_{2,n-5} + S_{3,n-7}) + S_{4,n-9} + \ldots] \\
&= O(G_{n-1}(i)) + O(G_{n-2}(i))
\end{aligned}$$

Example 2.31 When $n = 3$, $|G_n(i)| = 4 = 1 + 3$

When $n = 4$, $|G_n(i)| = 7 = 1 + 4 + 2 = 1 + (3+1) + 2 = (1+3) + (1+2)$

When $n = 5$, $|G_n(i)| = 11 = 1 + 5 + 5 = 1 + (4+1) + (2+3) = (1+4+2) + (1+3)$

When $n = 6$, $|G_n(i)| = 18 = 1 + 6 + 9 + 2 = 1 + (5+1) + (5+4) + 2 = (1+5+5) + (1+4+2)$

When $n = 7$, $|G_n(i)| = 29 = 1 + 7 + 14 + 7 = 1 + (6+1) + (9+5) + (2+5) = (1+6+9+12) + (1+5+5)$

When $n = 8$, $|G_n(i)| = 47 = 1 + 8 + 20 + 16 + 2 = 1 + (7+1) + (14+6) + (7+9) + 2 = (1+7+14+7) + (1+6+9+2)$

When $n = 9$,
$|G_n(i)| = 76 = 1+9+27+30+9 = 1+(8+1)+(20+7)+(16+14)+(2+7) = (1+8+20+16+2)+(1+7+1+7)$
and so on.

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