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Higher-Order Conjugate Gradient Method (HCGM) For Solving Continuous Optimal Control Problems

R.A. Raji[1] and M.O. Oke[2]
[1]Department of Mathematics and Statistics, Osun State Polytechnic, P. M. B. 301, Iree, Osun State, Nigeria.
[2]Department of Mathematical Sciences, Ekiti State University, P. M.B.5363, Ado - Ekiti, Ekiti State, Nigeria.

Abstract: *In this paper, we considered the role of penalty when the higher-order conjugate gradient method (HCGM) was used as a computational scheme for the minimization of penalised cost functions for optimal control problems described by linear systems and integral quadratic costs. For this family of commonly encountered problems, we find out that the conventional penalty methods require very large penalty constants for good constants satisfaction. Numerical results shows that, as the penalty constant tend to infinity, the convergence rate of the method becomes poor. To circumvent the poor convergence rate of the penalty method, the HCGM was often considered as a good substitute to accelerate the convergence.*

Keywords: *Conjugate Gradient Method, Optimal Control, Non-linear Systems, Positive Definite Matrix, Unconstrained Problems.*

I. Introduction

Optimal control theory, an extension of the calculus of variation, is a mathematical optimization method largely due to the work of Lev Pontryagin and his collaborators in the Sovient Union and Richard Bellman in the United States, Cannan et al. (1970), Dyer and Renolds (1970). It deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables.

An Optimal control is a set of differential equation describing the paths of the control variables that minimize the cost functional, Mehre and Davis (1972). It can be derived using Pontryagin's maximum principle or by solving the Halmitton-Jacobi-Bellman equation.

The Conjugate Gradient Method is an algorithm for solving large-scale unconstrained optimization problem with fastest rule of convergence. They are characterised by low memory requirements and have strong local and global convergence properties. The popularity of these methods is remarkable partially due to their simplicity both in their algebraic expression and in their implementation in computation codes, and partially due to their efficiency in solving large-scale unconstrained optimization problems.

Many reseachers, including Polak (1971), Bamigbola and Ejieji (2006), Hardorff (1976), Bamigbola and Nwaeze (2006), Naevalal (2004), Raji and Oke (2014) and Hagedorn (1988) just to list a few, have worked extensively on continuous time optimal control problems and conjugate gradient methods . But none of these researchers have considered the higher-order conjugate gradient method of solving continuous optimal control problems. This paper therefore, is concerned with the use of higher-order conjugate gradient method in solving continuous optimal control problems.

II. Problem Formulation

A special case of the general non-linear control problem is the quadratic optimal control problem stated as follows

$$\int_{t_0}^{t_f}[x^T(t)Qx(t) + U^T(t)RU(t)]dt \qquad (1)$$

subject to the first-order dynamic constraint

$$\frac{d}{dt}x(t) = Cx(t) + Du(t), t_0 \leq t \leq t_f \qquad (2)$$

with the initial condition

$$x(t_0) = x_0,$$

where x^T denotes the transpose of $x(t), \frac{d}{dt}x(t)$ is the derivative of $x(t)$ with respect to t. x is an $n \times 1$ state vector. u is a $q \times 1$ control vector. C and D are $n \times n$ and $n \times q$ constant matrices. Q and R are symmetric positive definite constant square matrices of dimensions $n \times q$ respectively.

In real life situation, equation (1) above can be written as

$$\int_{t_0}^{t_f} G(x,u,t)dt \tag{3}$$

subject to

$$\frac{d}{dt}x(t) = g(x,u,t) \tag{4}$$

with $G(x,u,t) = [x^T(t)Qx(t) + U^T(t)RU(t)]$

We introduced an objective functional or performance index J as

$$J(x,u,t,\mu) = \int_{t_0}^{t_f} G(x,u,t)dt + \mu \int_{t_0}^{t_f} \left\| \frac{dx}{dt} - g(x,u,t) \right\|^2 dt \tag{5}$$

We will now applied the higher-order conjugate gradient method algorithm, Raji and Oke (2014) to equation (5) which is a one-dimensional equality unconstrained control problem.

A model for the objective function can be expressed as an Nth degree polynomial of the form

$$J(x) = f(x^{(k)}) + df(x^{(k)}) + \frac{1}{2!}d^2f(x^{(k)}) + \frac{1}{3!}d^3f(x^{(k)}) + \cdots + \frac{1}{N!}d^Nf(x^{(k)}) \tag{6}$$

$$J(x) = f(x^{(k)}) + (x - x^{(k)})^T \nabla f(x^{(k)}) + \frac{1}{2!}(x - x^{(k)})^T \nabla^2 f(x^{(k)}) + \frac{1}{3!}(x - x^{(k)})^T \nabla^3 f(x^{(k)}) + \cdots + \frac{1}{N!}(x - x^{(k)})^T \nabla^N f(x^{(k)}) \tag{7}$$

for $N \geq 2,\ x^{(k)} \in R^N$.

which can compactly be expressed using Tensor notation as

$$J(x) = f(x^{(k)}) + \sum_{j=1}^{N} \frac{i}{j!} A_{i_1,i_2,i_3,\dots,i_j}(x^{(k)}) \prod_{p=1}^{j} (x - x^{(k)})_{i_p}^{z_p} \tag{8}$$

where $A_{i_1,i_2,i_3,\dots,i_j}(x^{(k)}) = \nabla^j f(x^{(k)}), N \geq 2,\ x^{(k)} \in R^N$. The term $\prod_{p=1}^{j}(x - x^{(k)})_{i_p}^{z_p}$ is the product of vector oriented along the coordinate axes $i_1, i_2, i_3, \dots, i_j$. The gradient of $J(x)$ is obtained as

$$GJ(x) = \nabla f(x^{(k)}) + \sum_{j=2}^{N} \frac{i}{(j-1)!} A_{i_2,i_3,\dots,i_j}(x^{(k)}) \prod_{p=1}^{j} (x - x^{(k)})_{i_p}^{z_p} \tag{9}$$

for $N \geq 2,\ x^{(k)} \in R^N$.

III. Computational Examples

In this section, the numerical experiment conducted on the higher-order conjugate gradient method algorithm for solving one-dimensional continuous optimal control problem will be reported. Below is a list of test problems used for the numerical experiment.

Example 1

Minimize $I(x,u) = \int_0^1 [x^2(t) - u^2(t)]dt$

Subject to the first-order dynamic constraint

$$\frac{d}{dt}x(t) = 2.095x(t) + 1.904u(t); 0 \leq t \leq 1$$

and the initial condition

$x(0) = 1, u(0) = 0.5;\ \Delta t = 0.1$

Example 2

Minimize $I(x,u) = \int_0^1 [x^2(t) - u^2(t)]dt$

Subject to the first-order dynamic constraint

$$\frac{d}{dt}x(t) = u(t); 0 \leq t \leq 1$$

and the initial condition

$x(0) = 1, u(0) = 0.5;\ \Delta t = 0.1$

Example 3

Minimize $I(x,u) = \int_0^1 [u^4(t)]dt$

Subject to the first-order dynamic constraint

$$\frac{d}{dt}x(t) = x(t) + u(t); 0 \leq t \leq 1$$

and the initial condition

$$x(0) = 1, x(1) = 0, u(0) = 0;\ \Delta t = 0.1$$

Example 4

Minimize $I(x,u) = \int_0^5 [x_1{}^2(t) + x_2{}^2(t) + u^2(t)]dt$

Subject to the first-order dynamic constraint

$$\frac{d}{dt}x(t) = x_1(t); 0 \leq t \leq 5$$

and the initial condition

$$x_1(0) = 1, x_2(1) = 0, u(0) = 0.5;\ \Delta t = 0.1$$

IV. Numerical Results

The higher-order conjugate gradient method algorithm was implemented using MATLAB 7.10.0 codes. The numerical results obtained is as tabulated below

Examples	No. of Iterations	HCGM Results (H)	Execution Time	Exact Solution (E)	Errors $\lvert(E-H)\rvert$
Example 1	6	1.3232	0.02	1.3225	7.0×10^{-4}
Example 2	6	0.7615	0.01	0.7613	2.0×10^{-4}
Example 3	10	0.7082	0.02	0.7067	1.5×10^{-3}
Example 4	14	1.2499	0.01	1.2509	1.0×10^{-3}

V. Conclusion

The higher-order conjugate gradient method has been applied in solving continuous optimal control problems. The result obtained using MATLAB 7.10.0 is very close to the exact solution. This shows that the convergence rate of the higher-order conjugate gradient method is very high and this makes it better than the conventional penalty methods.

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