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On Bernstein Polynomials

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Abstract: *We have defined a new polynomial on the interval* $[0,1+\frac{r}{n}]$ *for Lebesgue integral in* L_1 *norm as*

$$U^{\alpha}_{nr}(f,x)=(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t)dt\right\} q_{nr,k}(x;\alpha)$$

where

$q_{nr,k}(x;\alpha)=\binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$ and then proved the result of Voronowskaja

.

***Keywords**:Bernstein Polynomials, Convergence, Generalized Polynomial,* L_1 *norm, Lebesgue integrable function*

I. Introduction and Results

If $f(x)$ is a function defined [0,1], the Bernstein polynomial $B^{f}_{n}(x)$ of f is given as

$B^{f}_{n}(x)=\sum_{k=0}^{n} f(k/n)\,p_{n,k}(x)$……(1.1)

where

$p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ ………. (1.2)

Schurer [7] introduced an operator

S_{nr}: c [0, $1+\frac{r}{n}$] → c [0, 1]

Defined by

$S_{nr}(f,x)=\sum_{k=0}^{n+r} f(\frac{k}{n+r})\,p_{nr,k}(x)$ --------- (1.3)

where

$p_{nr,k}(x)=\binom{n+r}{k}x^k(1-x)^{n+r-k}$ --------- (1.4)

and r is a non-negative integer. In case r= 0 , this reduces to the well-known Bernstein operator

A slight modification of Bernstein polynomials due to Kantorovich[9] makes it possible to approximate Lebesgue integrable function in L_1-norm by the modified polynomials

$P^{f}_{n}(x)=(n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt\right\}p_{n,k}(x)$…(1.5)

where $P_{n,k}(x)$ is defined by (1.2)

By Abel's formula (see Jensen [8])

$(x+y)(x+y+n\alpha)^{n-1}=\sum_{k=0}^{n}\binom{n}{k}x(x+k\alpha)^{k-1}y(y+(n-k)\alpha)^{n-k-1}$…. (1.6)

which on substituting (n+r) for n becomes

$(x+y)(x+y+(n+r)\alpha)^{n+r-1}=\sum_{k=0}^{n+r}\binom{n+r}{k}x(x+k\alpha)^{k-1}y(y+(n+r-k)\alpha)^{n+r-k-1}$…. (1.7)

If we put $y=1-x$, we obtain (see Cheney and Sharma [5])

$1=\sum_{k=0}^{n+r}\binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$….. (1.8)

Thus defining

$q_{nr,k}(x;\alpha)=\binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$…..(1.9)

we have

$\sum_{k=0}^{n+r} q_{nr,k}(x;\alpha)=1$….(1.10)

For a finite interval [0,$1+\frac{r}{n}$] ,the operator is modified in a manner similar to that done to Bernstein's operator by Kantorovich [9] and thus we defined the operator as

U_{nr}: c[0,$1+\frac{r}{n}$] → c[0,1]

by

$U_{nr}^{\alpha}(f,x) = (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t)dt\right\} q_{nr,k}(x;\alpha)$----(1.11)

where$q_{nr,k}(x;\alpha)$same as (1.9) and r isis a non-negative integer. When r =0 &$\alpha = 0$, then it reduces to the well-known operator due to Kantorovich given by (1.5).

The function

(v, n+r, x, y)=$\sum_{k=0}^{n+r}\binom{n+r}{k}$ $(x + k\alpha)^{+v-1} y(y + n$+r $- k\,)\alpha)^{n^{+r}-k^{-1}}$ -------(1.12)

satisfies the reduction formula

S (v, n+r, x, y) = xS (v − 1, n+r, x, y) + (n+r)(v, n+r − 1, $x + \alpha$, y) -------- (1.13)

from (1.3) & (1.12) we can have

(0, n+r, x, y) = $(x + y)(x + y + (n$+r$)\alpha)^{n^{+r}-1}$,

by repeated use of reduction formula(1.13) and (1.6) we get

(1, n+r, x, y) =$\sum_{k=0}^{n+r}\binom{n+r}{k} k!\ \alpha^{n^{+r}}$ $(x + y)(x + y + (n$+r$)\alpha)^{n^{+r}-k^{-1}}$, -----(1.14)

S (2, n, x, y) =)=$\sum_{k=0}^{n+r}\binom{n+r}{k}$ $(x + k\alpha)k!\ \alpha^{n^{+r}}$(1, n+r − k, $x + k\alpha$, y) . ------(1.15)

Since$k!$ =$\int_0^\infty e^{-t}t^k dt$and using binomial expansion we obtain

S(1, n+r, x, y)=$\int_0^\infty e^{-t}(x + y)\ (x + y + (n$+r$)+ t\alpha)^{n^{+r-1}}dt$, ------(1.16)

(2, n+r, x, y)= $\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}[(x+\ )(x + y + (n$+r$)\alpha + t\alpha + s\alpha)^{n^{+r}-1}$+

(n+r)$\alpha^{n^{+r}}S(x + y + (n$+r$)+t\alpha + s\alpha)^{n^{+r}-2}$], --------------(1.17)

we, therefore , can show

S(1, n+r−1, $x + \alpha$, 1 −x)=$\int_0^\infty e^{-t}(1 + (n$+r$) + t\alpha)^{+r-1}d$, ------- (1.18)

S (1, n+r−2, $x + \alpha$, 1 − x+α) = $\int_0^\infty e^{-t}(1 + (n$+r$)+t\alpha)^{n^{+r-2}}dt$, ----------(1.19)

S(2, n+r − 2, $x + 2\alpha$, 1 −x)=$\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[(x + 2\alpha)\ (1 + (n$+r$)+ t\alpha + s\alpha)^{n^{+r-2}}$

+(n+r − 2)α^2s$(1 + (n$+r$)\alpha + t\alpha$+$s\alpha)^{n^{+r-3}}$], ---------(1.20)

S (2, n+r − 3, $x + 2\alpha$, 1 − x+α) = $\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[(x + 2\alpha)(1 + (n$+r$)\alpha + t\alpha$+ $s\alpha)^{n^{+r-3}}$

+ (n+r −3)^{2}s$(1 + (n$+r$)\alpha + t\alpha$+$s\alpha)^{n^{+r-4}}$], ---------(1.21)

Voronowskaja [6] proved his result by assuming $f(x)$ to be atleast twice differentiable at a point x of [0 1],

$$\lim_{n\to\infty} n[f(x) - B_n^f(x)] = -\frac{1}{2}x(1-x)f''(x)$$

In particular,if$f''(x) \neq 0$, difference$f(x) - B_n^f(x)$ is exactly of order n^{-1}.

In this paper we shall prove the corresponding result of Voronowskaja for Lebesgue integrable function in L_1 – norm by GeneralizedPolynomial (1.11) and hence we state our result as follows:

Theorem: Let f(x) be bounded Lebesgue integrable function in with its first derivative in [0, 1+$\frac{r}{n}$] and suppose second derivative $f''(x)$ exists at a certain point x of [0, 1+$\frac{r}{n}$] , then for $\alpha = \alpha_{nr} = 0\left(\frac{1}{n+r}\right)$,

$$\lim_{(n+r)\to\infty}(n+r)[f(x) - U_{nr}^{\alpha}(f,x)] = \frac{1}{2}[(1-2x)f'(x) - x(1-x)f''(x)]$$

II. Lemmas and their proofs

Lemma 2.1: For all values of x

$$\sum_{k=0}^{n+r} kq_{nr,k}(x;\alpha) \leq \frac{1+(n+r)\alpha}{1+\alpha}(n+r)x - \frac{(n+r)(n+r-1)x\alpha}{1+2\alpha}$$

Lemma 2.2: For all values of x

$$\sum_{k=0}^{n+r} k(k-1)q_{nr,k}(x;\alpha) \le (n+r)(n+r-1)(x+2\alpha)\{\frac{1+(n+r)\alpha}{(1+2\alpha)^2} - \frac{(n+r-2)\alpha}{(1+3\alpha)^3}$$

$$+(n+r-2)\alpha^2(\frac{1+(n+r)\alpha}{(1+3\alpha)^3} - \frac{(n+r-3)\alpha}{(1+4\alpha)^4})\}$$

Lemma 2.3: For all values of x of [0, 1+$\frac{r}{n}$] and for $\alpha = \alpha_{nr} = 0\left(\frac{1}{n+r}\right)$, we have

$$(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} ((t-x)^2)dt\right\} q_{nr,k}(x;\alpha) \le \frac{x(1-x)}{n+r}$$

Proof of Lemma 2.1:

$$\sum_{k=0}^{n+r} kq_{nr,k}(x;\alpha) = \sum_{k=0}^{n+r} k\binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$$

$$= (n+r)x\sum_{k=1}^{n+r}\binom{n+r-1}{k-1}\frac{(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$$

$$= (n+r)x\sum_{\mu=0}^{n+r-1}\binom{n+r-1}{\mu}\frac{(x+\mu\alpha+\alpha)^{\mu}(1-x)(1-x+(n+r-\mu-1)\alpha)^{n+r-\mu-2}}{(1+(n+r)\alpha)^{n+r-1}}$$

$$= \frac{(n+r)x}{(1+(n+r)\alpha)^{n+r-1}}\{\sum_{\mu=0}^{n+r-1}\binom{n+r-1}{\mu}(x+\mu\alpha+\alpha)^{\mu}(1-x+(n+r-\mu-1)\alpha)^{n+r-\mu-2}$$

$$-(n+r-1)\alpha\sum_{\mu=0}^{n+r-2}\binom{n+r-2}{\mu}(x+\mu\alpha+\alpha)^{\mu}(1-x+(n+r-\mu-2)\alpha)^{n+r-\mu-2}\}$$

$=\frac{(n+r)x}{(1+(n+r)\alpha)^{n+r-1}}[s(1,n+r-1,x+\alpha,1-x)-(n+r-1)\alpha s(1,n+r-2,x+\alpha,1-x+\alpha]$
by (1.15)

$=\frac{(n+r)x}{(1+(n+r)\alpha)^{n+r-1}}[\int_0^\infty e^{-t}(1+(n+r)\alpha+t\alpha)^{n+r-1}dt-(n+r-1)\alpha\int_0^\infty e^{-t}(1+(n+r)\alpha+t\alpha)^{n+r-2}dt]$
by (1.17)&(1.18)

$=\frac{(n+r)x}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}(1+(n+r)\alpha+t\alpha)^{n+r-1}dt-\frac{(n+r)(n+r-1)x\alpha}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}(1+(n+r)\alpha+t\alpha)^{n+r-2}dt$

$=\frac{(n+r)x}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}\left(1+\frac{t\alpha}{1+(n+r)\alpha}\right)^{n+r-1}(1+(n+r)\alpha)^{n+r-1}dt$

$$-\frac{(n+r)(n+r-1)x\alpha}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}\left(1+\frac{t\alpha}{1+(n+r)\alpha}\right)^{n+r-2}(1+(n+r)\alpha)^{n+r-2}dt$$

$=(n+r)x\int_0^\infty e^{-t}\left(1+\frac{t\alpha}{1+(n+r)\alpha}\right)^{n+r-1}dt$ -- $\frac{(n+r)(n+r-1)x\alpha}{1+(n+r)\alpha}\int_0^\infty e^{-t}\left(1+\frac{t\alpha}{1+(n+r)\alpha}\right)^{n+r-2}dt$

$= (n+r)x\int_0^\infty e^{-\frac{1+(n+r)\alpha}{\alpha}u}(1+u)^{n+r-1}\frac{1+(n+r)\alpha}{\alpha}du -$

$$\frac{(n+r)(n+r-1)x\alpha}{1+(n+r)\alpha}\int_0^\infty e^{-\frac{1+(n+r)\alpha}{\alpha}u}(1+u)^{n+r-2}\frac{1+(n+r)\alpha}{\alpha}du$$

$=\frac{(1+(n+r)\alpha)(n+r)x}{\alpha}\int_0^\infty e^{-(\frac{1}{\alpha}+n+r)u}(1+u)^{n+r-1}du$- $\frac{(n+r)(n+r-1)x\alpha}{\alpha}\int_0^\infty e^{-(\frac{1}{\alpha}+n+r)u}(1+u)^{n+r-2}du$

$\leq \frac{(1+(n+r)\alpha)(n+r)x}{\alpha}\int_0^{\infty} e^{-(\frac{1}{\alpha}+n+r)u} e^{(n+r-1)u}\, du - \frac{(n+r)(n+r-1)x\alpha}{\alpha}\int_0^{\infty} e^{-(\frac{1}{\alpha}+n+r)u} e^{(n+r-2)u}\, du$

$= \frac{(1+(n+r)\alpha)(n+r)x}{\alpha}\int_0^{\infty} e^{-(\frac{1}{\alpha}+1)u}\, du - \frac{(n+r)(n+r-1)x\alpha}{\alpha}\int_0^{\infty} e^{-(\frac{1}{\alpha}+2)u}\, du$

$= \frac{(1+(n+r)\alpha)(n+r)x}{1+\alpha}\int_0^{\infty} e^{-v} dv - \frac{(n+r)(n+r-1)x\alpha}{1+2\alpha}\int_0^{\infty} e^{-w} dw$

$= \frac{(1+(n+r)\alpha)(n+r)x}{1+\alpha} - \frac{(n+r)(n+r-1)x\alpha}{1+2\alpha}$

hence the proof of lemma 2.1.

Proof of Lemma 2.2:

$$\sum_{k=0}^{n+r} k(k-1) q_{nr,k}(x;\alpha)$$

$$\leq (n+r)(n+r-1)x \sum_{k=1}^{n+r} \binom{n+r-2}{k-2} \frac{(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$$

$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}} \sum_{k=1}^{n+r} \binom{n+r-2}{v} (x+k\alpha+2\alpha)^{v+1}(1-x)(1-x+(n+r-v-2)\alpha)^{n+r-k-3}$

$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}}[S(2, n+r-2, x+2\alpha, 1-x) - (n+r-2)\alpha\, S(2, n+r-3, x+2\alpha, 1-x+\alpha)]$

$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}}[S(2, n+r-2, x+2\alpha, 1-x)] - \frac{(n+r)(n+r-1)(n+r-2)x\alpha}{(1+(n+r)\alpha)^{n+r-1}}[S(2, n+r-3, x+2\alpha, 1-x+\alpha)]$

$= I_1 - I_2$-------------- (2.2.1)

$I_1 = \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}}[S(2, n+r-2, x+2\alpha, 1-x)]$

$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}}\int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-s} ds[(x+2\alpha)(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-2}$

$$+(n+r-2)\alpha^2 s(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-3}]$$

$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}}\int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-s} ds[(x+2\alpha)(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-2}]$

$+ \frac{(n+r)(n+r-1)(n+r-2)x\alpha^2}{(1+(n+r)\alpha)^{n+r-1}}[\int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-s} ds[s(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-3}]$

$= I_{1.1} + I_{1.2}$ ---------------(2.2.2)

$I_{1.1} = \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}}\int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-s} ds[(x+2\alpha)(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-2}]$

$= \frac{(n+r)(n+r-1)x}{(1+(n+r)\alpha)^{n+r-1}}\int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-s} ds[(x+2\alpha)\left(1+\frac{s\alpha+t\alpha}{1+(n+r)\alpha}\right)^{n+r-2}(1+(n+r)\alpha)^{n+r-2}]$

$= \frac{(n+r)(n+r-1)x}{1+(n+r)\alpha}\int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-s} ds[(x+2\alpha)\left(1+\frac{s\alpha+t\alpha}{1+(n+r)\alpha}\right)^{n+r-2}]$

$\leq \frac{n+r)(n+r-1)x(x+2\alpha)}{1+(n+r)\alpha}\int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-s} e^{(n+r-2)(\frac{s\alpha+t\alpha}{1+(n+r)\alpha})} ds$

$=\frac{(n+r)(n+r-1)x(x+2\alpha)}{1+(n+r)\alpha}\int_0^\infty e^{-t+(\frac{(n+r-2)t\alpha}{1+(n+r)\alpha})}dt\int_0^\infty e^{-s+(\frac{(n+r-2)s\alpha}{1+(n+r)\alpha})}ds$

$=\frac{(n+r)(n+r-1)x(x+2\alpha)}{1+(n+r)\alpha}\int_0^\infty e^{-t(\frac{1+2\alpha}{1+(n+r)\alpha})}dt\int_0^\infty e^{-s(\frac{1+2\alpha}{1+(n+r)\alpha})}ds$

$=\frac{(n+r)(n+r-1)x(x+2\alpha)}{1+(n+r)\alpha}\int_0^\infty e^{-u}du\frac{1+(n+r)\alpha}{1+2\alpha}\int_0^\infty e^{-v}dv\frac{1+(n+r)\alpha}{1+2\alpha}$

$=\frac{(n+r)(n+r-1)x(x+2\alpha)}{(1+2\alpha)^2}(1+(n+r)\alpha)\int_0^\infty e^{-u}du\int_0^\infty e^{-v}dv$

$=\frac{(n+r)(n+r-1)x(x+2\alpha)}{(1+2\alpha)^2}(1+(n+r)\alpha)$-------------(2.2.3)

$I_{1.2}=C\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[s(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-3}]$

$=\frac{(n+r)(n+r-1)x(x+2\alpha)}{1+(n+r)\alpha}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}\left(1+\frac{s\alpha+t\alpha}{1+(n+r)\alpha}\right)^{n-3}(1+(n+r)\alpha)^{n+r-3}ds$

$\leq\frac{(n+r)(n+r-1)(n+r-2)x\alpha^2}{(1+(n+r)\alpha)^2}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}e^{(n+r-3)(\frac{s\alpha+t\alpha}{1+(n+r)\alpha})}ds$

$=\frac{(n+r)(n+r-1)(n+r-2)x\alpha^2}{(1+(n+r)\alpha)^2}\int_0^\infty e^{-t(\frac{1+3\alpha}{1+n\alpha})}dt\int_0^\infty se^{-s(\frac{1+3\alpha}{1+n\alpha})}ds$

$=\frac{(n+r)(n+r-1)(n+r-2)x\alpha^2}{(1+3\alpha)^3}(1+(n+r)\alpha)\int_0^\infty e^{-u}du\int_0^\infty e^{-v}dv$

$=\frac{(n+r)(n+r-1)(n+r-2)x\alpha^2}{(1+3\alpha)^3}(1+(n+r)\alpha)$ ---------------- (2.2.4)

from (2.2.2) , (2.2.3) & (2.2.4) we have

$I_1\leq(1+(n+r)\alpha)(n+r)(n+r-1)x\{\frac{(x+2\alpha)}{(1+2\alpha)^2}+\frac{(n+r-2)\alpha^2}{(1+3\alpha)^3}$-------------(2.2.5)

Now we evaluate

$I_2=\frac{(n+r)(n+r-1)(n+r-2)x\alpha}{(1+(n+r)\alpha)^{n+r-1}}[S(2,n+r\text{-}3,x+2\alpha,1\text{-}x+\alpha)]$

$=\frac{(n+r)(n+r-1)(n+r-2)x\alpha}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds[(x+2\alpha)(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-3}$

$+(n+r-3)\alpha^2s(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-4}]$ by (1.19)

$=\frac{(n+r)(n+r-1)(n+r-2)x\alpha}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\,(x+2\alpha)(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-3}$

$+\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\,s(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-4}$

$=I_{2.1}+I_{2.2}$--------------------(2.2.6)

$$I_{2.1}=\frac{(n+r)(n+r-1)(n+r-2)x\alpha}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\,(x+2\alpha)\left(1+\frac{s\alpha+t\alpha}{1+(n+r)\alpha}\right)^{n+r-3}(1+(n+r)\alpha)^{n+r-3}$$

$\leq\frac{(n+r)(n+r-1)(n+r-2)x(x+2\alpha)\alpha}{(1+(n+r)\alpha)^2}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}e^{(n-3)(\frac{s\alpha+t\alpha}{1+(n+r)\alpha})}ds$

$=\frac{(n+r)(n+r-1)(n+r-2)x(x+2\alpha)\alpha}{(1+(n+r)\alpha)^2}\int_0^\infty e^{-t(\frac{1+3\alpha}{1+(n+r)\alpha})}dt\int_0^\infty se^{-s(\frac{1+3\alpha}{1+(n+r)\alpha})}ds$

$=\frac{(n+r)(n+r-1)(n+r-2)x(x+2\alpha)\alpha}{(1+(n+r)\alpha)^2}\int_0^\infty e^{-u}du\frac{1+(n+r)\alpha}{1+3\alpha}\int_0^\infty e^{-v}dv\frac{1+(n+r)\alpha}{1+3\alpha}$

$=\frac{(n+r)(n+r-1)(n+r-2)x(x+2\alpha)\alpha}{(1+3\alpha)^2}\int_0^\infty e^{-u}du\int_0^\infty e^{-v}dv$

$=\frac{(n+r)(n+r-1)(n+r-2)x(x+2\alpha)\alpha}{(1+3\alpha)^2}$--------------(2.2.7)

$I_{2.2}\quad=\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\,s(1+(n+r)\alpha+s\alpha+t\alpha)^{n+r-4}$

$=\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+(n+r)\alpha)^{n+r-1}}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}\left(1+\frac{s\alpha+t\alpha}{1+(n+r)\alpha}\right)^{n+r-3}ds(1+(n+r)\alpha)^{n+r-3}$

$\leq\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+(n+r)\alpha)^3}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}e^{(n-4)(\frac{s\alpha+t\alpha}{1+(n+r)\alpha})}ds$

$=\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+(n+r)\alpha)^3}\int_0^\infty e^{-t(\frac{1+4\alpha}{1+(n+r)\alpha})}dt\int_0^\infty se^{-s(\frac{1+4\alpha}{1+(n+r)\alpha})}ds$

$=\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+(n+r)\alpha)^3}\int_0^\infty e^{-u}du\frac{1+(n+r)\alpha}{1+4\alpha}\int_0^\infty v(\frac{1+(n+r)\alpha}{1+4\alpha})e^{-v}dv\frac{1+(n+r)\alpha}{1+4\alpha}$

$=\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+4\alpha)^3}\int_0^\infty e^{-u}du\int_0^\infty ve^{-v}dv$

$=\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(1+4\alpha)^3}$-----------------(2.2.8)

substituting the values of $I_{2.1}$from (2.2.7)&$I_{2.2}$ from (2.2.8) in (2.2.6) we get
$I_2\leq(n+r)(n+r-1)(n+r-2)x\{\frac{(x+2\alpha)}{(1+3\alpha)^2}+\frac{(n+r-3)\alpha^3}{(1+4\alpha)^3}\}$

Thereforesubstituting the values of I_1&I_2 in (2.2.1), weget

$\leq\ (1+(n+r)\alpha)(n+r)(n+r-1)x\{\frac{(x+2\alpha)}{(1+2\alpha)^2}+\frac{(n+r-2)\alpha^2}{(1+3\alpha)^3}\}\ -(n+r)(n+r-1)(n+r-2)x\{\frac{(x+2\alpha)}{(1+3\alpha)^2}+\frac{(n+r-3)\alpha^3}{(1+4\alpha)^3}\}$

$=(n+r)(n+r-1)x[(x+2\alpha)\{\frac{(1+(n+r)\alpha)}{(1+2\alpha)^2}-\frac{(n+r-2)\alpha}{(1+3\alpha)^2}\}+(n+r-2)\alpha^2\{\frac{(1+(n+r)\alpha)}{(1+3\alpha)^2}-\frac{(n+r-3)\alpha}{(1+4\alpha)^3}\}$

hence the proof of lemma 2.2.

Proof of Lemma2.3:

$$(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)}(t-x)^2dt\right\}q_{nr,k}(x;\alpha$$

$$=\sum_{k=0}^{n+r}[x^2-\frac{2kx+x}{n+r+1}+\frac{k^2+k}{(n+r+1)^2}+\frac{1}{3(n+r+1)^2}]q_{nR,k}(x;\alpha)$$

$$\leq x^2-\frac{1}{(n+r+1)}\left[\frac{2(1+(n+r)\alpha)(n+r)x}{1+\alpha}-\frac{2(n+r)(n+r-1)x^2\alpha}{1+2\alpha}\right]-\frac{x}{n+r+1}$$

$$+\frac{(n+r)(n+r-1)}{(n+r+1)^2}\ [(x+2\alpha)\{\frac{(1+(n+r)\alpha)}{(1+2\alpha)^2}-\frac{(n+r-2)\alpha}{(1+3\alpha)^2}\}$$

$$+(n+r-2)\alpha^2\left\{\frac{1+(n+r)\alpha}{(1+3\alpha)^3}-\frac{n+r-3}{(1+4\alpha)^3}\right\}+\frac{2(1+(n+r)\alpha)(n+)x}{(n+r+1)^2(1+\alpha)}$$

$-\frac{2(n+r)(n+r-1)x\alpha}{(n+r+1)^2(1+\alpha)}+\frac{1}{3(n+r+1)^2}$,by lemma 2.1 & lemma 2.2

$$= -\frac{x(1-x)}{n+r+1}+\frac{2(1+(n+r)\alpha)(n+r)}{(n+r+1)^2(1+\alpha)}x(1-x)-\frac{2(n+r)(n+r-1)\alpha}{(n+r+1)^2(1+\alpha)}x(1-x)+\frac{2(1+(n+r)\alpha)(n+r)^2x(1-x)\alpha}{(n+r+1)^2(1+2\alpha)^2}$$

$$-\frac{2(n+r)^2(n+r-1)x(1-x)\alpha^2}{(n+r+1)^2(1+3\alpha)^2}+\frac{(n+r)(n+r-1)(n+r-2)\alpha^2}{(n+r+1)^2(1+3\alpha)^3}x(1-x)+\frac{(n+r)x^2}{n+r+1}-\frac{2(1+(n+r)\alpha)(n+r)^2x^2(1+3\alpha+3\alpha^2)}{(n+r+1)^2(1+\alpha)(1+2\alpha)^2}$$

$$-\frac{2(1+(n+r)\alpha)(n+r)x\alpha}{(n+r+1)^2(1+2\alpha)^2}+\frac{2(n+r)^2(n+r-1)x^2\alpha(1+5\alpha+7\alpha^2)}{(n+r+1)^2(1+2\alpha)(1+3\alpha)^2}+\frac{4(n+r)(n+r-1)x\alpha^2}{(n+r+1)^2(1+3\alpha)^2}-\frac{(n+r)(n+r-1)(n+r-2)\alpha}{(n+r+1)^2(1+3\alpha)^3}$$

$$-x^2(1+2\alpha)+\frac{(n+r)(n+r-1)(n+r-2)x\alpha^3}{(n+r+1)^2(1+3\alpha)^3}+\frac{(n+r)(n+r-1)(1+(n+r)\alpha)x^2\alpha}{(n+r+1)(1+2\alpha)^2}-\frac{(n+r)(n+r-1)(n+r-2)(n+r-3)x\alpha^3}{(n+r+1)^2(1+4\alpha)^3}+\frac{1}{3(n+r+1)^2}$$

$$\leq\frac{x(1-x)}{n+r}\qquad \text{for } \alpha=\alpha_{nr}=o\left(\frac{1}{n+r}\right) \text{ and for large n}$$

hence the proof of Lemma 2.3.

III. Proof Of The Theorem

Proof :

The function $f(t)$can be expended by Taylor's Theorem at $t=x$ as

$f(t)=f(x)+(t-x)f^{'}(x)+(t-x)^2[\frac{1}{2}f''(x)+\eta(t-x)]$ ----------(3.1)

where$\eta(h)$ is bounded $|\eta(h)|\leq H$ for all h and converges to '0' with h.

Multiplying eqn. (3.1) by $(n+r+1)q_{nr,k}(x;\alpha)$and integrating it from k/(n+r+1) to (k+1)/(n+r+1),and then on summing ,we get

$$(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t)dt\right\}q_{nr,k}(x;\alpha)$$

$$=(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(x)dt\right\}q_{nr,k}(x;\alpha)$$

$$+(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t-x)f^{'}(x)dt\right\}q_{nr,k}(x;\alpha)$$

$$+\frac{1}{2}(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-x)^2f''(x)dt\right\}q_{nr,k}(x;\alpha)$$

$$+(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-x)^2\eta(t-x)dt\right\}q_{nr,k}(x;\alpha)$$

$=I_3+I_4+I_5+I_6$ (say) ----- --------- (3.2)

Now first we evaluate I_3:

$I_3 = (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{\frac{k}{n+r+1}}^{\frac{k+1}{n+r+1}} f(x)dt\right\} q_{nr,k}(x;\alpha) = f(x)$ ----- ---------(3.3)

and then

$$I_4 = (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t-x)f'(x)dt\right\} q_{nr,k}(x;\alpha)$$

$$= \sum_{k=0}^{n+r}\left(\frac{2k+1}{2(n+r+1)} - x\right) f'(x) q_{nr,k}(x;\alpha)$$

$\leq \frac{(1-2x)}{2(n+r)} f'(x)$ for $\alpha = \alpha_{nr} = o(1/(n+r))$(3.4)

Now we evaluate I_5:

$$I_5 = \frac{1}{2}(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-x)^2 f''(x)dt\right\} q_{nr,k}(x;\alpha)$$

$\leq x(1-x)f''(x)/2(n+r)$ (by lemma 2.3) ------------- (3.5)

and then in the last we evaluate I_6 :

$$I_6 = (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t-x)^2 \eta(t-x)dt\right\} q_{nr,k}(x;\alpha)$$

Let $\in > 0$ be arbitrary $\delta > 0$ such that $|\eta(h)| < \epsilon$ for $|h| < \delta$.
Thus breaking up the sum I_6 into two parts corresponding to those values of t for which | t - x |<δ, and those for which $|t-x| \geq \delta$ and since in the given range of t, $\left|\frac{k}{n+r} - x\right| \sim |t-x|$, we have

$$|I_6| \leq \in \sum_{\left|\left(\frac{k}{n+r}\right)-x\right|<\delta} (n+r+1)q_{nr,k}(x;\alpha)\left|\int_{\frac{k}{n+r+1}}^{\frac{k+1}{k+n+1}} (t-x)^2 dt\right|$$

$$+ H \sum_{\left|\left(\frac{k}{n+r}\right)-x\right|\geq\delta} (n+r+1)q_{nr,k}(x;\alpha)\left|\int_{\frac{k}{n+r+1}}^{\frac{k+1}{k+n+1}} dt\right|$$

$= I_{6.1} + I_{6.2}$ (say)

$|I_{6.1}| \leq \frac{\in}{n+r} \; |\{x(1-x)\}|$, for $\alpha = \alpha_{nr} = o(\frac{1}{n+r})$

$$I_{6.2} = (n+r+1)H \sum_{\left|\left(\frac{k}{n+r}\right)-x\right|\geq\delta} \left\{\int_{\frac{k}{n+r+1}}^{\frac{k+1}{n+r+1}} dt\right\} q_{nr,k}(x;\alpha)$$

$$= (n+r+1) \sum_{\left|\left(\frac{k}{n+r}\right)-x\right| \geq \delta} q_{nr,k}\,(x;\alpha) \frac{1}{n+r+1}$$

But if $\delta = (n+r)^{-\beta}$, 0 < β < 1/2 (see also Kantorovitch [9]),
then for $\alpha = \alpha_{nr} = o(\frac{1}{n+r})$

$\sum_{\left|\left(\frac{k}{n+r}\right)-x\right| \geq (n+r)^{-\beta}} q_{nr,k}\,(x;\alpha) \leq C(n+r)^{-\nu}$ for □ >0, the constant C= C(β,□).

whence $I_{6.2} < \frac{\epsilon}{n+r+1} < \epsilon/(n+r)$ for sufficiently large n ,
therefore itgives
$I_6 < \epsilon/(n+r)$, for all sufficiently large n ----- (3.6)

Hence from (3.2), (3.3), (3.4), (3.5) and (3.6), we have

$$(n+r+1) \sum_{k=0}^{n+r} \left\{ \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t)dt \right\} q_{nr,k}(x;\alpha)$$

$$= f(x) + [\{(1-2x)f'(x) + x(1-x)f''(x)\}/2(n+r)] + (\epsilon/(n+r)$$

and therefore, finally we get

$$\lim_{(n+r)\to\infty} (n+r) \left[U_{nr}^{\alpha}(f,x) - f(x) \right] = \frac{1}{2} [(1-2x)f'(x) - x(1-x)f''(x)]$$

where $\epsilon \to 0$ as $n \to \infty$
hencethe proof of the theorem.

IV. Conclusions

The result of Voronowskaja has been extended for Lebesgueintegrable function in L_1-norm by our newly defined Generalized Polynomials $U_{nr}^{\alpha}(f,x)$ on the interval $[\,0,1+\frac{r}{n}]$

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