IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 2 Ver. I (Mar - Apr. 2015), PP 31-34
www.iosrjournals.org

Fractional-Calculus Results Pertaining to *I*-function

Rajeev Shrivastava
Government Indira Gandhi Home Science Girls College, Shahdol-484001, M. P., India

Abstract: *Fractional Calculus and generalized hypergeometric functions have contributed a lot to the theory of science and engineering. In view of importance and usefulness of fractional-calculus operators in different directions, we present a number of key results for the product of two I-functions involving the Riemann-iouville, the Weyl and such other fractional-calculus operators as those based upon the Cauchy-Goursat Integral Formula. The results discussed here can be used to investigate a wide class of new and known results.*

I. Introduction

Fractional Calculus is the field of applied mathematics that deals with the derivatives and the integrals of arbitrary orders. During the last three decades Fractional Calculus has been applied to almost every field of science, engineering and mathematics. Many applications of Fractional Calculus can be found in fluid dynamics, Stochastic dynamical system, plasma physics and controlled thermonuclear fusion, image processing, nonlinear control theory, nonlinear biological system, astrophysics, etc.

Two of the most commonly encountered tools in the theory and applications of Fractional Calculus are provided by the Riemann-Liouville operator R_z^ν $(\nu \in C)$ and the Weyl W_z^ν $(\nu \in C)$ operator which are defined by [6, see also [1, 8, 10]. In this paper, we will define these operators as follows:

$$R_z^\nu\{f\,(z)\} = \begin{cases} \dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^z (z-\xi)^{\nu-1}\, f(\xi)\, d\xi & (Re\,(\nu) > 0) \\ \dfrac{d^n}{dz^n} R_z^{\nu+n}\{f(z)\} & (-n < Re\,(\nu) \le 0\,; n \in N) \end{cases} \tag{1.1}$$

and

$$W_z^\nu\{f\,(z)\} = \begin{cases} \dfrac{1}{\Gamma(\nu)}\displaystyle\int_z^\infty (\xi-z)^{\nu-1}\, f(\xi)\, d\xi & (Re\,(\nu) > 0) \\ \dfrac{d^n}{dz^n} W_z^{\nu+n}\{f(z)\} & (-n < Re\,(\nu) \le 0\,; n \in N) \end{cases} \tag{1.2}$$

provided that the defining integrals exist, N being the set of positive integers. The following definition of a fractional differintegral of order $\nu \in C$ is based essentially upon the familiar Cauchy-Goursat Integral Formula:

Definition (Cf. [4 and 7]) If the function $f(z)$ is analytic (regular) inside and on C, where

$$C = \{C^-, C^+\}, \tag{1.3}$$

C^- is a contour along the cut joining the points z and $-\infty + i\,\Im(z)$, which starts from the point at $-\infty$, encircles the point z once contour clockwise and returns to the point at $-\infty, C^+$ is a contour along the cut joining the points z and $\infty + i\,\Im(z)$, which starts from the point at ∞, encircles the point z once contour clockwise and returns to the point at ∞.

$$f_\nu(z) = (f(z))_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_c \frac{f(\xi)}{(\xi - z)^{\nu+1}}\, d\xi, \tag{1.4}$$

$$(\nu \in C\backslash z^-\,;\; z^- = \{-1, -2. -3, \dots\})$$

and

$$f_{-n}(z) = \lim_{\nu \to -n} \{f_\nu(z)\}\,, \quad (n \in N = \{1, 2, 3, \dots\}) \tag{1.5}$$

where $\xi \ne z$,

$$-\pi \le \arg(\xi - z) \le \pi \; for\; C^-, \tag{1.6}$$

and

$$0 \le \arg(\xi - z) \le 2\pi \; for\; C^+, \tag{1.7}$$

Then $f_\nu(z)$ $(Re(\nu) > 0)$ is said to be the fractional derivative of $f(z)$ of order ν and $f_\nu(z)$ $(Re(\nu) < 0)$ is said to be the fractional integral of $f(z)$ of order $-\nu$, provided that

$$|f_\nu(z)| < \infty \qquad (\nu \in R). \tag{1.8}$$

The I-function, defined by Saxena [2], has been further studied by other workers [3 and 5]. In this paper we will define and represent the I-function in the following manner:

$$I[x] = I^{M,N}_{P_i,Q_i,R}\left[x\left|\begin{matrix}\left[(c_j,\gamma_j)_{1,N},(c_{ji},\gamma_{ji})_{N+1,P_i}\right]\\\left[(d_j,\delta_j)_{1,M},(d_{ji},\delta_{ji})_{M+1,Q_i}\right]\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_{\mathcal{L}} \theta(s)\, x^s ds, \tag{1.9}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{M}\Gamma(d_j - \delta_j s)\ \prod_{j=1}^{N}\Gamma(1 - c_j + \gamma_j s)}{\sum_{i=1}^{R}\left\{\prod_{j=M+1}^{q_i}\Gamma(1 - d_{ji} + \delta_{ji} s)\ \prod_{j=N+1}^{p_i}\Gamma(c_{ji} - \gamma_{ji} s)\right\}}, \tag{1.10}$$

where $\mathcal{L}$ is a suitable contour, $\omega = \sqrt{-1}$ and all other conditions given in literature [9].

By summing up the residues at the simple poles of the integrand of (1.9), the following expression is obtained:

$$I[x] = \sum_{k=1}^{M}\sum_{h=0}^{\infty}\left\{\frac{(-1)^h\Theta(\zeta_{k,h})}{(h)!\,\delta_k}\, x^{\zeta_{k,h}}\right\}, \tag{1.11}$$

where

$$\zeta_{k,h} = \frac{d_k + h}{\delta_k}, \quad h = 0,1,2,\cdots \tag{1.12}$$

and

$$\Theta(\zeta_{k,h}) = \frac{\theta(t)}{\Gamma(d_k - \delta_k t)}, \tag{1.13}$$

provided that the series on the right hand side of (1.3) is absolutely convergent.

II. The Main Frational Differintegral Formulas

First of all, for the Riemann-Liouville operator R^ν_z defined by (1.1), we have

$$R^\nu_z\{z^{\rho-1} I^{m,n}_{p_i,q_i,r}(xz^\sigma)\, I^{M,N}_{P_i,Q_i,R}(yz^\tau)\}$$
$$= \frac{1}{\Gamma(\nu)}\int_0^z (z-\xi)^{\nu-1}\,\xi^{\rho-1} I^{m,n}_{p_i,q_i,r}(x\xi^\sigma)\, I^{M,N}_{P_i,Q_i,R}(y\xi^\tau)\, d\xi, \qquad (Re\,(\nu) > 0). \tag{2.1}$$

Now, expressing the one I-function in series form as given by (1.11) and another I-function in terms of Melline-Barnes type of contour integral given by (1.9), interchanging the orders of summation and integration and putting $\xi = zt$ in the resulting integral, we find that

$$R^\nu_z\{z^{\rho-1} I^{m,n}_{p_i,q_i,r}(xz^\sigma)\, I^{M,N}_{P_i,Q_i,R}(yz^\tau)\} = \frac{z^{\nu+\rho-1}}{\Gamma(\nu)}\sum_{k=1}^{M}\sum_{h=0}^{\infty}\left\{\frac{(-1)^h\Theta(\zeta_{k,h})}{(h)!\,\delta_k}\, x^{\zeta_{k,h}} z^{\sigma\zeta_{k,h}}\right\}$$

$$\bullet \frac{1}{2\pi\omega}\int_{\mathcal{L}} \theta(s)\,(yz^\tau)^s ds \int_0^1 t^{\rho+\sigma\zeta_{k,h}+\tau s-1}(1-t)^{\nu-1}\, dt, \qquad (Re\,(\nu) > 0). \tag{2.2}$$

Further, we evaluate the Eulerian integral in (2.2) by applying the following integral representation for the familiar Beta function $B(\alpha,\beta)$:

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\, dt = B(\beta,\alpha)$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \{Re\,(\alpha) > 0, Re\,(\beta) > 0\}. \tag{2.3}$$

and interpret the resulting integral in (2.2) as the I-function by means of the definition (1.9), we get

$$R_z^\nu\left\{z^{\rho-1} I_{p_i,q_i,r}^{m,n}(xz^\sigma)\, I_{P_i,Q_i,R}^{M,N}(yz^\tau)\right\} = z^{\nu+\rho-1}\sum_{k=1}^{M}\sum_{h=0}^{\infty}\left\{\frac{(-1)^h\Theta(\zeta_{k,h})}{(h)!\,\delta_k}\,x^{\zeta_{k,h}}z^{\sigma\zeta_{k,h}}\right\}$$

$$\bullet\, I_{P_i+1,Q_i+1,R}^{M,N+1}\left[yz^\tau\left|\begin{matrix}(1-\rho-\sigma\zeta_{k,h},\tau),(c_j,\gamma_j)_{1,N},(c_{ji},\gamma_{ji})_{N+1,P_i}\\(d_j,\delta_j)_{1,M},(d_{ji},\delta_{ji})_{M+1,Q_i},(1-\rho-\sigma\zeta_{k,h}-\nu,\tau)\end{matrix}\right.\right], \tag{2.4}$$

$$\left(\tau>0,\sigma>0; Re(\nu)>0, Re(\rho+\sigma\zeta_{k,h})+\tau\min_{1\le j\le m}\left\{Re\left(\frac{d_j}{\delta_j}\right)\right\}>0\right),$$

provided that the series on the right hand side of (2.4) is absolutely convergent.

In precisely the same manner as mentioned above, by applying the definition (1.2) with

$$\xi = z(1+t) \text{ and } d\xi = z\,dt,$$

and evaluating the resulting infinite integral as a Beta function by means of (2.3), we find that

$$W_z^\nu\left\{z^{-\rho} I_{p_i,q_i,r}^{m,n}(xz^{-\sigma})\, I_{P_i,Q_i,R}^{M,N}(yz^{-\tau})\right\} = z^{\nu-\rho}\sum_{k=1}^{M}\sum_{h=0}^{\infty}\left\{\frac{(-1)^h\Theta(\zeta_{k,h})}{(h)!\,\delta_k}\,x^{\zeta_{k,h}}z^{-\sigma\zeta_{k,h}}\right\}$$

$$\bullet\, I_{P_i+1,Q_i+1,R}^{M,N+1}\left[yz^{-\tau}\left|\begin{matrix}(1+\nu-\rho-\sigma\zeta_{k,h},\tau),(c_j,\gamma_j)_{1,N},(c_{ji},\gamma_{ji})_{N+1,P_i}\\(d_j,\delta_j)_{1,M},(d_{ji},\delta_{ji})_{M+1,Q_i},(1-\rho-\sigma\zeta_{k,h},\tau)\end{matrix}\right.\right], \tag{2.5}$$

$$\left(\tau>0,\sigma>0; Re(\nu)>0, Re(\rho+\sigma\zeta_{k,h})+\tau\min_{1\le j\le m}\left\{Re\left(\frac{c_j-1}{\gamma_j}\right)\right\}>0\right),$$

provided that each members of (2.5) exist.

Next, we make use of the definition (1.4) for which it is known that [6, Vol. I, p. 28]

$$(z^k)_\nu = e^{-i\pi\nu}\,\frac{\Gamma(\nu-k)}{\Gamma(-k)}\,z^{k-\nu}, \tag{2.6}$$

$$\left(k,\nu\in C;\ \left|\frac{\Gamma(\nu-k)}{\Gamma(-k)}\right|<\infty\right).$$

In this case, too, we choose to apply the series representation for the first I-function and for the second I-function in terms of Mellin-Barnes type of contour integral given by (1.9), and we thus obtain

$$\left(z^{-\rho} I_{p_i,q_i,r}^{m,n}(xz^{-\sigma})\, I_{P_i,Q_i,R}^{M,N}(yz^{-\tau})\right)_\nu = z^{-\nu-\rho}e^{-i\pi\nu}\sum_{k=1}^{M}\sum_{h=0}^{\infty}\left\{\frac{(-1)^h\Theta(\zeta_{k,h})}{(h)!\,\delta_k}\,x^{\zeta_{k,h}}z^{-\sigma\zeta_{k,h}}\right\}$$

$$\bullet\, I_{P_i+1,Q_i+1,R}^{M,N+1}\left[yz^{-\tau}\left|\begin{matrix}(1-\nu-\rho-\sigma\zeta_{k,h},\tau),(c_j,\gamma_j)_{1,N},(c_{ji},\gamma_{ji})_{N+1,P_i}\\(d_j,\delta_j)_{1,M},(d_{ji},\delta_{ji})_{M+1,Q_i},(1-\rho-\sigma\zeta_{k,h},\tau)\end{matrix}\right.\right], \tag{2.7}$$

$$(\tau>0,\sigma>0),$$

provided that each members of (2.7) exist.

III. Special Cases

(i) Taking $x=0, R=1$ in (2.4), we arrive at the following results involving Fox's H-function

$$R_z^\nu\left\{z^{\rho-1}\, H_{P,Q}^{M,N}(yz^\tau)\right\} = z^{\nu+\rho-1}H_{P+1,Q+1}^{M,N+1}\left[yz^\tau\left|\begin{matrix}(1-\rho,\tau),(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q},(1-\rho-\nu,\tau)\end{matrix}\right.\right], \tag{3.1}$$

$$\left(\tau>0, Re(\nu)>0, Re(\rho)+\tau\min_{1\le j\le m}\left\{Re\left(\frac{d_j}{\delta_j}\right)\right\}>0\right),$$

provided that each members of (3.1) exist.

(ii) Taking $x=0, R=1$ in (2.5), we arrive at the following results involving Fox's H-function

$$W_z^\nu\left\{z^{-\rho}\, H_{P,Q}^{M,N}(yz^{-\tau})\right\} = z^{\nu-\rho}H_{P+1,Q+1}^{M,N+1}\left[yz^{-\tau}\left|\begin{matrix}(1+\nu-\rho,\tau),(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q},(1-\rho,\tau)\end{matrix}\right.\right], \tag{3.2}$$

$$\left(\tau>0, Re(\rho)+\tau\min_{1\le j\le m}\left\{Re\left(\frac{c_j-1}{\gamma_j}\right)\right\}>Re(\nu)>0\right),$$

provided that each members of (3.2) exist.

(iii) Taking $x = 0, R = 1$ in (2.7), we arrive at the following results involving Fox's H-function

$$\left(z^{-\rho}\, H_{P,Q}^{M,N}(yz^{-\tau})\right)_{\nu} = z^{-\nu-\rho}\, e^{-i\pi\nu}\, H_{P+1,Q+1}^{M,N+1}\left[yz^{-\tau} \left| \begin{matrix} (1-\nu-\rho,\tau), (c_j,\gamma_j)_{1,P} \\ (d_j,\delta_j)_{1,Q}, (1-\rho,\tau) \end{matrix} \right. \right], \quad (3.3)$$

$$(\tau > 0, \sigma > 0),$$

provided that each members of (3.3) exist.

IV. Conclusion

The importance of our result lies in their manifold generality. In view of the generality of the I-function, on specializing the various parameters, we can obtain from our results, several results involving a remarkable wide variety of useful functions, which are expressible in terms of Fox's H-function, Meijer's G-function etc. and their special cases. Thus, the result established in this paper would at once yield a very large number of results involving a large variety of special functions occurring in the problem of science, engineering and mathematics.

References

Journal Papers:

[1] Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations to Methods of their Solutions and Some of Their Applications, Mathematics in Science and Engineering 198, Academic Press, New York, (1999).

[2] Saxena, V.P.: Proc. Nat. Acad. Sci. India 52A (1982): 366.

[3] Sharma, C.K. and Shrivastava, S.: Proc. Nat. Acad. Sci. India, 62 (A) (1992) : 235.

[4] Srivastava, H.M., Owa, S. and Nishimoto, K.: Some Fractional Differential Equations, J. Math. Anal. Appl. 106, (1985), 360-366.

[5] Vaishya, G.D., Jain, R. & Verma, R.C.: Proc. Nat. Acad. Sci. India 59A (1989): 329.

Books:

[6] Erdelyi, A., Magnus, W., Oberhittinger, F. and Tricomi, F.G.: Tables of Integral Transforms, Vol. II, McGraw-Hill Book Company, New York, (1954).

[7] Nishimoto, K.: Fractional Calculus, Vol. I-IV, Descartes Press, Koriyama, (1996).

[8] Samko, S.G., Kilbas, A.A. and Marichev, O.I.: Integrals and Derivatives of Fractional order and some of their Applications, Gordon and Breach Science publisher, (1993).

[9] Saxena, V.P.: The I-function. Anamaya Publishers, New Delhi (2008).

[10] Srivastava, H.M. and Buchman, R.G. : Theory and Applications of Convolution Integral Equations, Kluwer series on Mathematics and its Applications 79, Kluwer Academic Publishers, Dordrecht, (1992).