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A Class of Polynomials Associated with Differential Operator and with a Generalization of Bessel-Maitland Function

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Abstract:*The object of this paper is to present several classes of linear and bilateral generating relations by employing operational techniques, which reduces as a special case of (known or new) bilateral generating relations.At last some of generating functions associated with stirling number of second kind are also discussed.*
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I. Introduction

In literature several authors have discussed a number of polynomials defined by their Rodrigue's formula and gave several classes of linear, bilinear, bilateral and mixed multilateral generating functions.

In 1971, H.M. Srivastava and J.P.Singhal [1], introduced a general class of polynomial $G_n^{(\alpha)}(x,r,p,k)$ (of degree n in x^r and in α) as

$$G_n^{(\alpha)}(x,r,p,k)=\frac{x^{-\alpha-kn}}{n!}\exp(px^r)(x^{k+1}D)^n\{x^\alpha\exp(-px^r)\} \tag{1.1}$$

where Laguerre, Hermite, and Konhauser plynomials are the special case of (1.1).

In 2001, H.M. Srivastava [2], defined a polynomial set of degree n in x^r, α and η by

$$\tau_n^{(\alpha)}(x;r,\beta,k,\eta)=\frac{x^{-\alpha-kn}}{n!}\exp(\beta x^r)\{x^k(\eta+xD)\}^n\{x^\alpha\exp(-\beta x^r)\},\ \ n\in N_0 \tag{1.2}$$

A detailed account of (1.2) is present in the work of Chen et al. [3], who derived several new classes of linear, bilinear and mixed multilateral generating relations.

Recently in 2008, Shukla and Prajapati [4], introduced a class of polynomial which is connected by the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,q}(z)$, whose properties were discussed in his paper [5], is defined as

$$A_{qn}^{(\alpha,\beta,\gamma,q)}(x;a,k,s)=\frac{x^{-\delta-an}}{n!}E_{\alpha,\beta}^{\gamma,q}\{p_k(x)\}\{x^a(s+xD)\}^n\left[x^\delta E_{\alpha,\beta}^{\gamma,q}\{-p_k(x)\}\right] \tag{1.3}$$

where $\alpha,\beta,\gamma,\delta$ are real or complex numbers and a,k,s are constants.

In the present paper we have defined a polynomial by using the differential operator θ, which involves two parameters k and λ independent of x as

$$\theta=x^k(\lambda+xD_x),\qquad \left(D_x=\frac{d}{dx}\right) \tag{1.4}$$

In connection with the differential operator (1.4), we define the polynomial $M_{qn}^{(\mu,nu,\gamma,\xi)}(x;r,p,k,\lambda)$ of degree n in x^r, ξ and λ as

$$M_{qn}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)=\frac{x^{-\xi-kn}}{n!}J_{\nu,q}^{\mu,\gamma}(-px^r)\theta^n\{x^\xi J_{\nu,q}^{\mu,\gamma}(px^r)\} \tag{1.5}$$

In (1.5), μ,ν,γ,ξ are real or complex numbers and $J_{\nu,q}^{\mu,\gamma}(z)$ is the generalized Bessel's Maitland function for $\mu,\nu,\gamma\in C; Re(\mu)\geq 0, Re(\nu)\geq 0, Re(\gamma)\geq 0$ and $q\in(0,1)\cup N$ as

$$J_{\nu,q}^{\mu,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{n!\,\Gamma(\mu n+\nu+1)} \tag{1.6}$$

where $(\gamma)_{qn}=\frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol, which reduces to $q^{qn}\prod_{r=1}^{q}\left(\frac{\gamma+r-1}{q}\right)_n$, if $q\in N$. Certain properties of (1.6) have been discussed in [6].

Some special cases of the polynomial (1.5) are given below:

$$M_n^{(1,0,1,\alpha)}(x;r,p,k,0)=G_n^{(\alpha)}(x,r,p,k) \tag{1.7}$$

where $G_n^{(\alpha)}(x,r,p,k)$ is defined by equation (1.1).

$$M_n^{(1,0,1,\alpha)}(x;r,\beta,k,\eta)=\tau_n^{(\alpha)}(x;r,\beta,k,\eta) \tag{1.8}$$

where$\tau_n^{(\alpha)}(x;r,\beta,k,\eta)$ is defined by equation (1.2).

$$M_{qn}^{(\alpha,\beta-1,\gamma,\delta)}(x;k,p,a,s) = A_{qn}^{(\alpha,\beta,\gamma,q)}(x;a,k,s) \tag{1.9}$$

where $A_{qn}^{(\alpha,\beta,\gamma,q)}(x;a,k,s)$ is defined by equation (1.3).

II. Linear Generating Functions

By appealing the property of the operator of Patil and Thakre [7],

$$e^{t\theta}\{x^\alpha f(x)\} = \frac{x^\alpha}{(1-tkx^k)^{\frac{\alpha+\lambda}{k}}}.f\left\{\frac{x}{(1-tkx^k)^{\frac{1}{k}}}\right\} \tag{2.1}$$

and

$$e^{t\theta}\{x^{\alpha-n}f(x)\} = x^\alpha(1+kt)^{-1+\frac{\alpha+\lambda}{k}}f\left\{x(1+kt)^{\frac{1}{k}}\right\} \tag{2.2}$$

one obtains certain generating relations for the polynomials defined by (1.5) as given below:

$$\sum_{n=0}^{\infty} M_{qn}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)t^n = (1-kt)^{-\frac{\xi+\lambda}{k}} J_{\nu,q}^{\mu,\gamma}(-px^r) J_{\nu,q}^{\mu,\gamma}\left[p\left\{x(1-kt)^{-\frac{1}{k}}\right\}^r\right] \tag{2.3}$$

$$(|t| < |k|^{-1}; k \neq 0)$$

$$\sum_{n=0}^{\infty} M_{qn}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)t^n = (1+kt)^{-1+\frac{\xi+\lambda}{k}} J_{\nu,q}^{\mu,\gamma}(-px^r) J_{\nu,q}^{\mu,\gamma}[p\{x\left(1+kt\right)^{\frac{1}{k}}\}^r] \tag{2.4}$$

$$(|t| < |k|^{-1}; k \neq 0)$$

$$\sum_{n=0}^{\infty}\binom{m+n}{n} M_{q(m+n)}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)t^n$$

$$= (1-kt)^{-m-(\xi+\lambda)/k}\frac{J_{\nu,q}^{\mu,\gamma}(-px^r)}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1-kt)^{-\frac{1}{k}}\}^r]} M_{qm}^{(\mu,\nu,\gamma,\xi)}(x\left(1-kt\right)^{-\frac{1}{k}};r,p,k,\lambda) \tag{2.5}$$

$$(n \in N_0; |t| < |k|^{-1}; k \neq 0)$$

$$\sum_{n=0}^{\infty}\binom{m+n}{n} M_{q(m+n)}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)t^n$$

$$= (1+kt)^{-1+(\xi+\lambda)/k}\frac{J_{\nu,q}^{\mu,\gamma}(-px^r)}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1+kt)^{1/k}\}^r]} M_{qm}^{(\mu,\nu,\gamma,\xi)}(x(1+kt)^{1/k};r,p,k,\lambda) \tag{2.6}$$

$$(n \in N_0; |t| < |k|^{-1}; k \neq 0)$$

Proof of (2.5): From (1.5), we write

$$\sum_{n=0}^{\infty}\binom{m+n}{n} M_{q(m+n)}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)t^n = \sum_{n=0}^{\infty}\frac{x^{-\xi-(m+n)k}}{m!\,n!} J_{\nu,q}^{\mu,\gamma}(-px^r)\theta^{(m+n)}\{x^\xi J_{\nu,q}^{\mu,\gamma}(px^r)\}t^n$$

$$= \frac{x^{-\xi-mk}}{m!} J_{\nu,q}^{\mu,\gamma}(-px^r)\sum_{n=0}^{\infty}\frac{x^{-nk}\,t^n\theta^n}{n!}.\theta^m\{x^\xi J_{\nu,q}^{\mu,\gamma}(px^r)\}$$

$$= \frac{x^{-\xi-mk}}{m!} J_{\nu,q}^{\mu,\gamma}(-px^r)\exp(x^{-k}t\theta)\theta^m\{x^\xi J_{\nu,q}^{\mu,\gamma}(px^r)\}$$

Again by using (1.5) and (2.1), we obtain the generating relation (2.5).

Proof of (2.6): Multiplying equation (2.4) by $\frac{x^\xi}{J_{\nu,q}^{\mu,\gamma}(-px^r)}$ and then operating upon both sides by the differential operator θ^m, we get

$$\sum_{n=0}^{\infty}\theta^m\left\{\frac{x^\xi}{J_{\nu,q}^{\mu,\gamma}(-px^r)} M_{qn}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)\right\}t^n = (1+kt)^{-1+\frac{\xi+\lambda}{k}}\theta^m\left\{x^\xi J_{\nu,q}^{\mu,\gamma}[p\{x(1+kt)^{\frac{1}{k}}\}^r]\right\} \tag{2.7}$$

Now replacing n by m in (1.5), we obtain

$$m!\frac{x^{\xi+mk}}{J_{\nu,q}^{\mu,\gamma}(-px^r)} M_{qm}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda) = \theta^m\{x^\xi J_{\nu,q}^{\mu,\gamma}(px^r)\} \tag{2.8}$$

Again replacing m by $m+n$ in equation (2.8), we obtain

$$(m+n)!\frac{x^{\xi+mk+nk}}{J_{\nu,q}^{\mu,\gamma}(-px^r)} M_{q(m+n)}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda) = \theta^{(m+n)}\{x^\xi J_{\nu,q}^{\mu,\gamma}(px^r)\}$$

$$= \theta^m\left[\theta^n\{x^\xi J_{\nu,q}^{\mu,\gamma}(px^r)\}\right] = n!\left\{\frac{x^{\xi+nk}}{J_{\nu,q}^{\mu,\gamma}(-px^r)} M_{qn}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)\right\} \tag{2.9}$$

Further replacing ξ by $(\xi - kn)$ in (2.9), we obtain

$$\left\{\frac{x^\xi}{J_{\nu,q}^{\mu,\gamma}(-px^r)} M_{qn}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)\right\} = \frac{(m+n!)}{n!}\frac{x^{\xi+mk}}{J_{\nu,q}^{\mu,\gamma}(-px^r)} M_{q(m+n)}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda) \tag{2.10}$$

Substituting the value of equation (2.10) in equation (2.7) and using equation (2.8), we obtain

$$\sum_{n=0}^{\infty}\frac{(m+n!)}{n!}\frac{x^{\xi+kn}}{J_{\nu,q}^{\mu,\gamma}(-px^r)} M_{q(m+n)}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)t^n$$

$$= (1+kt)^{-1+\frac{\xi+\lambda}{k}} m!\frac{x^{\xi+kn}}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1+kt)^{\frac{1}{k}}\}^r]} M_{qm}^{(\mu,\nu,\gamma,\xi)}\left(\mathrm{x}(1+kz)^{\frac{1}{k}};r,p,k,\lambda\right)$$

which gives the generating relation (2.6).

The theorem by H.M. Srivastava on mixed generating functions (cf. [8], p. 378, Th. 12), plays an important role to derive several other generating relation for the polynomial $M_{qn}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)$ defined by (1.5).

Theorem 2.1: *Let each of the functions $A(z)$,$B_j(z)(j=1,\ldots,m)$, and $z^{-1}C_l(z)(l=1,\ldots,s)$ be analytic in a neighborhood of the origin, and assume that*

$$A(0).B(0).C'_l(0) \neq 0 (j=1,\ldots,m; l=1,\ldots,s). \tag{2.11}$$

Define the sequence of function

$$\left\{g_n^{(\alpha_1,\ldots,\alpha_m)}(x_1,\ldots,x_s)\right\}_{n=0}^{\infty}$$

by

$$A(z)\prod_{j=1}^{m}\{[B_j(z)]^{\alpha_j}\}.\exp\left(\sum_{l=1}^{s}x_l C_l(z)\right) = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\ldots,\alpha_m)}(x_1,\ldots,x_s)\frac{z^n}{n!} \tag{2.12}$$

where $\alpha_1,\ldots,\alpha_m$ and $x_1,\ldots,x_s$ are arbitrary complex numbers independent of z.

Then, for arbitrary parameters $\lambda_1,\ldots,\lambda_m$ and $y_1,\ldots,y_s$ independent of z,

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n,\ldots,\alpha_m+\lambda_m n)}(x_1+ny_1,\ldots,x_s+ny_s)\frac{t^n}{n!} = \frac{A(\zeta)\prod_{j=1}^{m}\{[B_j(\zeta)]^{\alpha_j}\}.\exp(\sum_{l=1}^{s}x_l C_l(\zeta))}{1-\zeta\left(\sum_{j=1}^{m}\lambda_j[B'_j(\zeta)/B_j(\zeta)]+\sum_{l=1}^{s}y_l C'_l(\zeta)\right)}$$

$$\left(\zeta = t\prod_{j=1}^{m}\{[B_j(\zeta)]^{\lambda_j}\}\exp\left(\sum_{l=1}^{s}y_l C_l(\zeta)\right); m,s \in N\right) \tag{2.13}$$

By putting $\mu=1,\nu=0,\gamma=1$ and $q=1$ in the generating relation (2.3), reduces into the form

$$\sum_{n=0}^{\infty} M_n^{(1,0,1,\xi)}(x;r,p,k,\lambda)t^n = (1-kt)^{-\frac{\xi+\lambda}{k}}\exp\left[px^r\{1-(1-kt)^{-\frac{r}{k}}\}\right] \tag{2.14}$$

The generating function (2.14) is equivalently of the type (2.12), with of course,

$m-1=s=1, A(z)=1, B_1(z)=B_2(z)=(1-kt)^{-1/k}, x_1=p,\ C_1(z)=px^r[1-(1-kt)^{-\frac{r}{k}}$ and $g_n^{(\xi,\lambda)}(p) = n!\,M_n^{(1,0,1,\xi)}(x;r,p,k,\lambda); (n\in N)$.

Thus applying the above theorem to the generating function (2.14), one obtains

$$\sum_{n=0}^{\infty} M_n^{(1,0,1,\xi+\delta n)}(x;r,p+un,k,\lambda+vn)t^n = \frac{(1-k\zeta)^{-(\xi+\lambda)/k}\exp\left[px^r\{1-(1-k\zeta)^{-r/k}\}\right]}{1-\zeta(1-k\zeta)^{-1}\{\delta+v-urx^r(1-k\zeta)^{-r/k}\}} \tag{2.15}$$

where,

$$\zeta = t(1-k\zeta)^{-(\delta+v)/k}\exp\left[ux^r\{1-(1-k\zeta)^{-r/k}\}\right]; k\neq 0$$

By setting ζ by $\frac{\zeta}{k}$ $(k\neq 0)$, in the generating relation (2.15), one obtains

$$\sum_{n=0}^{\infty} M_n^{(1,0,1,\xi+\delta n)}(x;r,p+un,k,\lambda+vn)t^n = \frac{(1-\zeta)^{-(\xi+\lambda)/k}\exp\left[px^r\{1-(1-\zeta)^{-r/k}\}\right]}{1-k^{-1}\zeta(1-\zeta)^{-1}\{\delta+v-urx^r(1-\zeta)^{-r/k}\}} \tag{2.16}$$

where,

$$\zeta = kt(1-\zeta)^{-(\delta+v)/k}\exp\left[ux^r\{1-(1-\zeta)^{-r/k}\}\right]; k\neq 0$$

By replacing ζ by $\frac{\zeta}{1+\zeta}$ in the generating function (2.16), one immediately obtains the generating function

$$\sum_{n=0}^{\infty} M_n^{(1,0,1,\xi+\delta n)}(x;r,p+un,k,\lambda+vn)t^n = \frac{(1+\zeta)^{(\xi+\lambda)/k}\exp[px^r\{1-(1+\zeta)^{r/k}\}]}{1-k^{-1}\zeta\{\delta+v-urx^r(1+\zeta)^{r/k}\}} \quad (2.17)$$

where,

$$\zeta = kt(1+\zeta)^{1+(\delta+v)/k}\exp[ux^r\{1-(1+\zeta)^{r/k}\}]; k \neq 0$$

Again by setting $\delta = -v-k$ and $u = 0$ in (2.17), one obtains

$$\sum_{n=0}^{\infty} M_n^{(1,0,1,\xi-(v+k)n)}(x;r,p,k,\lambda+vn)t^n = (1+kt)^{-1+\frac{\xi+\lambda}{k}}\exp\left[px^r\{1-(1+kt)^{\frac{r}{k}}\}\right] \quad (2.18)$$

$$(|t| < |k|^{-1}; k \neq 0)$$

when, $v = 0$, (2.18) readily gives us

$$\sum_{n=0}^{\infty} M_n^{(1,0,1,\xi-kn)}(x;r,p,k,\lambda)t^n = (1+kt)^{-1+\frac{\xi+\lambda}{k}}\exp\left[px^r\{1-(1+kt)^{\frac{r}{k}}\}\right] \quad (2.19)$$

$$(|t| < |k|^{-1}; k \neq 0)$$

The generating relation obtained (2.19) is the particular case of (2.4) at $\mu = 1, v = 0, \gamma = 1$ and $q = 1$.

With the help of (1.8), the generating relation (2.15), (2.16), (2.17), (2.18), and (2.19) reduce to the generating function (2.16), (2.19), (2.22), (2.24) and (2.51) of Chen et al. [3].

III. Bilateral Generating Functions

A class of function $\{S_n(x), n = 0,1,2,\dots\}$ (see [8], p. 411) generated by

$$\sum_{n=0}^{\infty} A_{m,n}S_{m+n}t^n = f(x,t)\{g(x,t)\}^{-m}S_m(h(x,t)) \quad (3.1)$$

where $m \geq 0$ is an integer, the coefficient $A_{m,n}$ are arbitrary constant and f, g, h are suitable functions of x and t.

Numerous classes of bilinear (or bilateral) generating functions were obtained for the function $S_n(x)$ generated by (3.1).

Theorem 3.1: (cf. [8], p. 412, Th. 13) *For the sequence $\{S_n(x)\}$ generated by (3.1), let*

$$F(x,t) = \sum_{n=0}^{\infty} a_n S_n(x)t^n \quad (3.2)$$

where $a_n \neq 0$ are arbitrary constant,

Then,

$$f(x,t)F\left[h(x,t), \frac{yt}{g(x,t)}\right] = \sum_{n=0}^{\infty} S_n(x)\sigma_n(y)t^n \quad (3.3)$$

where $\sigma_n(y)$ is a polynomial (of degree n in y) defined by

$$\sigma_n(y) = \sum_{k=0}^{n} a_k A_{k,n-k}y^k \quad (3.4)$$

By using the theorem 3.1, we obtain the bilateral generating function for the polynomial (1.5) as

$$\sum_{n=0}^{\infty} M_{qn}^{(\mu,v,\gamma,\xi)}(x;r,p,k,\lambda)\sigma_n(y)t^n$$

$$= (1-kt)^{-\frac{\xi+\lambda}{k}} \frac{J_{v,q}^{\mu,\gamma}(-px^r)}{J_{v,q}^{\mu,\gamma}[-p\{x(1-kt)^{-\frac{1}{k}}\}^r]} F\left[x(1-kt)^{-\frac{1}{k}}, yt(1-kt)^{-1}\right] \quad (3.5)$$

where,

$$F(x,t) = \sum_{n=0}^{\infty} (a)_n M_{qn}^{(\mu,v,\gamma,\xi)}(x;r,p,k,\lambda)t^n$$

Proof of (3.5): With suitable replacement in theorem 3.1, we obtain

$$\sum_{n=0}^{\infty} M_{qn}^{(\mu,v,\gamma,\xi)}(x;r,p,k,\lambda)\sigma_n(y)t^n = \sum_{n=0}^{\infty} M_{qn}^{(\mu,v,\gamma,\xi)}(x;r,p,k,\lambda)\sum_{m=0}^{n}\binom{n}{m}(a)_m(y)^m t^n$$

$$= \sum_{m=0}^{\infty} (a)_m (yt)^m \sum_{n=0}^{\infty} \binom{m+n}{m} M_{q(m+n)}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda) t^n$$

Making use of generating relation (2.5), yields the bilateral generating function (3.5).

Particular cases of 3.5:

(i) If we put $\mu = 1, \nu = 0, \gamma = 1, \lambda = 0, q = 1, \xi = \alpha$ and $a = \mu$ in (3.5), which in conjunction with the generating function (cf. [1]),

$$\sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\alpha)}(x,r,p,k) t^n = (1-kt)^{-m-\frac{\alpha}{k}} \exp\left[px^r\{1-(1-kt)^{-\frac{r}{k}}\}\right] G_m^{(\alpha)}\left\{x(1-kt)^{-\frac{1}{k}}, r, p, k\right\} \quad (3.6)$$

and the use of (1.7), we get bilinear generating function obtained by Srivastava and Singhal [1],

$$\sum_{n=0}^{\infty} n!\, G_n^{(\alpha)}(x,r,p,k) \sigma_n(y) t^n = (1-kt)^{-\frac{\alpha}{k}} \exp\left[px^r\{1-(1-kt)^{-\frac{r}{k}}\}\right] F\left[x(1-kt)^{-\frac{1}{k}}, yt(1-kt)^{-1}\right] \quad (3.7)$$

where,

$$F(x,t) = \sum_{n=0}^{\infty} (\mu)_n G_n^{(\alpha)}(x,r,p,k) t^n$$

(ii) Again, putting $\mu = \alpha, \nu = \beta - 1, px^r = p_k(x), \xi = \delta, k = a, \lambda = s$ and $a = \mu$ in (3.5), which in conjunction with the generating function (cf. [4]; p. 26. Eq. (1.6))

$$\sum_{m=0}^{\infty} \binom{m+n}{n} A_{q(m+n)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s) t^m$$

$$= (1-at)^{-n-\frac{\delta+s}{a}} \frac{E_{\alpha,\beta}^{\gamma,q}\{p_k(x)\}}{E_{\alpha,\beta}^{\gamma,q}\left[p_k\{x(1-at)^{-\frac{1}{a}}\}\right]} A_{qn}^{(\alpha,\beta,\gamma,\delta)}\left\{x(1-at)^{-\frac{1}{a}}; a,k,s\right\} \quad (3.8)$$

and use of (1.9), we obtain a new bilateral generating function for the polynomial defined by Shukla and Prajapati [4] as

$$\sum_{n=0}^{\infty} A_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s) \sigma_n(y) t^n$$

$$= (1-at)^{-\frac{\delta+s}{a}} \frac{E_{\alpha,\beta}^{\gamma,q}\{p_k(x)\}}{E_{\alpha,\beta}^{\gamma,q}\left[p_k\{x(1-at)^{-\frac{1}{a}}\}\right]} F\left[x(1-at)^{-\frac{1}{a}}, yt(1-at)^{-1}\right] \quad (3.9)$$

where,

$$F(x,t) = \sum_{n=0}^{\infty} (\mu)_n A_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s) t^n$$

In 1990, Hubble and Srivastava [9] generalize the theorem.

Theorem 3.2: *Corresponding to the function $S_n(x)$, generated by (3.1), let*

$$\Theta_N[x,y,t] = \sum_{n=0}^{\infty} a_n S_n(x) L_N^{(\lambda+n)}(y) t^n, (a_n \neq 0) \quad (3.10)$$

where λ is an arbitrary (real or complex) parameter. Suppose also that

$$\theta_{m,n}(z,\omega,x) = \sum_{k=0}^{\min\{m,n\}} \frac{(-1)^k}{k} a_{m-k} A_{m-k,n-k} z^{m-k} \omega^{n-k} S_{m+n-2k}(x) \quad (3.11)$$

Then,

$$\sum_{m,n=0}^{\infty} \theta_{m,n}(z,\omega,x) L_N^{(\lambda+m)}(y) t^m = \exp(-t) f(x,\omega) \Theta_N\left[h(x,\omega), y+t, \frac{zt}{g(x,\omega)}\right] \quad (3.12)$$

Provided that each member exists.

In view of the equation (2.5) and (2.6) and the well known identity (cf.[10], p. 142, Eq. (18); see also [8], p.172, prob.22(ii))

$$e^{-t} L_N^{(\alpha)}(x+t) = \sum_{n=0}^{\infty} L_N^{(\alpha+n)}(x) \frac{(-t)^n}{n!} \quad (3.13)$$

which follows immediately from the Taylor's expansion, since

$$D_x^n\{e^{-x}L_N^{(\alpha)}(x)\} = (-1)^n e^{-x} L_N^{(\alpha+n)}(x) \tag{3.14}$$

Thus, above theorem 3.2 yields,

Corollary 3.2.1: If

$$\Phi_N[x,y,t] = \sum_{n=0}^{\infty} a_n M_{qn}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda) L_N^{(\lambda+n)}(y)t^n, \quad (a_n \neq 0) \tag{3.15}$$

and

$$\Psi_{m,n}(z,\omega,x) = \sum_{k=0}^{\min\{m,n\}} \frac{(-1)^k}{k!}\binom{m+n-2k}{n-k} a_{m-k} Z^{m-k}\omega^{n-k} M_{q(m+n-2k)}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda), \tag{3.16}$$

then,

$$\sum_{m,n=0}^{\infty} \Psi_{m,n}(z,\omega,x) L_N^{(\lambda+m)}(y)t^m$$

$$= (1-k\omega)^{-\frac{\xi+\lambda}{k}} e^{-t} \frac{J_{\nu,q}^{\mu,\gamma}(-px^r)}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1-k\omega)^{-\frac{1}{k}}\}^r]} \Phi_N\left[x\left(1-k\omega\right)^{-\frac{1}{k}}, y+t, \frac{zt}{1-k\omega}\right)\Big], \qquad (k \neq 0). \tag{3.17}$$

Corollary 3.2.2: If

$$\Xi_N[x,y,t] = \sum_{n=0}^{\infty} a_n M_{qn}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda) L_N^{(\lambda+n)}(y)t^n, \qquad (a_n \neq 0) \tag{3.18}$$

and

$$\Lambda_{m,n}(z,\omega,x) = \sum_{k=0}^{\min\{m,n\}} \frac{(-1)^k}{k!}\binom{m+n-2k}{n-k} a_{m-k} Z^{m-k}\omega^{n-k} M_{q(m+n-2k)}^{(\mu,\nu,\gamma,\xi-k(m+n-2k))}(x;r,p,k,\lambda), \tag{3.19}$$

then,

$$\sum_{m,n=0}^{\infty} \Lambda_{m,n}(z,\omega,x) L_N^{(\lambda+m)}(y)t^m$$

$$= (1+k\omega)^{-1+(\xi+\lambda)/k} e^{-t} \frac{J_{\nu,q}^{\mu,\gamma}(-px^r)}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1+k\omega)^{1/k}\}^r]} \Xi_N\left[x(1+k\omega)^{\frac{1}{k}}, y+t, \frac{zt}{1+k\omega})\right], \qquad (k \neq 0). \tag{3.20}$$

Particular cases of Corollary 3.2.1

(i) By using (1.7), which in conjunction with (3.6) yields the result obtained by Hubble and Srivastava [9],

$$\sum_{m,n=0}^{\infty} \Psi_{m,n}(z,\omega,x) L_N^{(\lambda+m)}(y)t^m$$

$$= (1-k\omega)^{-\alpha/k} \exp\left[px^r\{1-(1-k\omega)^{-r/k}\} - t\right] \Phi_N\left[x(1-k\omega)^{-1/k}, y+t, zt/(1-k\omega)\right]. \tag{3.21}$$

(ii) By using (1.8), which in conjunction with the generating function (cf. [3]; p. 348, eq. (2.8)),

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \tau_{m+n}^{(\alpha)}(x;r,\beta,k,\eta)t^n$$

$$= (1-kt)^{-m-(\alpha+\eta)/k} \exp\left[\beta x^r\{1-(1-kt)^{-r/k}\}\right] \tau_m^{(\alpha)}\{x(1-kt)^{-\frac{1}{k}}; r,\beta,k,\eta\}, \tag{3.22}$$

we get

$$\sum_{m,n=0}^{\infty} \Psi_{m,n}(z,\omega,x) L_N^{(\lambda+m)}(y)t^m$$

$$= (1-k\omega)^{-(\alpha+\eta)/k} \exp\left[\beta x^r\{1-(1-k\omega)^{-r/k}\} - t\right] \Phi_N\left[x(1-k\omega)^{-1/k}, y+t, zt/(1-k\omega)\right]. \tag{3.23}$$

(iii) By using (1.9), which in conjunction with (3.8), yields the generating relation

$$\sum_{m,n=0}^{\infty} \Psi_{m,n}(z,\omega,x) L_N^{(\lambda+m)}(y)t^m$$

$$= (1-a\omega)^{-(\delta+s)/a} e^{-t} \frac{E_{\alpha,\beta}^{\gamma,q}\{p_k(x)\}}{E_{\alpha,\beta}^{\gamma,q}[p_k\{x(1-a\omega)^{-1/a}\}]} \Phi_N\left[x(1-a\omega)^{-1/a}, y+t, zt/(1-a\omega)\right] \tag{3.24}$$

Particular cases of Corollary 3.2.2:

(i) By using (1.7), which in conjunction with the generating function (cf. [1]; p. 239)

$$\sum_{n=0}^{\infty}\binom{m+n}{n}G_{m+n}^{(\alpha-kn)}(x,r,p,k)t^n=(1+kt)^{-1+\frac{\alpha}{k}}\exp\left[px^r\{1-(1+kt)^{\frac{r}{k}}\}\right]G_m^{(\alpha)}\{x(1+kt)^{\frac{1}{k}},r,p,k\},\quad(3.25)$$

we get the result obtained by Hubble and Srivastava [9],

$$\sum_{m,n=0}^{\infty}\Lambda_{m,n}(z,\omega,x)L_N^{(\lambda+m)}(y)t^m$$
$$=(1+k\omega)^{-1+\alpha/k}\exp\left[px^r\{1-(1+k\omega)^{r/k}\}-t\right]\Xi_N\left[x(1+k\omega)^{1/k},y+t,zt/(1+k\omega)\right]\quad(3.26)$$

(ii) By using (1.8), which in conjunction with the generating function ([3]; p. 359),

$$\sum_{n=0}^{\infty}\binom{m+n}{n}\tau_{m+n}^{(\alpha-kn)}(x;r,\beta,k,\eta)t^n$$
$$=(1+kt)^{-1+(\alpha+\eta)/k}\exp\left[\beta x^r\{1-(1+kt)^{r/k}\}\right]\tau_m^{(\alpha)}\{x(1+kt)^{\frac{1}{k}};r,\beta,k,\eta\},\quad(3.27)$$

we obtain,

$$\sum_{m,n=0}^{\infty}\Lambda_{m,n}(z,\omega,x)L_N^{(\lambda+m)}(y)t^m$$
$$=(1+k\omega)^{-1+(\alpha+\eta)/k}\exp\left[\beta x^r\{1-(1+k\omega)^{r/k}\}-t\right]\Xi_N\left[x(1+k\omega)^{1/k},y+t,zt/(1+k\omega)\right].\quad(3.28)$$

(iii) By using (1.9), which in conjunction with the generating function ([2]; p. 26),

$$\sum_{n=0}^{\infty}\binom{m+n}{n}A_{q(m+n)}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)t^n$$
$$=(1+at)^{-1+\frac{\delta+s}{a}}\frac{E_{\alpha,\beta}^{\gamma,q}\{p_k(x)\}}{E_{\alpha,\beta}^{\gamma,q}\left[p_k\{x(1+at)^{\frac{1}{a}}\}\right]}A_{qm}^{(\alpha,\beta,\gamma,\delta)}\left\{x(1+at)^{\frac{1}{a}};a,k,s\right\},\quad(3.29)$$

we obtain,

$$\sum_{m,n=0}^{\infty}\Lambda_{m,n}(z,\omega,x)L_N^{(\lambda+m)}(y)t^m$$
$$=(1+a\omega)^{-1+(\delta+s)/a}e^{-t}\frac{E_{\alpha,\beta}^{\gamma,q}\{p_k(x)\}}{E_{\alpha,\beta}^{\gamma,q}[p_k\{x(1+a\omega)^{1/a}\}]}\Xi_N\left[x(1+a\omega)^{1/a},y+t,zt/(1+a\omega)\right].\quad(3.30)$$

The results (3.23), (3.24), (3.28), and (3.30) are believed to be new.

Further we use to recall here the theorem of (see [4]) to obtain another bilateral generating relation as follows:

Theorem 3.3: *If sequence $\{\Delta_\mu(x):\mu$ is a complex number$\}$ is generated by*

$$\sum_{n=0}^{\infty}\gamma_{\mu,n}\Delta_{\mu+n}(x)t^n=\theta(x,t)\{\phi(x,t)\}^{-\mu}\Delta_\mu(\psi(x,t)).\quad(3.31)$$

where $\gamma_{\mu,n}$ are arbitrary constants and θ,ϕ and ψ are arbitrary functions of x and t.

Let

$$\Phi_{q,v}[x,t]=\sum_{n=0}^{\infty}\delta_{v,n}\Delta_{v+qn}(x)t^n,\quad\delta_{v,n}\neq0\quad(3.32)$$

q is a positive integer and v is an arbitrary complex number, then

$$\sum_{n=0}^{\infty}\Delta_{\mu+n}(x)R_{n,v}^q(y)t^n=\theta(x,t)\{\phi(x,t)\}^{-\mu}\Phi_{q,v}\left[\psi(x,t),y\left\{\frac{t}{\phi(x,t)}\right\}^q\right]\quad(3.33)$$

where $R_{n,v}^q(y)$ is a polynomial of degree $[n/q]$ in y, which is defined as

$$R_{n,v}^q(y)=\sum_{k=0}^{\left[\frac{n}{q}\right]}\gamma_{v+qk,n-qk}\delta_{v,k}y^k\quad(3.34)$$

In view of the relation (2.5), if $\mu=m,\gamma=\binom{m+n}{n}$, $\Delta_m(\psi(x,t)=M_{qm}^{(\mu,\nu,\gamma,\xi)}\{x(1-kt)^{-\frac{1}{k}};r,p,k,\lambda\}$, $\phi(x,t)=1,\theta(x,t)=(1-kt)^{-m-(\xi+\lambda)/k}J_{v,q}^{\mu,\gamma}(-px^r)/J_{v,q}^{\mu,\gamma}[-p\{x(1-kt)^{-1/k}\}^r]$and $\psi(x,t)=x(1-kt)^{-1/k}$.

Then,

$$\sum_{n=0}^{\infty} M_{q(v+n)}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)R_{n,v}^{b}(y)t^{n}$$

$$= (1-kt)^{-m-\frac{\xi+\lambda}{k}} \frac{J_{\nu,q}^{\mu,\gamma}(-px^{r})}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1-kt)^{-\frac{1}{k}}\}^{r}]} \Phi_{b,y}\left[x(1-kt)^{-\frac{1}{k}}, yt^{b}\right] \tag{3.35}$$

where,

$$\Phi_{b,v}[x,t] = \sum_{n=0} \delta_{v,n} M_{q(v+n)}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)t^{n}, \left(\delta_{v,n} \neq 0\right)$$

$$R_{n,v}^{b}(y) = \sum_{k=0}^{[n/b]} \binom{v+n}{v+bk} \delta_{v,k} y^{k},$$

is a polynomial of degree $[n/b]$ in y and b is a positive integer and v is an arbitrary complex number.

IV. Generating Functions Involving Stirling Number of Second Kind

In his work Riodan [11], denoted a Stirling number of second kind by $S(n,k)$ and is defined as

$$S(n,k) = \frac{1}{k!}\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^{n} \tag{4.1}$$

so that,

$$S(n,0) = \begin{cases} 1, n=0 \\ 0, n \in N \end{cases} \tag{4.2}$$

$$S(n,1) = S(n,n) = 1 \text{ and } S(n,n-1) = \binom{n}{2}. \tag{4.3}$$

Recently, several authors have developed a number of families of generating function associated with Stirling number of second kind $S(n,k)$ defined by (4.1).

To derive the generating function for $M_{qn}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)$ defined by (1.5), we use the theorem of Srivastava [12] as

Theorem 4.1: *Let the sequence $\{\xi_n(x)\}_{n=0}^{\infty}$ be generated by*

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \xi_{n+k}(x)t^{k} = f(x,t)\{g(x,t)\}^{-n} \xi_{n}\left(h(x,t)\right) \tag{4.4}$$

where f, g and h are suitable functions of x and t.

Then, in terms of the Stirling number $S(n,k)$ defined by equation (4.1), the following family of generating function

$$\sum_{k=0}^{\infty} k^{n} \xi_{k}(h(x,-z)) \left(\frac{z}{g(x,-z)}\right)^{k} = \{f(x,-z)\}^{-1} \sum_{k=0}^{n} k!\, S(n,k) \xi(x) z^{k} \tag{4.5}$$

holds true provided that each member of equation (4.5) exists.

The generating function (2.5) and (2.6) relates to the family given by (4.4). Now by comparing (2.5) and (4.4), it is easily observed that

$$f(x,t) = (1-kt)^{-m-(\xi+\lambda)/k} \frac{J_{\nu,q}^{\mu,\gamma}(-px^{r})}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1-kt)^{-1/k}\}^{r}]}$$

$$g(x,t) = (1-kt), h(x,t) = x(1-kt)^{-\frac{1}{k}}$$

and

$$\xi_{k}(x) = M_{qk}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)$$

Then the equation (4.5) of theorem 4.1, yields the generating function

$$\sum_{k=0}^{\infty} M_{qk}^{(\mu,\nu,\gamma,\xi)}(x(1+kz)^{-1/k};r,p,k,\lambda) \left(\frac{z}{1+kz}\right)^{k}$$

$$= (1+kz)^{(\xi+\lambda)/k} \frac{J_{\nu,q}^{\mu,\gamma}[-p\{x(1+kz)^{-1/k}\}^{r}]}{J_{\nu,q}^{\mu,\gamma}(-px^{r})} \sum_{k=0}^{n} k!\, S(n,k) M_{qk}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda) z^{k} \tag{4.6}$$

$$(n \in N_0; |z| < |k|^{-1}; k \neq 0)$$

Replacing z by $\frac{z}{1-kz}$ and x by $\frac{x}{(1-kz)^{1/k}}$ the above equation immediately yields,

$$\sum_{k=0}^{\infty} M_{qk}^{(\mu,\nu,\gamma,\xi)}(x;r,p,k,\lambda)z^k$$
$$= (1-kz)^{\frac{-\xi+\lambda}{k}} \frac{J_{\nu,q}^{\mu,\gamma}(-px^r)}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1-kz)^{-\frac{1}{k}}\}^r]} \sum_{k=0}^{n} k!\,S(n,k) M_{qk}^{(\mu,\nu,\gamma,\xi)}\left(\mathrm{x}(1-kz)^{-\frac{1}{k}};r,p,k,\lambda\right)\left(\frac{z}{1-kz}\right)^k \qquad (4.7)$$
$$(n \in N_0; |z| < |k|^{-1}; k \neq 0)$$

Similarly, theorem 4.1, applied to the generating function (2.6), would readily gives us the generating function

$$\sum_{k=0}^{\infty} k^n M_{qk}^{(\mu,\nu,\gamma,\xi-kn)}(x(1-kz)^{1/k};r,p,k,\lambda)\left(\frac{z}{1-kz}\right)^k$$
$$= (1-kz)^{1-(\xi+\lambda)/k} \frac{J_{\nu,q}^{\mu,\gamma}[-p\{x(1-kz)^{1/k}\}^r]}{J_{\nu,q}^{\mu,\gamma}(-px^r)} \sum_{k=0}^{n} k!\,S(n,\mathrm{k}) M_{qk}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)z^k \qquad (4.8)$$
$$(n \in N_0; |z| < |k|^{-1}; k \neq 0)$$

Thus for replacing$z$ by $\frac{z}{1+kz}$and x by $\frac{x}{(1+kz)^{1/k}}$ the above equation immediately yields,

$$\sum_{k=0}^{\infty} k^n M_{qk}^{(\mu,\nu,\gamma,\xi-kn)}(x;r,p,k,\lambda)z^k$$
$$= (1+kz)^{-1+\frac{\xi+\lambda}{k}} \frac{J_{\nu,q}^{\mu,\gamma}(-px^r)}{J_{\nu,q}^{\mu,\gamma}[-p\{x(1+kz)^{\frac{1}{k}}\}^r]} \sum_{k=0}^{n} k!\,S(n,k) M_{qk}^{(\mu,\nu,\gamma,\xi-kn)}\left(\mathrm{x}(1+kz)^{\frac{1}{k}};r,p,k,\lambda\right)\left(\frac{z}{1+kz}\right)^k \qquad (4.9)$$
$$(n \in N_0; |z| < |k|^{-1}; k \neq 0)$$

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