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Pseudospherical Planes and Evolution Equations in Higher Dimensions II

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Abstract : *In this paper, the study of evolution equations with two independent variables which are related to pseudospherical surfaces in R^3, is extended to evolution equations with more than two independent variables. Equations of the type*

$$u_t = \psi(u, u_x, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}})$$

are studied and characterized. Some features and results on properties of these equations are given via this study.
Keywords: *Evolution equations, Pseudospherical surfaces, Riemannian manifold, Solitons and* differential equations.

I. Introduction

As well known now, he study of non-linear evolution equations has been closely related to the study of soliton phenomena. The traced properties of 2-dimensional (one spatial variable and the time variable) soliton equations: such as having Bäcklund transformation $[2-10, 14, 21, 26]$being solvable by the inverse scattering method $[1, 7]$, having infinite number of conservation laws $[7, 22]$, satisfying the painleve $[9]$ and describing Pseudospherical surfaces$[5,12,15-21]$; have been under extensivem studies till now. The interrelations between these properties also are well established, $[3, 11, 14, 19]$. However, in higher dimensions, the studies of solitons $[13, 23-26]$, accordingly of the non- linear evolution equations with two or more spatial variables are less developed and remain one of the interesting present and future field of studies .In fact the studies in higher dimensions take the form of studying each single traced property in itself, then searching for relations to other properties, $[7, 9, 19, 22]$. From geometric point of view, the properties of describing Pseudospherical plane as well as Bäcklund transformations interest us more. In a previous study, $[12]$, EI- Sabbagh et al showed what necessary and sufficient conditions for the evolution equations

$$u_{xt} = \psi(u, u_x, u_{xx}, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, u_{yy}, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}})$$

To describe a two-parameter3-dimensional Pseudospherical plane (P.S.P) in R^5, (i.e 3-dim plane with constant sectional curvature -1 isometrically imbedded in R^5).

In the present paper, we carry a similar study for other types of evolution equations with two or more spatial variables which have the form

$$u_t = \psi(u, u_x, u_{xx}, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, u_{yy}, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}}) \qquad (1)$$

In section II, we give necessary definitions and notations to lay the ground for the main study in section III, where conditions on equations (1) to describe(η, ξ) 3-dim. P.S.P in R^5 are given. Some features and comments are also given.

II. Basic Notations And Definitons

For the present paper to be self contained, we give a simple review of needed geometry. Let M be an n-dimensional Riemannian manifold with constant negative sectional curvature K isometrically imbedded in the (2n-1) Eculidian spaceR^{2n-1}.The dimension (2n-1) is the least possible dimension so that an isometric imbedding can exist[7.19]. Let $e_1, e_2, \ldots, e_{2n-1}$ be an orthonormal frame on an open set of R^{2n-1}so that at points of M,$e_1, e_2, \ldots, e_n$are tangents toM.

Let $\omega_A$be the dual orthonormal coframe and consider ω_{AB}defined by

$$de_A = \sum_B \omega_{AB}\, e_B$$

Thus for R^{2n-1} we have

$$d\omega_A = \sum_B \omega_B \wedge \omega_{BA} \quad , \quad \omega_{AB} + \omega_{BA} = 0 \qquad (2)$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} \quad \text{with} 1 \leq A, B, C \leq 2n-1 \qquad (3)$$

Restricting these forms to M we have $\omega_\alpha = 0$,so(2) gives with $n+1 \leq \alpha,\beta,\gamma \leq 2n-1$ and $1 \leq I, J, L \leq n$,

$$d\omega_\alpha = \sum_I \omega_I \wedge \omega_{I\alpha} = 0 \qquad (4)$$

$$d\omega_I = \sum_J \omega_J \wedge \omega_{JI} \quad (5)$$

from(3) we obtain, Gauss equation

$$d\omega_{IJ} = \sum_L \omega_{IL} \wedge \omega_{LJ} + \sum_\alpha \omega_{I\alpha} \wedge \omega_{\alpha J} \quad (6)$$

andCodazzi equation

$$d\omega_{I\alpha} = \sum_A \omega_{IA} \wedge \omega_{A\alpha} \quad (7)$$

Mhas constant sectional curvature K if and only if its curvature 2-forms have the form

$$\Omega_{IJ} = d\omega_{IJ} - \sum_L \omega_{IL} \wedge \omega_{LJ} = -K\, \omega_I \wedge \omega_J \quad (8)$$

The normal curvature forms of M are

$$\Omega_{\alpha\beta} = \sum_I \omega_{\alpha I} \wedge \omega_{I\beta} \quad (9)$$

While its first fundamental form is $I = \sum_I (\omega_I)^2$.

For our purpose in this paper, we write these equations when M is a 3-dimensional submanifold with constant sectional curvature $K = -1$(i.e. pseudo spherical 3-plane in R^5).

These equations take the forms

$$\left.\begin{aligned} d\omega_1 &= \omega_4 \wedge \omega_2 + \omega_5 \wedge \omega_3 \\ d\omega_2 &= -\omega_4 \wedge \omega_1 + \omega_6 \wedge \omega_3 \\ d\omega_3 &= -\omega_5 \wedge \omega_1 - \omega_6 \wedge \omega_2 \\ d\omega_4 &= \omega_1 \wedge \omega_2 \\ d\omega_5 &= \omega_1 \wedge \omega_3 \\ d\omega_6 &= \omega_2 \wedge \omega_3 \end{aligned}\right\} (10)$$

where we have written

ω4 = ω 12ω5 = ω13, and

ω6 = ω23 with $\omega_{ij} = -\omega_{ji}$, $i, j = 1,2,3$, $\omega_{ii} = 0$

We shall recall here the definition of a differential equation to describe a pseudospherical surface, introduced in [6.12] and modify it in order to suit our purposes here.

Definition 2.1.

A differential equation E- for a real function $u(x, y, t)$ describes a 3-dimensional pseudospherical plane in R^5 (simply p.s.p.) if it is the necessary and sufficient condition for the existence of differentiable functions $f_{\alpha i}$, $1 \leq \alpha \leq 6$ and $1 \leq i \leq 3$, depending on u and its derivatives, such that the 1-forms

$$\omega_\alpha = f_{\alpha 1} dx + f_{\alpha 2} dy + f_{\alpha 3} dt \qquad (11)$$

satisfy the structure equations of a 3-plane of constant sectional curvature -1 in R^5 i.e. equations(10).

Definition 2.2.

We shall define such 3-dimensional P.S.P to be a two-parameters 3-dimensional P.S.P $f_{31} = f_{41} = \eta$and $f_{22} = f_{42} = \xi$, with ηand ξ constant parameters. In Fact, one can see that when $u(x, y, t)$ is a generic solution of E, it provides a metric defined on an open subset of R3, whose sectional curvature is -1 and the lengths of the vector fields $\frac{\partial}{\partial x}$and $\frac{\partial}{\partial y}$ satisfy $\left|\frac{\partial}{\partial x}\right|2 \geq \eta^2$, $\left|\frac{\partial}{\partial y}\right|2 \geq \xi^2$.

III. Equations Of Type $\mathbf{u_t = \psi(u, u_x, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}})}$

To study equation (1).we first write

$z_0 = u$, $z_1 = u_x$, $z_2 = u_{xx}, \ldots\ldots, z_k = \frac{\partial^k u}{\partial x^k}$and

$z_{1'} = u_y, z_{2'} = u_{yy}, \dots\dots\dots, z_{k'} = \frac{\partial^{k'} u}{\partial y^{k'}}$ thus equation(1) becomes

$$z_{1,t} = \psi(z_0, z_1, \dots, z_k, z_{1'}, \dots\dots, z_{k'}) \tag{12}$$

We shall consider equation (12) with the following assumptions

$$\left.\begin{array}{l} z_{i,t} = z_{i',t} = 0 \\ z_{i',x} = z_{i,y} = 0 \quad \text{for} \quad 1 \le i \le k \;, 1 \le i' \le k' \end{array}\right\} \tag{13}$$

where the comma denotes partial differentiation with respect to the shown variable. Now consider the following ideal I of forms on the space of variables $x, y, t, z_0, z_1, \dots\dots\dots, z_k, z_{1'}, z_{2'}, \dots\dots, z_{k'}$:

$$\left.\begin{array}{l} \Omega_i = dz_i \wedge dt - z_{i+1} dx \wedge dt, \qquad 0 \le i \le k-1 \\ \Omega_{i'} = dz_{i'} \wedge dt - z_{i'+1} dy \wedge dt, \qquad 0 \le i' \le k'-1 \\ \Omega_k = dz_k \wedge dx \wedge dt + dz_{k'} \wedge dy \wedge dt \\ \Omega = dz_i \wedge dx \wedge dy - \psi\, dx \wedge dy \wedge dt \end{array}\right\} (14)$$

Note that assumptions (13) mean that u has no (xy) terms. Now, if we apply Cartan-Kahler theory, [6] for equation (12) and using the notation above we can obtain the following result which relates solutions of the differential equation (1) with integral manifolds of the ideal I formed by the forms in (14).

Proposition 3.1

The ideal I is a closed differential ideal. Moreover, if $u(x, y, t)$ is a solution of equation (1), then with the given notations, the map

$$\phi(x, y, t) = (x, y, t, z_0(x, y, t), z_1(x, y, t), \dots, z_k(x, y, t), z_{1'}(x, y, t), \dots, z_{k'}(x, y, t)) \qquad (15)$$

defines an integral manifold of I Conversely, any 3-dimensional integral manifold of I given by

$$\phi(a, b, c) = \big(x(a, b, c), y(a, b, c), t(a, b, c), z_0(a, b, c), \dots, z_{k'}(a, b, c)\big)$$

with dx, dy and dt are linearly independent, determines a. local solution of equation (12).

Proof:

It is easy to show that I is a closed differential ideal, where

$d\Omega_i = \Omega_{i+1} \wedge dx \quad \in I$, $d\Omega_{i'} = \Omega_{i'+1} \wedge dy \quad \in I$

$d\Omega_k = 0 \quad \in I$, $\quad d\Omega = -d\psi \wedge dx \wedge dy \wedge dt \quad$ and also

$$d\Omega = -\psi_{z_0} dz_0 \wedge dx \wedge dy \wedge dt - \psi_{z_1} dz_1 \wedge dx \wedge dy \wedge dt - \dots\dots - \psi_{z_k} dz_k \wedge dx \wedge dy \wedge dt - \psi_{z_{1'}} dz_{1'} \wedge dx \wedge dy \wedge dt - \cdots\cdots\cdots - \psi_{z_{k'}} dz_{k'} \wedge dx \wedge dy \wedge dt$$

From equation (14) we have

$d\Omega = -\psi_{z_0} dx \wedge dy \wedge \Omega_0 - \psi_{z_1} dx \wedge dy \wedge \Omega_1 - \cdots\cdots - \psi_{z_k} dy \wedge \Omega_k - \psi_{z_{1'}} dx \wedge dy \wedge \Omega_{1'} - \cdots\cdots - \psi_{z_{k'}} d \wedge \Omega_k \in I$ then $d\Omega \in I$

Also, suppose $u(x, y, t)$ is a solution of (1) we need to show that for ϕ defined by (15), we have $\phi^* I = 0$, where ϕ^* is the pullback map of ϕ . From the definition of ϕ one can easily see that

$$\phi^*\Omega_k = 0 \quad , \quad \phi^*\Omega = 0$$

$\phi^*\Omega_i = 0 \quad , \quad \phi^*\Omega_{i'} = 0$ then $\phi^* I = 0$

Conversely,sectioning these forms in I,one can show that the map

$$\phi: (a, b, c) \to (x(a, b, c), y(a, b, c), t(a, b, c), z_0(a, b, c), \cdots, z_{k'}(a, b, c))$$

is a solution of equation (12). i.e. if this map ϕ is an integral manifold of I such that $dx \wedge dy \wedge dt \neq 0$, then we locally have $(a, b, c) = g(x, y, t)$.

Taking ϕ as $\phi = \phi \, o g$ we get $\phi^*\Omega_i = g^*\phi^*\Omega_i = 0$

$, \phi^*\Omega_{i'} = g^*\phi^*\Omega_{i'} = 0$

Similarly $\phi^*\Omega_k = 0$ and $\phi^*\Omega = 0$ so we can write

$dz_i \wedge dt - z_{i+1} dx \wedge dt = 0, \qquad 0 \le i \le k-1$

$dz_{i'} \wedge dt - z_{i'+1} dy \wedge dt = 0 \qquad 0 \le i' \le k'-1$

and $\big(-z_{0,t} + \psi(x, y, t, z_0, z_1, \cdots\cdots, z_k, z_{1'}, \cdots, z_{k'})\big) dx \wedge dy \wedge dt = 0$

So, with $z_0 = u$, $z_1 = u_x, \cdots\cdots, z_k = \frac{\partial^k u}{\partial x^k}, z_{1'} = u_y, \ , \cdots, z_{k'} = \frac{\partial^{k'} u}{\partial y^{k'}}$

the function $u(x, y, t)$ is a solution of equation (1).

Now to characterize equation (1), we first give the following result:

Lemma 3.1

Let $z_{0,t} = \psi(z_0, z_1, \cdots\cdots, z_k, z_{1'}, \cdots\cdots, z_{k'})$,be a differential equation which describe an (η, ξ) 3-dirnensional P.S.P with the associated 1-forms $\omega_\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dy + f_{\alpha 3} dt, \alpha = 1,2,\dots,6$ where $f_{\alpha i}$ and ψ are real differentiable (C^∞) functions defined on an open connected subset $U \subset R^{k+k'+1}$ with no explicit dependence on x, y and t.Then

$$\left.\begin{array}{l} f_{11,z_i} = f_{12,z_i} = f_{21,z_i} = f_{22,z_i} = f_{51,z_i} = f_{52,z_i} = f_{61,z_i} = f_{62,z_i} = 0 \quad ,1 \le i \le k \\ f_{11,z_{k'}} = f_{13,z_{k'}} = f_{21,z_{k'}} = f_{23,z_{k'}} = f_{33,z_{k'}} = f_{43,z_{k'}} = f_{51,z_{k'}} = f_{53,z_{k'}} = f_{61,z_{k'}} = f_{63,z_{k'}} = 0 \\ f_{33,z_{k'-1}} = f_{43,z_{k'-1}} = 0 \\ f_{12,z_k} = f_{13,z_k} = f_{22,z_k} = f_{23,z_k} = f_{33,z_k} = f_{43,z_k} = f_{52,z_k} = f_{53,z_k} = f_{62,z_k} = f_{63,z_k} = 0 \\ f_{33,z_{k-1}} = f_{43,z_{k-1}} = 0 \\ f_{11,z_0}^2 + f_{12,z_0}^2 + f_{21,z_0}^2 + f_{22,z_0}^2 + f_{51,z_0}^2 + f_{52,z_0}^2 + f_{61,z_0}^2 + f_{62,z_0}^2 \neq 0 \end{array}\right\} \quad (16)$$

In U, and

$$-\sum_{i'=0}^{k'-1} z_{i'+1} f_{11,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} = \eta f_{22} - \xi f_{21} + \xi f_{51} + \eta f_{52} \quad (17)$$

$$-\psi f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{13,z_i} = \eta\, f_{23} - f_{43} f_{21} + f_{51} f_{33} - \eta f_{53} \quad (18)$$

$$-\psi f_{12,z_0} + \sum_{i=0}^{k'-1} z_{i'+1} f_{13,z_{i'}} = \xi f_{23} - f_{43} f_{22} + f_{52} f_{33} - \xi f_{53} \quad (19)$$

$$-\sum_{i'=0}^{k'-1} z_{i'+1} f_{21,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{22,z_i} = \xi f_{11} - \eta f_{12} + \xi f_{61} - \eta f_{62} \quad (20)$$

$$-\psi f_{21,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{23,z_i} = f_{11} f_{43} - \eta f_{13} + f_{61} f_{33} - \eta f_{63} \quad (21)$$

$$-\psi f_{22,z_0} + \sum_{i'=0}^{k'-1} z_{i'+1} f_{23,z_{i'}} = f_{12} f_{43} - \eta f_{13} + f_{62} f_{33} - \eta f_{63} \quad (22)$$

$f_{11} f_{52} - f_{12} f_{51} = f_{22} f_{61} - f_{21} f_{62}$ (23)

$$\sum_{i=0}^{k-2} z_{i+1} f_{33,z_i} = f_{11} f_{53} - f_{13} f_{51} + f_{21} f_{63} - f_{23} f_{61} \quad (24)$$

$$\sum_{i'=0}^{k'-2} z_{i'+1} f_{33,z_{i'}} = f_{12} f_{53} - f_{13} f_{52} + f_{22} f_{63} - f_{23} f_{62} \quad (25)$$

$$f_{11} f_{22} - f_{12} f_{21} = 0 \quad (26)$$

$$\sum_{i=0}^{k-2} z_{i+1} f_{43,z_i} = f_{11} f_{23} - f_{13} f_{21} \quad (27)$$

$$\sum_{i'=0}^{k'-2} z_{i'+1} f_{43,z_{i'}} = f_{12} f_{23} - f_{13} f_{22} \quad (28)$$

$$-\sum_{i'=0}^{k'-1} z_{i'+1} f_{51,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{52,z_i} = \xi f_{11} - \eta f_{12} \quad (29)$$

$$-\psi f_{51,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{53,z_i} = f_{11} f_{33} - \eta f_{13} \quad (30)$$

$$-\psi f_{52,z_0} + \sum_{i'=0}^{k'-1} z_{i'+1} f_{53,z_{i'}} = f_{12} f_{33} - \xi f_{13} \quad (31)$$

$$-\sum_{i'=0}^{k'-1} z_{i'+1} f_{61,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{62,z_i} = f_{21} f_{32} - f_{22} f_{31} \quad (32)$$

$$-\psi f_{61,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{63,z_i} = f_{21} f_{33} - \eta f_{23} \quad (33)$$

$$-\psi f_{62,z_0}+\sum_{i'=0}^{k'-1} z_{i'+1} f_{63,z_{i'}} = f_{22}f_{33}-\xi f_{23} \tag{34}$$

with the assumptions (13).
Proof:
In the space of variables $(x,y,t,z_0,z_1,\cdots\cdots,z_k,z_{1'},\cdots\cdots,z_{k'})$ we consider the ideal I generated by $\Omega_i,\Omega_{i'},\Omega_k$, and Ω defined by equations (14) with ψ given by equation(12) It follows from proposition (12) that $\Omega_i=\Omega_{i'}=\Omega_k=\Omega=0$,when restricted to each integral manifold of E.
Hence, for $z_0,z_1,\ldots.,z_k,z_{1'},\ldots\ldots,z_{k'}$ satisfying (12), we have

$$\left.\begin{array}{ll} dz_i\wedge dt = z_{i+1}dx\wedge dt, & i = 0,1,\ldots\ldots,k-1\\ dz_{i'}\wedge dt = z_{i'+1}dy\wedge dt, & i' = 0,1,\ldots\ldots,k'-1\\ dz_0\wedge dx\wedge dy - \psi\, dx\wedge dy\wedge dt = 0 & \end{array}\right\} \tag{35}$$

At the beginning by using assumptions (13) and (16) we have
The 1-forms ω_α satisfy the structure equations (10) therefore,

$$\sum_{i=0}^{k} f_{11,z_i}dz_i\wedge dx+\sum_{i'=0}^{k'} f_{11,z_{i'}}dz_{i'}\wedge dx+\sum_{i=0}^{k} f_{12,z_i}dz_i\wedge dy+\sum_{i'=0}^{k'} f_{12,z_{i'}}dz_{i'}\wedge dy+\sum_{i=0}^{k} f_{13,z_i}dz_i\wedge dt$$
$$+\sum_{i'=0}^{k'} f_{13,z_{i'}}dz_{i'}\wedge dt$$
$$=(\eta f_{22}-\xi f_{21}+\xi f_{51}-\eta f_{52})dx\wedge dy+(\eta f_{23}-f_{43}f_{21}+f_{51}f_{33}-\eta f_{53})dx\wedge dt$$
$$+(\xi f_{23}-f_{43}f_{22}+f_{52}f_{33}-\xi f_{53})dy\wedge dt \qquad (*)$$

From the above equation$(*)$we can obtain equation (17),(18) and (19) by simple calculations and by using equations (35)
In similar way by using assumptions (13) and (16) we have the1-forms ω_α satisfy the structure equations (10) then

$$\sum_{i=0}^{k} f_{21,z_i}dz_i\wedge dx+\sum_{i'=0}^{k'} f_{21,z_{i'}}dz_{i'}\wedge dx+\sum_{i=0}^{k} f_{22,z_i}dz_i\wedge dy$$
$$+\sum_{i'=0}^{k'} f_{22,z_{i'}}dz_{i'}\wedge dy+\sum_{i=0}^{k} f_{23,z_i}dz_i\wedge dt+\sum_{i'=0}^{k'} f_{23,z_{i'}}dz_{i'}\wedge dt$$
$$=(-\eta f_{12}+\xi f_{11}+\xi f_{61}-\eta f_{62})dx\wedge dy+(-\eta f_{13}+f_{43}f_{11}+f_{61}f_{33}-\eta f_{63})dx\wedge dt$$
$$+(-\xi f_{13}+f_{43}f_{12}+f_{62}f_{33}-\xi f_{63})dy\wedge dt \qquad (**)$$

From the above equation$(**)$we can obtain equation(20),(21) and (22) by simple calculations and by using equations (35)
In similar way by using assumptions(13) and(16) we have the 1-forms $\omega_\alpha$satisfy the structure equations(10) then

$$\sum_{i=0}^{k} f_{31,z_i}dz_i\wedge dx+\sum_{i'=0}^{k'} f_{31,z_{i'}}dz_{i'}\wedge dx+\sum_{i=0}^{k} f_{32,z_i}dz_i\wedge dy$$
$$+\sum_{i'=0}^{k'} f_{32,z_{i'}}dz_{i'}\wedge dy+\sum_{i=0}^{k} f_{33,z_i}dz_i\wedge dt+\sum_{i'=0}^{k'} f_{33,z_{i'}}dz_{i'}\wedge dt$$
$$=(-f_{51}f_{12}+f_{52}f_{11}-f_{61}f_{22}+f_{62}f_{21})dx\wedge dy+(-f_{52}f_{13}+f_{53}f_{12}-f_{62}f_{23}+f_{22}f_{63})dy\wedge dt$$
$$+(-f_{51}f_{13}+f_{53}f_{11}-f_{61}f_{23}-f_{63}f_{21})dx\wedge dt \qquad (***)$$

From the above equation$(***)$we can obtain equation(23),(24) and (25) by simple calculations and by using equations (35)
Similarly by using assumptions(13) and(16) we have the1-forms ω_α satisfy the structure equations (10) then

$$\sum_{i=0}^{k} f_{41,z_i}dz_i\wedge dx+\sum_{i'=0}^{k'} f_{41,z_{i'}}dz_{i'}\wedge dx+\sum_{i=0}^{k} f_{42,z_i}dz_i\wedge dy$$
$$+\sum_{i'=0}^{k'} f_{42,z_{i'}}dz_{i'}\wedge dy+\sum_{i=0}^{k} f_{43,z_i}dz_i\wedge dt+\sum_{i'=0}^{k'} f_{43,z_{i'}}dz_{i'}\wedge dt$$
$$=(f_{11}f_{22}-f_{12}f_{21})dx\wedge dy+(f_{12}f_{23}-f_{13}f_{22})dy\wedge dt+(f_{11}f_{23}-f_{13}f_{21})dx\wedge dt \qquad (\cdot)$$

From the above equation$(\cdot)$we can obtain equation(26),(27) and (28) by simple calculations and by using equations (35)
Similarly by using assumptions (13) and (16) we have the 1-forms $\omega_\alpha$satisfy the structure equations(10) then

$$\sum_{i=0}^{k} f_{51,z_i} dz_i \wedge dx + \sum_{i'=0}^{k'} f_{51,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^{k} f_{52,z_i} dz_i \wedge dy$$
$$+ \sum_{i'=0}^{k'} f_{52,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^{k} f_{53,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{53,z_{i'}} dz_{i'} \wedge dt$$
$$= (f_{11}f_{32} - f_{12}f_{31}) dx \wedge dy + (f_{12}f_{33} - f_{13}f_{32}) dy \wedge dt + (f_{11}f_{33} - f_{13}f_{31}) dx \wedge dt \qquad (\cdot\cdot)$$

From the above equation$(\cdot\cdot)$we can obtain equation(29),(30) and (31) by simple calculations and by using equations (35)

Finally by using assumptions (13) and (16) we have the 1-forms ω_α satisfy the structure equations (10) then

$$\sum_{i=0}^{k} f_{61,z_i} dz_i \wedge dx + \sum_{i'=0}^{k'} f_{61,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^{k} f_{62,z_i} dz_i \wedge dy$$
$$+ \sum_{i'=0}^{k'} f_{62,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^{k} f_{63,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{63,z_{i'}} dz_{i'} \wedge dt$$
$$= (f_{21}f_{32} - f_{22}f_{31}) dx \wedge dy + (f_{22}f_{33} - f_{23}f_{32}) dy \wedge dt + (f_{21}f_{33} - f_{23}f_{31}) dx \wedge dt \qquad (\cdots)$$

From the above equation$(\cdots)$we can obtain equation(32),(33) and (34) by simple calculations and by using equations (35)

Now by taking the z_k derivative of equations (24) and (27), and the $z_{k'}$derivative of equations (25) and (28), we obtain by using equations (16), $f_{33,z_{k-1}} \equiv f_{43,z_{k-1}} \equiv 0$; $f_{33,z_{k'-1}} \equiv f_{43,z_{k'-1}} = 0$

Hence we have obtained the relations (16) and relations from (17)→(34).Finally, we observe that if$f_{51,z_1}, f_{52,z_1}, f_{61,z_1}$and f_{62,z_1}

vanish simultaneously, then the equation(12)cannot be the necessary and sufficient condition for$\omega\alpha$to satisfy the structure equations (10)

for a 3-dimensional P.S.P.This completes the proof of the lemma.□

Now, based on the above lemma, we will try to formulate the following in a simple form. So, we introduce these notations

$$\left.\begin{array}{ll} L_1 = f_{11}f_{51,z_0} - f_{51}f_{11,z_0} , & L_2 = f_{21}f_{61,z_0} - f_{61}f_{21,z_0} \\ H_1 = f_{11}f_{11,z_0} - f_{51}f_{51,z_0} , & H_2 = f_{21}f_{21,z_0} - f_{61}f_{61,z_0} \\ P_1 = f_{11,z_0}f_{51,z_0z_0} - f_{51,z_0}f_{11,z_0z_0} , & P_2 = f_{21,z_0}f_{61,z_0z_0} - f_{61,z_0}f_{21,z_0z_0} \\ M_1 = f_{51,z_0}^2 - f_{11,z_0}^2 , & M_2 = f_{61,z_0}^2 - f_{21,z_0}^2 \end{array}\right\} \qquad (36)$$

and

$$A_1 = \sum_{i=0}^{k-2} z_{i+1} f_{33,z_i} , \qquad A_2 = \sum_{i=0}^{k-2} z_{i+1} f_{43,z_i} \qquad (37)$$

Also, we consider the following

$$\left.\begin{array}{ll} L_1' = f_{12}f_{52,z_0} - f_{52}f_{12,z_0} , & L_2' = f_{22}f_{62,z_0} - f_{62}f_{22,z_0} \\ H_1' = f_{12}f_{12,z_0} - f_{52}f_{52,z_0} , & H_2' = f_{22}f_{22,z_0} - f_{62}f_{62,z_0} \\ P_1' = f_{12,z_0}f_{52,z_0z_0} - f_{52,z_0}f_{12,z_0z_0} , & P_2' = f_{22,z_0}f_{62,z_0z_0} - f_{62,z_0}f_{22,z_0z_0} \\ M_1' = f_{52,z_0}^2 - f_{12,z_0}^2 , & M_2' = f_{62,z_0}^2 - f_{22,z_0}^2 \end{array}\right\} \qquad (38)$$

And

$$B_1 = \sum_{i'=0}^{k'-2} z_{i'+1} f_{33,z_{i'}} , \qquad B_2 = \sum_{i'=0}^{k'-2} z_{i'+1} f_{43,z_{i'}} \qquad (39)$$

Moreover whenever $L_1 \neq 0$, $L_2 \neq 0$, $L_1' \neq 0$and $L_2' \neq 0$, we define R^j and $R^{j'}$respectively as follows: $R^{k-1} = 0$

$$R^j = -\sum_{i=0}^{k-1} z_{i+1} R^{j+1}_{z_i} + f_{33,z_{j+1}}(H_1 + H_2) + f_{43,z_{j+1}}(H_1 + H_2) + \frac{R^{j+1}}{L_1}\left(z_1 L_{1,z_0} + \eta H_1\right)$$
$$+ \frac{R^{j+1}}{L_2}\left(z_1 L_{2,z_0} + \eta H_2\right) + \frac{1}{L_1}(A_1)_{z_{j+1}}(\eta M_1 - z_1 P_1) + \frac{1}{L_2}(A_2)_{z_{j+1}}(\eta M_2 - z_1 P_2) \qquad (40)$$

Where $0 \leq j \leq k-2$ and $R^{k'-1} = 0$

$$R^{j'} = -\sum_{i'=0}^{k'-1} z_{i'+1} R^{j'+1}_{z_{i'}} + f_{33,z_{j'+1}}(H'_1 + H'_2) + f_{43,z_{j'+1}}(H'_1 + H'_2) + \frac{R^{j'+1}}{L'_1}\left(z_{1'}L'_{1,z_0} + \eta H'_1\right)$$
$$+ \frac{R^{j'+1}}{L'_2}\left(z_{1'}L'_{2,z_0} + \eta H'_2\right) + \frac{1}{L'_1}(B_1)_{z_{j'+1}}\left(\eta M'_1 - z_{1'}P'_1\right) + \frac{1}{L'_2}(B_2)_{z_{j'+1}}\left(\eta M'_2 - z_{1'}P'_2\right) \quad (41)$$

Where $0 \le j' \le k' - 2$ .Thus we have this theorem

Theorem 3.1

Let $f_{\alpha 1}, f_{\alpha 2}$ and $f_{\alpha 3}$, $1 \le \alpha \le 6$ be differentiable functions of $z_0, z_1, z_{1'}, \ldots, z_k, z_{k'}$ such that the relations (16) hold and $f_{31} = f_{41} = \eta$, $f_{32} = f_{42} = \xi$ two non zero parameters. Suppose $H_1 L_1 \neq 0$, $H_2 L_2 \neq 0$, $H'_1 L'_1 \neq 0$ and $H'_2 L'_2 \neq 0$.Then the equation $z_{0,t} = \psi(z_0, z_1, z_{1'}, \ldots, z_k, z_{k'})$ describes a two-parameters 3-dimensional P.S.P with associated 1-forms $\omega_\alpha = f_{\alpha 1}dx + f_{\alpha 2}dy + f_{\alpha 3}dt$, $1 \le \alpha \le 6$ If and only if the function ψ is given by

$$\psi = \left(\frac{1}{L_1} + \frac{1}{L_2}\right) + \sum_{i=0}^{k-1} z_{i+1}(A_1)_{z_i} + \left(\frac{1}{L'_1} + \frac{1}{L'_2}\right) + \sum_{i'=0}^{k'-1} z_{i'+1}(B_1)_{z_{i'}} + \frac{1}{H_1 L_1}\left(-z_1\frac{L_1}{\eta} + f_{51}^2 - f_{11}^2\right)\sum_{i=0}^{k-2} z_{i+1}R^i$$
$$+ \frac{1}{H_2 L_2}\left(-z_1\frac{L_2}{\eta} + f_{61}^2 - f_{21}^2\right)\sum_{i=0}^{k-2} z_{i+1}R^i + \frac{1}{H'_1 L'_1}\left(-z_{1'}\frac{L'_1}{\xi} + f_{52}^2 - f_{12}^2\right)\sum_{i=0}^{k-2} z_{i'+1}R^{i'}$$
$$+ \frac{1}{H'_2 L'_2}\left(-z_{1'}\frac{L'_2}{\xi} + f_{62}^2 - f_{22}^2\right)\sum_{i'=0}^{k'-2} z_{i'+1}R^{i'} + \frac{A_1}{H_1 L_1}(z_1 M_1 - \eta L_1) + \frac{A_1}{H_2 L_2}(z_1 M_2 - \eta L_2)$$
$$+ \frac{B_1}{H'_1 L'_1}\left(z_{1'}M'_1 - \xi L'_1\right) + \frac{B_1}{H'_2 L'_2}\left(z_{1'}M'_2 - \xi L'_2\right) + 2z_1\frac{f_{33}}{\eta} + 2z_{1'}\frac{f_{33}}{\xi} \quad (42)$$

Moreover

$$f_{13} = \frac{f_{11}f_{33}}{\eta} + \frac{1}{H_1}\left[-\frac{f_{11}}{\eta}\sum_{i=0}^{k-2} z_{i+1}R^i + f_{51,z_0}A_1\right]$$
$$+ \frac{1}{H_1}\left[\frac{f_{51,z_0}f_{11}}{\eta}(f_{43}f_{21} - \eta f_{23}) + f_{23}f_{61}f_{51,z_0} - f_{21}f_{63}f_{51,z_0}\right] \quad (43)$$

$$f_{53} = \frac{f_{51}f_{33}}{\eta} + \frac{1}{H_1}\left[-\frac{f_{51}}{\eta}\sum_{i=0}^{k-2} z_{i+1}R^i + f_{11,z_0}A_1\right]$$
$$+ \frac{1}{H_1}\left[\frac{f_{51,z_0}f_{51}}{\eta}(f_{43}f_{21} - \eta f_{23}) + f_{23}f_{61}f_{11,z_0} - f_{21}f_{63}f_{11,z_0}\right] \quad (44)$$

$$f_{23} = \frac{f_{21}f_{33}}{\eta} + \frac{1}{H_2}\left[-\frac{f_{21}}{\eta}\sum_{i=0}^{k-2} z_{i+1}R^i + f_{61,z_0}A_1\right]$$
$$+ \frac{1}{H_2}\left[\frac{f_{61,z_0}f_{21}}{\eta}(\eta f_{13} - f_{11}f_{43}) + f_{13}f_{51}f_{61,z_0} - f_{11}f_{53}f_{61,z_0}\right] \quad (45)$$

$$f_{63} = \frac{f_{61}f_{33}}{\eta} + \frac{1}{H_2}\left[-\frac{f_{61}}{\eta}\sum_{i=0}^{k-2} z_{i+1}R^i + f_{21,z_0}A_1\right]$$
$$+ \frac{1}{H_2}\left[\frac{f_{61,z_0}f_{61}}{\eta}(\eta f_{13} - f_{11}f_{43}) + f_{13}f_{51}f_{21,z_0} - f_{11}f_{53}f_{21,z_0}\right] \quad (46)$$

Remark :It is noted that by similar construction, one may obtain

$$f_{13} = \frac{f_{12}f_{33}}{\xi} + \frac{1}{H'_1}\left[-\frac{f_{12}}{\xi}\sum_{i'=0}^{k'-2} z_{i'+1}R^{i'} + f_{52,z_0}B_1\right]$$
$$+ \frac{1}{H'_1}\left[\frac{f_{52,z_0}f_{12}}{\xi}(f_{43}f_{22} - \xi f_{23}) + f_{23}f_{62}f_{52,z_0} - f_{22}f_{63}f_{52,z_0}\right] \quad (47)$$

$$f_{53} = \frac{f_{52}f_{33}}{\xi} + \frac{1}{H'_1}\left[-\frac{f_{52}}{\xi}\sum_{i'=0}^{k'-2} z_{i'+1}R^{i'} + f_{12,z_0}B_1\right]$$
$$+ \frac{1}{H'_1}\left[\frac{f_{52,z_0}f_{52}}{\xi}(f_{43}f_{22} - \xi f_{23}) + f_{23}f_{62}f_{12,z_0} - f_{22}f_{63}f_{12,z_0}\right] \quad (48)$$

$$f_{23}=\frac{f_{22}f_{33}}{\xi}+\frac{1}{H_2'}\left[-\frac{f_{22}}{\xi}\sum_{i'=0}^{k'-2}z_{i'+1}R^{i'}+f_{62,z_0}B_1\right]$$
$$+\frac{1}{H_2'}\left[\frac{f_{62,z_0}f_{22}}{\xi}(\xi f_{13}-f_{12}f_{43})+f_{13}f_{52}f_{62,z_0}-f_{12}f_{53}f_{62,z_0}\right] \quad (49)$$
$$f_{63}=\frac{f_{62}f_{33}}{\xi}+\frac{1}{H_2'}\left[-\frac{f_{62}}{\xi}\sum_{i'=0}^{k'-2}z_{i'+1}R^{i'}+f_{22,z_0}B_1\right]$$
$$+\frac{1}{H_2'}\left[\frac{f_{62,z_0}f_{62}}{\xi}(\xi f_{13}-f_{12}f_{43})+f_{13}f_{52}f_{22,z_0}-f_{12}f_{53}f_{22,z_0}\right] \quad (50)$$

Thus we have the following result:

Corollary 3.1

If f_{α^i} are as in theorem(3.1)then following equations hold: R.H.S of eqn(43)=R.H.S of(47),R.H.S of eqn(44)=R.H.S of (48),R.H.S of eqn(45) = R.H.S of (49), R.H.S of eqn(46) =R.H.S of (50).

Proof of Theorem (3.1)

Suppose that equation(12)describes as(η,ξ)3-dimensional P.S.P inR^5then it follows from lemma(3.1)that equations(16)and equations (17-34)are satisfied where

$$f_{11,z_0}^2+f_{12,z_0}^2+f_{21,z_0}^2+f_{22,z_0}^2+f_{51,z_0}^2+f_{52,z_0}^2+f_{61,z_0}^2+f_{62,z_0}^2\neq 0$$

therefore equations(17)-(34) are equivalent to the following

$$\sum_{i=0}^{k-1}z_{i+1}\left(f_{13,z_i}f_{51,z_0}-f_{53,z_i}f_{11,z_0}\right)+\eta\left(f_{53}f_{51,z_0}-f_{13}f_{11,z_0}\right)+f_{33}H_1+f_{51,z_0}(f_{43}f_{21}-\eta f_{23})=0 \quad (51)$$
$$\sum_{i=0}^{k-1}z_{i+1}\left(f_{23,z_i}f_{61,z_0}-f_{63,z_i}f_{21,z_0}\right)+\eta\left(f_{63}f_{61,z_0}-f_{23}f_{21,z_0}\right)+f_{33}H_2+f_{61,z_0}(\eta f_{13}-f_{11}f_{43})=0 \quad (52)$$
$$\sum_{i'=0}^{k'-1}z_{i'+1}\left(f_{13,z_{i'}}f_{52,z_0}-f_{53,z_{i'}}f_{12,z_0}\right)+\xi\left(f_{53}f_{52,z_0}-f_{13}f_{12,z_0}\right)+f_{33}H_1'+f_{52,z_0}(f_{43}f_{22}-\eta f_{23})=0 \quad (53)$$
$$\sum_{i'=0}^{k'-1}z_{i'+1}\left(f_{23,z_{i'}}f_{62,z_0}-f_{63,z_{i'}}f_{22,z_0}\right)+\xi\left(f_{63}f_{62,z_0}-f_{23}f_{22,z_0}\right)+f_{33}H_2'+f_{62,z_0}(\xi f_{13}-f_{12}f_{43})=0 \quad (54)$$
$$A_1-f_{11}f_{53}+f_{13}f_{51}-f_{21}f_{63}+f_{23}f_{61}=0 \quad (55)$$
$$A_2-f_{11}f_{23}+f_{13}f_{21}=0 \quad (56)$$
$$B_1-f_{12}f_{53}+f_{13}f_{52}-f_{22}f_{63}+f_{23}f_{62}=0 \quad (57)$$
$$B_2-f_{12}f_{23}+f_{13}f_{22}=0 \quad (58)$$
$$\psi L_1+\sum_{i=0}^{k-1}z_{i+1}\left(f_{13,z_i}f_{51}-f_{53,z_i}f_{11}\right)+\eta(f_{53}f_{51}-f_{13}f_{11})-f_{33}(f_{51}^2-f_{11}^2)+f_{51}(f_{43}f_{21}-\eta f_{23})=0 \quad (59)$$
$$\psi L_2+\sum_{i=0}^{k-1}z_{i+1}\left(f_{23,z_i}f_{61}-f_{63,z_i}f_{21}\right)+\eta(f_{63}f_{61}-f_{23}f_{21})-f_{33}(f_{61}^2-f_{21}^2)+f_{61}(\eta f_{13}-f_{11}f_{43})=0 \quad (60)$$
$$\psi L_1'+\sum_{i'=0}^{k'-1}z_{i'+1}\left(f_{13,z_{i'}}f_{52}-f_{53,z_{i'}}f_{12}\right)+\xi(f_{53}f_{52}-f_{13}f_{12})-f_{33}(f_{52}^2-f_{12}^2)+f_{52}(f_{43}f_{22}-\xi f_{23})=0 \quad (61)$$
$$\psi L_2'+\sum_{i'=0}^{k'-1}z_{i'+1}\left(f_{23,z_{i'}}f_{62}-f_{63,z_{i'}}f_{22}\right)+\xi(f_{63}f_{62}-f_{23}f_{22})-f_{33}(f_{62}^2-f_{22}^2)+f_{62}(\xi f_{13}-f_{12}f_{43})=0 \quad (62)$$

Now, taking the z_k derivative of equations (51),(52) and take the $z_{k'}$, derivative of equations (53), (54) and also taking the z_{k-1} derivative of equation (55)and $z_{k'-1}$ derivative of equations (56) then by using equations (16) we can obtain

$$f_{13,z_{k-1}}f_{51,z_0}-f_{53,z_{k-1}}f_{11,z_0}=-R^{k-1}=0 \quad (63)$$
$$f_{23,z_{k-1}}f_{61,z_0}-f_{63,z_{k-1}}f_{21,z_0}=-R^{k-1}=0 \quad (64)$$
$$f_{13,z_{k'-1}}f_{52,z_0}-f_{53,z_{k'-1}}f_{12,z_0}=-R^{k'-1}=0 \quad (65)$$
$$f_{23,z_{k'-1}}f_{62,z_0}-f_{63,z_{k'-1}}f_{22,z_0}=-R^{k'-1}=0 \quad (66)$$
$$f_{13,z_{k-1}}f_{51}-f_{53,z_{k-1}}f_{11}+(A_1)_{z_{k-1}}+f_{23,z_{k-1}}f_{61}-f_{63,z_{k-1}}f_{21}=0 \quad (67)$$

$$f_{13,z_{k'-1}}f_{52} - f_{53,z_{k'-1}}f_{12} + (B_1)_{z_{k'-1}} + f_{23,z_{k'-1}}f_{62} - f_{63,z_{k'-1}}f_{22} = 0 \quad (68)$$

Therefore

$$f_{13,z_{k-1}} = \frac{f_{11,z_0}}{L_1}\left[(A_1)_{z_{k-1}} + f_{23,z_{k-1}}f_{61} - f_{63,z_{k-1}}f_{21}\right] \quad (69)$$

$$f_{53,z_{k-1}} = \frac{f_{51,z_0}}{L_1}\left[(A_1)_{z_{k-1}} - f_{63,z_{k-1}}f_{21} + f_{23,z_{k-1}}f_{61}\right] \quad (70)$$

$$f_{23,z_{k-1}} = \frac{f_{21,z_0}}{L_2}\left[(A_1)_{z_{k-1}} + f_{13,z_{k-1}}f_{51} - f_{53,z_{k-1}}f_{11}\right] \quad (71)$$

$$f_{63,z_{k-1}} = \frac{f_{61,z_0}}{L_2}\left[(A_1)_{z_{k-1}} + f_{13,z_{k-1}}f_{51} - f_{53,z_{k-1}}f_{11}\right] \quad (72)$$

$$f_{13,z_{k'-1}} = \frac{f_{12,z_0}}{L_1'}\left[(B_1)_{z_{k'-1}} + f_{23,z_{k'-1}}f_{62} - f_{63,z_{k'-1}}f_{22}\right] \quad (73)$$

$$f_{53,z_{k'-1}} = \frac{f_{52,z_0}}{L_1'}\left[(B_1)_{z_{k'-1}} + f_{23,z_{k'-1}}f_{62} - f_{63,z_{k'-1}}f_{22}\right] \quad (74)$$

$$f_{23,z_{k'-1}} = \frac{f_{22,z_0}}{L_2'}\left[(B_1)_{z_{k'-1}} + f_{13,z_{k'-1}}f_{52} - f_{53,z_{k'-1}}f_{12}\right] \quad (75)$$

$$f_{63,z_{k'-1}} = \frac{f_{62,z_0}}{L_2'}\left[(B_1)_{z_{k'-1}} + f_{13,z_{k'-1}}f_{52} - f_{53,z_{k'-1}}f_{12}\right] \quad (76)$$

Takingthe z_{j+1} derivative of equations(51),(52) and take the $z_{j'+1}$, derivative of equations(53),(54)moreover taking the$z_j$derivative of equation(55)and $z_{j'}$ derivative of equations(57)with $1 \le j \le k-1$ And$1 \le j' \le k'-1$ we can obtain

$$f_{13,z_j} = -\frac{1}{L_1}\left[f_{11}R^j - f_{11,z_0}(A_1)_{z_j} - f_{23,z_j}f_{61}f_{11,z_0} + f_{63,z_j}f_{21}f_{11,z_0}\right]$$

$$f_{53,z_j} = -\frac{1}{L_1}\left[f_{51}R^j - f_{51,z_0}(A_1)_{z_j} + f_{63,z_j}f_{21}f_{51,z_0} - f_{23,z_j}f_{61}f_{51,z_0}\right]$$

$$f_{23,z_j} = -\frac{1}{L_2}\left[f_{21}R^j - f_{21,z_0}(A_1)_{z_j} - f_{13,z_j}f_{51}f_{21,z_0} + f_{53,z_j}f_{11}f_{21,z_0}\right]$$

$$f_{63,z_j} = -\frac{1}{L_2}\left[f_{61}R^j - f_{61,z_0}(A_1)_{z_j} - f_{13,z_j}f_{51}f_{61,z_0} + f_{53,z_j}f_{11}f_{61,z_0}\right]$$

$$f_{13,z_{j'}} = -\frac{1}{L_1'}\left[f_{12}R^{j'} - f_{12,z_0}(B_1)_{z_{j'}} - f_{23,z_{j'}}f_{62}f_{12,z_0} + f_{63,z_{j'}}f_{22}f_{12,z_0}\right]$$

$$f_{53,z_{j'}} = -\frac{1}{L_1'}\left[f_{52}R^{j'} - f_{52,z_0}(B_1)_{z_{j'}} - f_{23,z_{j'}}f_{62}f_{52,z_0} + f_{63,z_{j'}}f_{22}f_{52,z_0}\right]$$

$$f_{23,z_{j'}} = -\frac{1}{L_2'}\left[f_{22}R^{j'} - f_{22,z_0}(B_1)_{z_{j'}} - f_{13,z_{j'}}f_{52}f_{22,z_0} + f_{53,z_{j'}}f_{12}f_{22,z_0}\right]$$

$$f_{63,z_{j'}} = -\frac{1}{L_2'}\left[f_{62}R^{j'} - f_{62,z_0}(B_1)_{z_{j'}} - f_{13,z_{j'}}f_{52}f_{62,z_0} + f_{53,z_{j'}}f_{12}f_{62,z_0}\right]$$

Also, taking thez_1 derivative of (51), (52)and we get the$z_{1'}$ derivative of (53), (54)

$$f_{13,z_0}f_{51,z_0} - f_{53,z_0}f_{11,z_0} = -R^0 \quad (77)$$

$$f_{23,z_0}f_{61,z_0} - f_{63,z_0}f_{21,z_0} = -R^0 \quad (78)$$

$$f_{13,z_0}f_{52,z_0} - f_{53,z_0}f_{12,z_0} = -R^0 \quad (79)$$

$$f_{23,z_0}f_{62,z_0} - f_{63,z_0}f_{22,z_0} = -R^0 \quad (80)$$

From equations(51)-(54) we can get by using equations (63- 66)and(77- 80) the following equations

$$f_{53}f_{51,z_0} - f_{13}f_{11,z_0} - \tfrac{1}{\eta}\textstyle\sum_{i=0}^{k-2} z_{i+1}R^i + \tfrac{f_{23}}{\eta}H_1 + \tfrac{f_{51,z_0}}{\eta}\left(f_{43}f_{21} - \eta f_{23}\right) = 0 \quad (81)$$

$$f_{63}f_{61,z_0} - f_{23}f_{21,z_0} - \frac{1}{\eta}\sum_{i=0}^{k-2} z_{i+1}R^i + \frac{f_{33}}{\eta}H_2 + \frac{f_{61,z_0}}{\eta}(\eta f_{13} - f_{11}f_{43}) = 0 \quad (82)$$

$$f_{53}f_{52,z_0} - f_{13}f_{12,z_0} - \frac{1}{\xi}\sum_{i'=0}^{k'-2} z_{i'+1}R^{i'} + \frac{f_{33}}{\xi}H_1' + \frac{f_{52,z_0}}{\xi}(f_{43}f_{22} - \xi f_{23}) = 0 \quad (83)$$

$$f_{63}f_{62,z_0} - f_{23}f_{22,z_0} - \frac{1}{\xi}\sum_{i'=0}^{k'-2} z_{i'+1}R^{i'} + \frac{f_{33}}{\xi}H_2' + \frac{f_{62,z_0}}{\xi}(\xi f_{13} - f_{12}f_{43}) = 0 \quad (84)$$

Now from equations(55)and(56)and equations (81)-(84)one can obtain the functions f_{13}, f_{23}, f_{53} and f_{63} as given in theorem (3.1)

Moreover, from equations$(81-84),(55)-(62)$it follows that ψ is given by(42)Conversely, if f_{13}, f_{23}, f_{53} and f_{63} are given by $(43-46)$, it follows by straight forward computation that the $1-$forms

$\omega_{\alpha 1} = f_{\alpha 1}\,\mathrm{dx} + f_{\alpha 2}\,\mathrm{dy} + f_{\alpha 3}\,\mathrm{dt}, 1 \le \alpha \le 6$ satisfy the structure equations of an (η, ξ) 3-dim. P.S.P if $z_{0,t} = \psi(z_0, z_1, z_{1'}, \ldots, z_k, z_{k'})$.

This completes proof of theorem. It is worth mentioning that other types of evolution equations in higher dimensions shall be considered in other papers as well as the associated Backlund transformation equations, conservation laws and the associated linear systems.

IV. Conclusion

In this paper, we extended the notion of P.S.P to higher dimensions i.e. 3-dim plane of constant sectional curvature-1 imbedded in R^5 and we studied the change in the results and properties.

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