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Analytical Approximate Solutions of Fifth Order More Critically Damped Systems in the case of Smaller Triply Repeated Roots

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Abstract: *In order to look into the transient behavior of vibrating systems, the Krylov-Bogoliubov-Mitropolskii (KBM) method is extensively used. The method was initially devised to obtain the periodic solutions of second order nonlinear differential systems with small nonlinearities. In this article, the method has been modified to investigate the solutions of fifth order more critically damped nonlinear systems. A fifth order more critically damped nonlinear differential system is considered and asymptotic solutions are studied when the triply eigenvalues are small and the other two equal eigenvalues are large. The results obtained by the presented modified KBM method agree with those obtained by the fourth order Runge-Kuttamethod satisfactorily.*
Keywords: *KBM, nonlinearity, more critically damped system, asymptotic solution, eigenvalues*

I. Introduction

The Krylov-Bogoliubov-Mitropolskii (KBM) ([1],[2]) method is devised to obtain the approximate solutions of weakly nonlinear systems. Popov [3] extended the method to damped oscillatory nonlinear systems despite the fact that it was initially designed for approximating periodic solutions of second order nonlinear differential systems with small nonlinearities. Murty and Deekshatulu [4] examined over-damped nonlinear systems using Bogoliubov's method. Sattar [5] found an asymptotic solution of a second order critically damped nonlinear system. Shamsul [6] suggested a technique for obtaining approximate solutions of second order over-damped and critically damped nonlinear systems. Osiniskii [7] studied solution of third order nonlinear systems by bogoliubov's method imposing some restrictions on the parameters. As a result, the solution was over-simplified and presented incorrect results. Mulholland [8] removed these restrictions imposed by Osiniskii and obtained desired solutions. Bojadziv [9] considered solutions of nonlinear systems by transforming it to a three dimensional differential system. Shamsul and Sattar [10] presented a unified KBM method for solving third order nonlinear system. Sattar [11] investigated solutions of third order over-damped nonlinear systems. Shamsul [12] proposed solutions of third order over-damped systems whose unequal eigenvalues are integral multiple. Shamsul and Sattar [13] presented a unified KBM method for obtaining approximate solutions of third order damped and over-damped nonlinear systems. Kawser and Ali Akbar [14] explored an asymptotic solution for the third order critically damped nonlinear system in the case for equal eigenvalues. Kawser and Sattar [15] propounded an asymptotic solution of a fourth order critically damped nonlinear system with pair wise equal eigenvalues. Akbar and Tanzer [16] extended the KBM method for solving the fifth order over-damped nonlinear systems with cubic nonlinearity.

In this study, we have investigated the solution of fifth order more critically damped systems in the case of smaller triply repeated roots. For different sets of initial conditions as well as for different sets of eigenvalues the solutions show excellent coincident with the numerical solutions.

II. The Method

Consider a fifth order weakly nonlinear ordinary differential system

$$x^{v}+k_1x^{iv}+k_2\dddot{x}+k_3\ddot{x}+k_4\dot{x}+k_5x=-\varepsilon f(x,\dot{x},\ddot{x},\dddot{x},x^{iv}) \qquad (1)$$

Where x^{v} denotes the fifth derivative, x^{iv} denotes the fourth derivative of *x* and over dots are used to denote the first, second and third derivatives of *x* with respect to *t*; k_1, k_2, k_3, k_4, k_5 are characteristic parameters, ε is a small parameter and *f* (*x*) is the given nonlinear function. As the equation is fifth order, so there are five real negative Eigen values, and three of the Eigen values are equal (for more critically damped). Suppose the Eigen values are –λ, -λ, -λ, -μ, - μ. When $\varepsilon=0$, the equation (1) becomes linear and the solution of the corresponding linear equation is

$$x(t,0)=(a_0+b_0t+c_0t^2)e^{-\lambda t}+(d_0+h_0t)e^{-\mu t} \qquad (2)$$

Where a_0, b_0, c_0, d_0, h_0 are constants of integration.

Where $\varepsilon\neq 0$ following Shamsul [17] an asymptotic solution of the equation (1) is sought in the form

$$x(t,\varepsilon)=(a+bt+ct^2)e^{-\lambda t}+(d+ht)e^{-\mu t}+\varepsilon u_1(a,b,c,d,h,t)+... \qquad (3)$$

Where a, b, c, d, h are the functions of t and satisfy the first order differential equations

$$\begin{aligned}\dot{a}&=\varepsilon A_1(a,b,c,d,h,t)+...\\ \dot{b}&=\varepsilon B_1(a,b,c,d,h,t)+...\\ \dot{c}&=\varepsilon C_1(a,b,c,d,h,t)+...\\ \dot{d}&=\varepsilon D_1(a,b,c,d,h,t)+...\\ \dot{h}&=\varepsilon H_1(a,b,c,d,h,t)+...\end{aligned} \qquad (4)$$

Now differentiating (3) five times with respect to t, substituting the value of x and the derivatives x^{v}, x^{iv}, $\dddot{x}$, $\ddot{x}$, $\dot{x}$ in the original equation (1) utilizing the relations presented in (4) and finally extracting the coefficients of ε, we obtain

$$\begin{aligned}&e^{-\lambda t}(D+\mu-\lambda)^2\left(\frac{\partial^2 A_1}{\partial t^2}+\frac{\partial^2 B_1}{\partial t^2}t+3\frac{\partial B_1}{\partial t}+\frac{\partial^2 C_1}{\partial t^2}t^2+6\frac{\partial C_1}{\partial t}t+6C_1\right)+\\ &e^{-\mu t}(D+\lambda-\mu)^3\left(\frac{\partial D_1}{\partial t}+2H_1+\frac{\partial H_1}{\partial t}t\right)+(D+\lambda)^3(D+\mu)^2u_1=\\ &\qquad -f^0(a,b,c,d,h,t)\end{aligned} \qquad (5)$$

where $f^{(0)}(a,b,c,d,h,t)=f(x,\dot{x},\ddot{x},\dddot{x},x^{iv})$ *and* $x(t,0)=(a_0+b_0t+c_0t^2)e^{-\lambda t}+(d_0+h_0t)e^{-\mu t}$

We have expanded the function $f^{(0)}$ in the Taylor's series (Sattar [5], Shamsul [18], Shamsul and Sattar [13]) about the origin in power of t. Therefore, we obtain:

$$f^{(0)}=\sum_{q=0}^{\infty}\left\{t^q\sum_{i,j,k=0}^{\infty}F_{q,k}(a,b,c,d,h)e^{-(i\lambda+j\mu)t}\right\} \qquad (6)$$

Thus, using (6), the equation (5) becomes

Following the KBM method, Murty and Deekshatulu [4], Sattar [5], Shamsul and Sattar ([10], [13]) imposed the condition that u_1 does not contain the fundamental terms of f (0). Therefore, equation (7) can be separated for unknown functions A_1, B_1, C_1, D_1, H_1 and u_1 in the following way:

$$\begin{aligned}&e^{-\lambda t}(D+\mu-\lambda)^2\left(\frac{\partial^2 A_1}{\partial t^2}+\frac{\partial^2 B_1}{\partial t^2}t+3\frac{\partial B_1}{\partial t}+\frac{\partial^2 C_1}{\partial t^2}t^2+6\frac{\partial C_1}{\partial t}t+6C_1\right)+\\ &e^{-\mu t}(D+\lambda-\mu)^3\left(\frac{\partial D_1}{\partial t}+2H_1+\frac{\partial H_1}{\partial t}t\right)+(D+\lambda)^3(D+\mu)^2u_1=\\ &\qquad -\sum_{q=0}^{\infty}\left\{t^q\sum_{i,j,k=0}^{\infty}F_{q,k}(a,b,c,d,h)e^{-(i\lambda+j\mu)t}\right\}\end{aligned} \qquad (7)$$

$$\begin{aligned}&e^{-\lambda t}(D+\mu-\lambda)^2\left(\frac{\partial^2 A_1}{\partial t^2}+\frac{\partial^2 B_1}{\partial t^2}t+3\frac{\partial B_1}{\partial t}+\frac{\partial^2 C_1}{\partial t^2}t^2+6\frac{\partial C_1}{\partial t}t+6C_1\right)+\\ &e^{-\mu t}(D+\lambda-\mu)^3\left(\frac{\partial D_1}{\partial t}+2H_1+\frac{\partial H_1}{\partial t}t\right)=\\ &\qquad -\sum_{q=0}^{1}\left\{t^q\sum_{i,j,k=0}^{\infty}F_{q,k}(a,b,c,d,h)e^{-(i\lambda+j\mu)t}\right\}\end{aligned} \qquad (8)$$

And

$$(D+\lambda)^3(D+\mu)^2 u_1 = -\sum_{q=2}^{\infty}\left\{t^q \sum_{i,j,k=0}^{\infty} F_{q,k}(a,b,c,d,h)e^{-(i\lambda+j\mu)t}\right\} \tag{9}$$

Now equating the coefficients of t^0, t^1, t^2 from equation (8), we obtain

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)^2\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+e^{-\mu t}(D+\lambda-\mu)^3\left(\frac{\partial D_1}{\partial t}+2H_1\right) \\ &= -\sum_{i,j,k=0}^{\infty} F_{0,K}(a,b,c,d,h)e^{-(i\lambda+j\mu)t} \end{aligned} \tag{10}$$

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)^2\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+e^{-\mu t}(D+\lambda-\mu)^3\frac{\partial H_1}{\partial t} \\ &= -\sum_{i,j,k=0}^{\infty} F_{1,K}(a,b,c,d,h)e^{-(i\lambda+j\mu)t} \end{aligned} \tag{11}$$

$$e^{-\lambda t}(D+\mu-\lambda)^2\frac{\partial^2 C_1}{\partial t^2} = -\sum_{i,j,k=0}^{\infty} F_{1,K}(a,b,c,d,h)e^{-(i\lambda+j\mu)t} \tag{12}$$

Here, we have only three equations (10), (11) and (12) for determining the unknown functions A_1, B_1, C_1 , D_1 and H_1 Thus, to obtain the unknown functions A_1, B_1, C_1, D_1 and H_1 we need to impose some conditions (Shamsul[19], [20], [21]) between the eigenvalues. Different authors imposed different conditions according to the behavior of the systems; such as Shamsul [12] imposed the condition $i_1\lambda_1+i_2\lambda_2+...+i_n\lambda_n \le (i_1+i_1+...+i_1)(\lambda_1+\lambda_2+...+\lambda_n)$. In this study, we have investigated solutions for the case $\mu \gg \lambda$. Therefore, we shall be able to separate the equation (12) for unknown functions C_1 and solving them for B_1 and H_1 substituting the values of C_1 into the equation (11) and applying the condition $\mu \gg \lambda$: we can separate the equation (11) for two unknown functions B_1 and H_1; and solving them for A_1 and D_1. Since $\dot{a},\dot{b},\dot{c},\dot{d},\dot{h}$ are proportional to small parameter, they are slowly varying functions of time t and for first approximate solution, we may consider them as constants in the right hand side. This assumption was first made by Murty and Deekshatulu [22]. Thus the solutions of the equation (4) become

$$\begin{aligned} a &= a_0+\varepsilon\int_0^t A_1(a,b,c,d,h,t)dt \\ b &= b_0+\varepsilon\int_0^t B_1(a,b,c,d,h,t)dt \\ c &= c_0+\varepsilon\int_0^t C_1(a,b,c,d,h,t)dt \\ d &= d_0+\varepsilon\int_0^t D_1(a,b,c,d,h,t)dt \\ h &= h_0+\varepsilon\int_0^t H_1(a,b,c,d,h,t)dt \end{aligned} \tag{13}$$

Equation (9) is a non-homogeneous linear ordinary differential equation; therefore, it can be solved by the well-known operator method. Substituting the values of *a, b, c,d,h* and u_1 in the equation (3), we shall get the complete solution of (1). Therefore, the determination of the first approximate solution is complete.

III. Example

As an example of the above method, we have considered the Duffing type equation of fifth order weakly-nonlinear oscillatory system:

$$x^{v}+k_1x^{iv}+k_2\dddot{x}+k_3\ddot{x}+k_4\dot{x}+k_5x=-\varepsilon x^3 \tag{14}$$

Comparing (13) and (1), we obtain $f(x,\dot{x},\ddot{x},\dddot{x},x^{iv})=x^3$

$$\begin{aligned}f^{(0)}=\{&(a^3+3a^2bt+3ab^2t^2+b^3t^3+3a^2ct^2+6abct^3+3b^2ct^4+3ac^2t^4+3bc^2t^5+\\&c^3t^6)e^{-3\lambda t}+3(a^2d+2abdt+b^2dt^2+2acdt^2+2bcdt^3+c^2dt^4+a^2ht+2abht^2+\\&b^2ht^3+2acht^3+2bcht^4+c^2ht^5)e^{-(2\lambda+\mu)t}+3(ad^2+bd^2t+cd^2t^2+2adht+2bdht^2\\&+2cdht^3+ah^2t^2+bh^2t^3+ch^2t^4)e^{-(\lambda+2\mu)t}+(d^3+3d^2ht+3dh^2t^2+h^3t^3)e^{-3\mu t}\}\end{aligned} \tag{15}$$

For example equation (14), the equations (9)-(12) respectively become

$$\begin{aligned}&e^{-\lambda t}(D+\mu-\lambda)^2\left(\frac{\partial^2A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+e^{-\mu t}(D+\lambda-\mu)^3\left(\frac{\partial D_1}{\partial t}+2H_1\right)=-a^3e^{-3\lambda t}\\&\quad-3a^2de^{-(2\lambda+\mu)t}-3ad^2e^{-(\lambda+2\mu)t}-d^3e^{-3\mu t}\end{aligned} \tag{16}$$

$$\begin{aligned}&e^{-\lambda t}(D+\mu-\lambda)^2\left(\frac{\partial^2B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+e^{-\mu t}(D+\lambda-\mu)^3\frac{\partial H_1}{\partial t}=-3a^2be^{-3\lambda t}-3(2abd+a^2h)\\&\quad e^{-(2\lambda+\mu)t}-3(bd^2+2adh)e^{-(\lambda+2\mu)t}-3d^2he^{-3\mu t}\end{aligned} \tag{17}$$

$$\begin{aligned}&e^{-\lambda t}(D+\mu-\lambda)^2\frac{\partial^2C_1}{\partial t^2}=-3(ab^2+a^2c)e^{-3\lambda t}-3(b^2d+2acd+2abh)e^{-(2\lambda+\mu)t}\\&\quad-3(cd^2+2bdh+ah^2)e^{-(\lambda+2\mu)t}-3dh^2e^{-3\mu t}\end{aligned} \tag{18}$$

And

$$\begin{aligned}(D+\lambda)^3(D+\mu)^2u_1=\{&(b^3t^3+6abct^3+3b^2ct^4+3ac^2t^4+3bc^2t^5+c^3t^6)e^{-3\lambda t}\\&+3(2bcdt^3+c^2dt^4+b^2ht^3+2acht^3+2bcht^4+c^2ht^5)e^{-(2\lambda+\mu)t}+3(2cdht^3+\\&bh^2t^3+ch^2t^4)e^{-(\lambda+2\mu)t}+h^3t^3e^{-3\mu t}\}\end{aligned} \tag{19}$$

Solution of equation (18) is

$$\begin{aligned}C_1=&\,l_1(ab^2+a^2c)e^{-2\lambda t}+l_2(b^2d+2acd+2abh)e^{-(\lambda+\mu)t}+l_3(cd^2+2bdh+ah^2)e^{-2\mu t}\\&+l_4dh^2e^{-(3\mu-\lambda)t}\end{aligned} \tag{20}$$

Where $l_1=-\dfrac{3}{4\lambda^2(\mu-3\lambda)^2}$, $l_2=-\dfrac{3}{4\lambda^2(\mu+\lambda)^2}$, $l_3=-\dfrac{3}{4\mu^2(\mu+\lambda)^2}$, $l_4=-\dfrac{3}{4\mu^2(3\mu-\lambda)^2}$

Now differentiating equation (20) with respect to t and putting the value of $\dfrac{\partial C_1}{\partial t}$ in equation (17) and solving B_1 and H_1 imposing the condition $\mu \gg \lambda$, we get

$$B_1=r_1a^2be^{-2\lambda t}+r_2(ab^2+a^2c)e^{-2\lambda t} \tag{21}$$

Where $r_1=-\dfrac{3}{4\lambda^2(\mu-3\lambda)^2}$, $r_2=-\dfrac{9}{4\lambda^3(\mu-3\lambda)^2}$

And

$$\begin{aligned}H_1=&\,p_1(2abd+a^2h)e^{-2\lambda t}+p_2(bd^2+2adh)e^{-(\lambda+\mu)t}+p_3d^2he^{-2\mu t}+p_4(b^2d+\\&2acd+2abh)e^{-2\lambda t}+p_5(cd^2+2bdh+ah^2)e^{-(\lambda+\mu)t}+p_6dh^2e^{-2\mu t}\end{aligned} \tag{22}$$

Where $p_1 = -\frac{3}{2\lambda(\lambda+\mu)^3}$, $p_2 = -\frac{3}{8\mu^3(\lambda+\mu)}$, $p_3 = -\frac{3}{2\mu(3\mu-\lambda)^3}$, $p_4 = -\frac{9}{\lambda(\lambda+\mu)^4}$,
$p_5 = -\frac{9}{8\mu^4(\lambda+\mu)}$, $p_6 = -\frac{9}{\mu(3\mu-\lambda)^4}$.

Again, putting the value $\frac{\partial B_1}{\partial t}$, C_1 and H_1 in Eq. (10) and solving A_1 and D_1 and imposing the condition $\mu >> \lambda$ we get

$$A_1 = q_1 a^3 e^{-2\lambda t} + q_2 a^2 b e^{-2\lambda t} + (q_3 + q_4)(a^2 c + ab^2) e^{-2\lambda t} \tag{23}$$

Where $q_1 = -\frac{1}{4\lambda^2(\mu-3\lambda)^2}$, $q_2 = -\frac{9}{8\lambda^3(\mu-3\lambda)^2}$, $q_3 = -\frac{27}{8\lambda^4(\mu-3\lambda)^2}$, $q_4 = \frac{9}{8\lambda^4(\mu-3\lambda)^2}$

And

$$\begin{aligned} D_1 &= n_1 a^2 d e^{-2\lambda t} + n_2 a d^2 e^{-(\lambda+\mu)t} + n_3 d^3 e^{-2\mu t} + n_4(b^2 d + 2acd + 2abh)e^{-2\lambda t} + n_5(cd^2 + 2bdh \\ &+ ah^2)e^{-(\lambda+\mu)t} + n_6 dh^2 e^{-2\mu t} + n_7(2abd + a^2 h)e^{-2\lambda t} + n_8(bd^2 + 2adh)e^{-(\lambda+\mu)t} + n_9 d^2 h e^{-2\mu t} \\ &+ n_{10}(b^2 d + 2acd + 2abh)e^{-2\lambda t} + n_{11}(cd^2 + 2bdh + ah^2)e^{-(\lambda+\mu)t} + n_{12} dh^2 e^{-2\mu t} \end{aligned} \tag{24}$$

Where $n_1 = -\frac{3}{2\lambda(\lambda+\mu)^3}$, $n_2 = -\frac{3}{8\mu^3(\lambda+\mu)}$, $n_3 = -\frac{1}{2\mu(3\mu-\lambda)^3}$, $n_4 = -\frac{9}{\lambda(\lambda+\mu)^5}$,
$n_5 = \frac{9}{16\mu^5(\lambda+\mu)}$, $n_6 = \frac{9}{\mu(3\mu-\lambda)^5}$, $n_7 = -\frac{3}{2\lambda^2(\lambda+\mu)^3}$, $n_8 = -\frac{3}{8\mu^3(\lambda+\mu)^2}$,
$n_9 = -\frac{3}{2\mu^2(3\mu-\lambda)^3}$, $n_{10} = -\frac{9}{\lambda^2(\lambda+\mu)^4}$, $n_{11} = -\frac{9}{4\mu^4(\lambda+\mu)^2}$, $n_{12} = -\frac{9}{\mu^2(\lambda-3\mu)^4}$

The solution of the equation (19) for u_1 is

$$\begin{aligned} u_1 &= e^{-3\lambda t}\{(b^3 + 6abc)(m_1 t^3 + m_2 t^2 + m_3 t + m_4) + (b^2 c + ac^2)(m_5 t^4 + m_6 t^3 + m_7 t^2 + \\ &m_8 t + m_9) + bc^2(m_{10} t^5 + m_{11} t^4 + m_{12} t^3 + m_{13} t^2 + m_{14} t + m_{15}) + c^3(m_{16} t^6 + m_{17} t^5 + \\ &m_{18} t^4 + m_{19} t^3 + m_{20} t^2 + m_{21} t + m_{22})\} + e^{-(2\lambda+\mu)t}\{(2bch + 2ach + b^2 h)(m_{23} t^3 + m_{24} t^2 \\ &+ m_{25} t + m_{26}) + (c^2 d + 2bch)(m_{27} t^4 + m_{28} t^3 + m_{29} t^2 + m_{30} t + m_{31}) + c^2 h(m_{32} t^5 + m_{33} t^4 \\ &+ m_{34} t^3 + m_{35} t^2 + m_{36} t + m_{37})\} + e^{-(\lambda+2\mu)t}\{(2cdh + bh^2)(m_{38} t^3 + m_{39} t^2 + m_{40} t + m_{41}) + \\ &ch^2(m_{42} t^4 + m_{43} t^3 + m_{44} t^2 + m_{45} t + m_{46})\} + e^{-3\mu t} h^3(m_{47} t^3 + m_{48} t^2 + m_{49} t + m_{50}) \end{aligned} \tag{25}$$

Where

$m_1 = \frac{e^{-3\lambda t}}{8\lambda^3(3\lambda-\mu)^2}$, $m_2 = \frac{3e^{-3\lambda t}}{8\lambda^3(3\lambda-\mu)^2}(\frac{3}{2\lambda} + \frac{2}{3\lambda-\mu})$,

$m_3 = \frac{3e^{-3\lambda t}}{4\lambda^3(3\lambda-\mu)^2}(\frac{3}{2\lambda^2} + \frac{3}{(3\lambda-\mu)^2} + \frac{3}{\lambda(3\lambda-\mu)})$,

$m_4 = \frac{3e^{-3\lambda t}}{4\lambda^3(3\lambda-\mu)^2}(\frac{5}{4\lambda^3} + \frac{4}{(3\lambda-\mu)^3} + \frac{9}{2\lambda(3\lambda-\mu)^2} + \frac{3}{\lambda^2(3\lambda-\mu)})$, $m_5 = \frac{3e^{-3\lambda t}}{8\lambda^3(\mu-3\lambda)^2}$,

$m_6 = \frac{3e^{-3\lambda t}}{2\lambda^3(\mu-3\lambda)^2}(\frac{3}{2\lambda} + \frac{2}{3\lambda-\mu})$, $m_7 = \frac{9e^{-3\lambda t}}{2\lambda^3(\mu-3\lambda)^2}(\frac{3}{2\lambda^2} + \frac{3}{(3\lambda-\mu)^2} + \frac{3}{\lambda(3\lambda-\mu)})$,

$$m_8 = \frac{9e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{5}{4\lambda^3}+\frac{4}{(3\lambda-\mu)^3}+\frac{9}{2\lambda(3\lambda-\mu)^2}+\frac{3}{\lambda^2(3\lambda-\mu)}),$$

$$m_9 = \frac{9e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{15}{16\lambda^4}+\frac{5}{(3\lambda-\mu)^4}+\frac{9}{2\lambda^2(3\lambda-\mu)^2}+\frac{6}{\lambda(3\lambda-\mu)^3}+\frac{20}{\lambda^3(3\lambda-\mu)}),$$

$$m_{10} = \frac{3e^{-3\lambda t}}{8\lambda^3(\mu-3\lambda)^2},\ m_{11} = \frac{15e^{-3\lambda t}}{8\lambda^3(\mu-3\lambda)^2}(\frac{3}{2\lambda}+\frac{2}{3\lambda-\mu}),$$

$$m_{12} = \frac{15e^{-3\lambda t}}{2\lambda^3(\mu-3\lambda)^2}(\frac{3}{2\lambda^2}+\frac{3}{(3\lambda-\mu)^2}+\frac{3}{\lambda(3\lambda-\mu)}),$$

$$m_{13} = \frac{45e^{-3\lambda t}}{2\lambda^3(\mu-3\lambda)^2}(\frac{5}{8\lambda^3}+\frac{4}{(3\lambda-\mu)^3}+\frac{9}{2\lambda(3\lambda-\mu)^2}+\frac{3}{\lambda^2(3\lambda-\mu)}),$$

$$m_{14} = \frac{45e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{15}{16\lambda^4}+\frac{5}{(3\lambda-\mu)^4}+\frac{5}{2\lambda^2(3\lambda-\mu)^2}+\frac{6}{\lambda(3\lambda-\mu)^3}+\frac{9}{2\lambda^2(3\lambda-\mu)^2}),$$

$$m_{15} = \frac{45e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{21}{32\lambda^5}+\frac{6}{(3\lambda-\mu)^5}+\frac{15}{8\lambda^4(3\lambda-\mu)}+\frac{15}{2\lambda(3\lambda-\mu)^4}+\frac{15}{4\lambda^3(3\lambda-\mu)^2}+\frac{6}{\lambda^2(3\lambda-\mu)^3}),$$

$$m_{16} = \frac{e^{-3\lambda t}}{8\lambda^3(\mu-3\lambda)^2},m_{17} = \frac{3e^{-3\lambda t}}{4\lambda^3(\mu-3\lambda)^2}(\frac{3}{2\lambda}+\frac{2}{3\lambda-\mu}),$$

$$m_{18} = \frac{15e^{-3\lambda t}}{4\lambda^3(\mu-3\lambda)^2}(\frac{3}{2\lambda^2}+\frac{3}{(3\lambda-\mu)^2}+\frac{3}{\lambda(3\lambda-\mu)}),$$

$$m_{19} = \frac{15e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{5}{4\lambda^3}+\frac{4}{(3\lambda-\mu)^3}+\frac{9}{2\lambda(3\lambda-\mu)^2}+\frac{3}{\lambda^2(3\lambda-\mu)}),$$

$$m_{20} = \frac{45e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{15}{16\lambda^4}+\frac{5}{(3\lambda-\mu)^4}+\frac{9}{2\lambda^2(3\lambda-\mu)^2}+\frac{6}{\lambda(3\lambda-\mu)^3}+\frac{20}{\lambda^3(3\lambda-\mu)}),$$

$$m_{21} = \frac{90e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{21}{32\lambda^5}+\frac{6}{(3\lambda-\mu)^5}+\frac{15}{4\lambda^3(3\lambda-\mu)^2}+\frac{6}{\lambda^2(3\lambda-\mu)^3}+\frac{15}{2\lambda(3\lambda-\mu)^4}+\frac{15}{8\lambda^4(3\lambda-\mu)}),$$

$$m_{22} = \frac{90e^{-3\lambda t}}{\lambda^3(\mu-3\lambda)^2}(\frac{7}{16\lambda^6}+\frac{7}{(3\lambda-\mu)^6}+\frac{21}{16\lambda^5(3\lambda-\mu)}+\frac{9}{\lambda(3\lambda-\mu)^5}+\frac{45}{16\lambda^4(3\lambda-\mu)^2}$$

$$+\frac{15}{2\lambda^2(3\lambda-\mu)^4}+\frac{5}{4\lambda^3(3\lambda-\mu)^3}),$$

$$m_{23} = \frac{3e^{-(2\lambda+\mu)t}}{4\lambda^2(\mu+\lambda)^3},m_{24} = \frac{9e^{-(2\lambda+\mu)t}}{4\lambda^2(\mu+\lambda)^3}(\frac{1}{\lambda}+\frac{3}{\lambda+\mu}),$$

$$m_{25} = \frac{9e^{-(2\lambda+\mu)t}}{2\lambda^2(\mu+\lambda)^3}(\frac{3}{4\lambda^2}+\frac{3}{\lambda(\lambda+\mu)}+\frac{6}{(\lambda+\mu)^2}),$$

$$m_{26} = \frac{9e^{-(2\lambda+\mu)t}}{2\lambda^2(\mu+\lambda)^3}(\frac{1}{2\lambda^3}+\frac{6}{\lambda(\lambda+\mu)^2}+\frac{10}{(\lambda+\mu)^3}+\frac{9}{4\lambda^2(\lambda+\mu)}),$$

$$m_{27} = \frac{3e^{-(2\lambda+\mu)t}}{4\lambda^2(\mu+\lambda)^3},\ m_{28} = \frac{3e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}(\frac{1}{\lambda}+\frac{3}{\lambda+\mu}),$$

$$m_{29} = \frac{9e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}(\frac{3}{4\lambda^2}+\frac{3}{\lambda(\lambda+\mu)}+\frac{6}{(\lambda+\mu)^2}),$$

$$m_{30}=\frac{18e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}\left(\frac{1}{2\lambda^3}+\frac{6}{\lambda(\lambda+\mu)^2}+\frac{10}{(\lambda+\mu)^3}+\frac{9}{4\lambda^2(\lambda+\mu)}\right),$$

$$m_{31}=\frac{18e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}\left(\frac{15}{16\lambda^4}+\frac{10}{\lambda(\lambda+\mu)^3}+\frac{15}{(\lambda+\mu)^4}+\frac{3}{2\lambda^3(\lambda+\mu)}+\frac{9}{2\lambda^2(\lambda+\mu)^2}\right),$$

$$m_{32}=\frac{3e^{-(2\lambda+\mu)t}}{4\lambda^2(\mu+\lambda)^3},\quad m_{33}=\frac{15e^{-(2\lambda+\mu)t}}{4\lambda^2(\mu+\lambda)^3}\left(\frac{1}{\lambda}+\frac{3}{\lambda+\mu}\right),$$

$$m_{34}=\frac{15e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}\left(\frac{3}{4\lambda^2}+\frac{3}{\lambda(\lambda+\mu)}+\frac{6}{(\lambda+\mu)^2}\right),$$

$$m_{35}=\frac{45e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}\left(\frac{1}{2\lambda^3}+\frac{6}{\lambda(\lambda+\mu)^2}+\frac{10}{(\lambda+\mu)^3}+\frac{9}{4\lambda^2(\lambda+\mu)}\right),$$

$$m_{36}=\frac{90e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}\left(\frac{15}{16\lambda^4}+\frac{10}{\lambda(\lambda+\mu)^3}+\frac{15}{(\lambda+\mu)^4}+\frac{3}{2\lambda^3(\lambda+\mu)}+\frac{9}{2\lambda^2(\lambda+\mu)^2}\right),$$

$$m_{37}=\frac{90e^{-(2\lambda+\mu)t}}{\lambda^2(\mu+\lambda)^3}\left(\frac{3}{16\lambda^5}+\frac{15}{\lambda(\lambda+\mu)^4}+\frac{15}{16\lambda^4(\lambda+\mu)}+\frac{21}{(\lambda+\mu)^5}+\frac{15}{2\lambda^2(\lambda+\mu)^3}+\frac{3}{\lambda^3(\lambda+\mu)^2}\right),$$

$$m_{38}=\frac{3e^{-(\lambda+2\mu)t}}{8\mu^3(\mu+\lambda)^2},\quad m_{39}=\frac{9e^{-(\lambda+2\mu)t}}{8\mu^3(\mu+\lambda)^2}\left(\frac{3}{2\lambda}+\frac{2}{\lambda+\mu}\right),$$

$$m_{40}=\frac{9e^{-(\lambda+2\mu)t}}{4\mu^3(\mu+\lambda)^2}\left(\frac{3}{2\lambda^2}+\frac{3}{(\lambda+\mu)^2}+\frac{3}{\lambda(\lambda+\mu)}\right),$$

$$m_{41}=\frac{9e^{-(\lambda+2\mu)t}}{4\mu^3(\mu+\lambda)^2}\left(\frac{5}{4\lambda^3}+\frac{4}{(\lambda+\mu)^3}+\frac{3}{\lambda^2(\lambda+\mu)}+\frac{9}{2\lambda(\lambda+\mu)^2}\right),$$

$$m_{42}=\frac{3e^{-(\lambda+2\mu)t}}{8\mu^3(\mu+\lambda)^2},\quad m_{43}=\frac{3e^{-(\lambda+2\mu)t}}{2\mu^3(\mu+\lambda)^2}\left(\frac{3}{2\lambda}+\frac{2}{\lambda+\mu}\right),$$

$$m_{44}=\frac{9e^{-(\lambda+2\mu)t}}{2\mu^3(\mu+\lambda)^2}\left(\frac{3}{2\lambda^2}+\frac{3}{(\lambda+\mu)^2}+\frac{3}{\lambda(\lambda+\mu)}\right),$$

$$m_{45}=\frac{9e^{-(\lambda+2\mu)t}}{\mu^3(\mu+\lambda)^2}\left(\frac{5}{4\lambda^3}+\frac{4}{(\lambda+\mu)^3}+\frac{3}{\lambda^2(\lambda+\mu)}+\frac{9}{2\lambda(\lambda+\mu)^2}\right),$$

$$m_{46}=\frac{9e^{-(\lambda+2\mu)t}}{\mu^3(\mu+\lambda)^2}\left(\frac{15}{16\lambda^4}+\frac{5}{(\lambda+\mu)^4}+\frac{5}{2\lambda^3(\lambda+\mu)}+\frac{6}{\lambda(\lambda+\mu)^3}+\frac{9}{2\lambda^2(\lambda+\mu)^2}\right),$$

$$m_{47}=\frac{e^{-3\mu t}}{4\mu^2(3\mu-\lambda)^3},\quad m_{48}=\frac{3e^{-3\mu t}}{4\mu^2(3\mu-\lambda)^3}\left(\frac{3}{3\mu-\lambda}+\frac{1}{\mu}\right),$$

$$m_{49}=\frac{3e^{-3\mu t}}{2\mu^2(3\mu-\lambda)^3}\left(\frac{6}{(3\mu-\lambda)^2}+\frac{3}{4\mu^2}+\frac{3}{\mu(3\mu-\lambda)}\right),$$

$$m_{50}=\frac{3e^{-3\mu t}}{2\mu^2(3\mu-\lambda)^3}\left(\frac{10}{(3\mu-\lambda)^3}+\frac{1}{2\mu^3}+\frac{9}{4\mu^2(3\mu-\lambda)}+\frac{6}{\mu(3\mu-\lambda)^2}\right)$$

Substituting the values of A_1, B_1, C_1, D_1 and H_1 and from equation (23), (21), (20), (24) and (22) into equation (4), we obtain:

$$\dot{a}=\varepsilon\{q_1a^3e^{-2\lambda t}+q_2a^2be^{-2\lambda t}+(q_3+q_4)(a^2c+ab^2)e^{-2\lambda t}\}$$

$$\dot{b}=\varepsilon\{r_1a^2be^{-2\lambda t}+r_2(ab^2+a^2c)e^{-2\lambda t}\}$$

$$\dot{c}=\varepsilon\{l_1(ab^2+a^2c)e^{-2\lambda t}+l_2(b^2d+2acd+2abh)e^{-(\lambda+\mu)t}+l_3(cd^2+2bdh+ah^2)e^{-2\mu t}+l_4dh^2e^{-(3\mu-\lambda)t}\}$$

$$\dot{d} = \varepsilon\{n_1 a^2 d e^{-2\lambda t} + n_2 a d^2 e^{-(\lambda+\mu)t} + n_3 d^3 e^{-2\mu t} + n_4(b^2 d + 2acd + 2abh)e^{-2\lambda t} + n_5(cd^2 + 2bdh + ah^2)e^{-(\lambda+\mu)t} + n_6 dh^2 e^{-2\mu t} + n_7(2abd + a^2 h)e^{-2\lambda t} + n_8(bd^2 + 2adh)e^{-(\lambda+\mu)t} + n_9 d^2 h e^{-2\mu t} + n_{10}(b^2 d + 2acd + 2abh)e^{-2\lambda t} + n_{11}(cd^2 + 2bdh + ah^2)e^{-(\lambda+\mu)t} + n_{12} dh^2 e^{-2\mu t}\}$$

$$\dot{h} = \varepsilon\{p_1(2abd + a^2 h)e^{-2\lambda t} + p_2(bd^2 + 2adh)e^{-(\lambda+\mu)t} + p_3 d^2 h e^{-2\mu t} + p_4(b^2 d + 2acd + 2abh)e^{-2\lambda t} + p_5(cd^2 + 2bdh + ah^2)e^{-(\lambda+\mu)t} + p_6 dh^2 e^{-2\mu t}\} \tag{26}$$

Here all of the equation of (26) have no exact solutions, but since $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{h}$ are proportional to the small parameter ε, so they are slowly varying functions of time *t*. Consequently, it is possible to replace *a*, *b*, *c*, *d*, *h* by their respective values obtained in linear case (i.e., the values of *a*, *b*, *c*, *d*, *h* obtained when $\varepsilon = 0$) in the right hand side of equation (26). This type of replacement was first introduced by Murty and Deekshatulu ([4], [22]) to solve similar type of nonlinear equations.

Therefore, the solution of (26) is:

$$\dot{a} = a_0 + \varepsilon\{q_1 a^3 \frac{1-e^{-2\lambda t}}{2\lambda} + q_2 a^2 b \frac{1-e^{-2\lambda t}}{2\lambda} + (q_3 + q_4)(a^2 c + ab^2)\frac{1-e^{-2\lambda t}}{2\lambda}\}$$

$$\dot{b} = b_0 + \varepsilon\{r_1 a^2 b \frac{1-e^{-2\lambda t}}{2\lambda} + r_2(ab^2 + a^2 c)\frac{1-e^{-2\lambda t}}{2\lambda}\}$$

$$\dot{c} = c_0 + \varepsilon\{l_1(ab^2 + a^2 c)\frac{1-e^{-2\lambda t}}{2\lambda} + l_2(b^2 d + 2acd + 2abh)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + l_3(cd^2 + 2bdh + ah^2)\frac{1-e^{-2\mu t}}{2\mu} + l_4 dh^2 \frac{1-e^{-(3\mu-\lambda)t}}{3\mu-\lambda}\} \tag{27}$$

$$\dot{d} = d_0 + \varepsilon\{n_1 a^2 d \frac{1-e^{-2\lambda t}}{2\lambda} + n_2 a d^2 \frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + n_3 d^3 \frac{1-e^{-2\mu t}}{2\mu} + n_4(b^2 d + 2acd + 2abh)\frac{1-e^{-2\lambda t}}{2\lambda} + n_5(cd^2 + 2bdh + ah^2)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + n_6 dh^2 \frac{1-e^{-2\mu t}}{2\mu} + n_7(2abd + a^2 h)\frac{1-e^{-2\lambda t}}{2\lambda} + n_8(bd^2 + 2adh)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + n_9 d^2 h \frac{1-e^{-2\mu t}}{2\mu} + n_{10}(b^2 d + 2acd + 2abh)\frac{1-e^{-2\lambda t}}{2\lambda} + n_{11}(cd^2 + 2bdh + ah^2)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + n_{12} dh^2 \frac{1-e^{-2\mu t}}{2\mu}\}$$

$$\dot{h} = h_0 + \varepsilon\{p_1(2abd + a^2 h)\frac{1-e^{-2\lambda t}}{2\lambda} + p_2(bd^2 + 2adh)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + p_3 d^2 h \frac{1-e^{-2\mu t}}{2\mu} + p_4(b^2 d + 2acd + 2abh)\frac{1-e^{-2\lambda t}}{2\lambda} + p_5(cd^2 + 2bdh + ah^2)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + p_6 dh^2 \frac{1-e^{-2\mu t}}{2\mu}\}$$

Hence, we obtain the first approximate solution of the equation (14) as:

$$x(t,\varepsilon) = (a + bt + ct^2)e^{-\lambda t} + (d + ht)e^{-\mu t} + \varepsilon u_1 \tag{28}$$

where*a*, *b*, *c*,*d* and *h* are given by the equation (27) and u_1 is given by (25).

IV. Figures and Tables

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we have compared the approximate solution to the numerical solution.We have computed $x(t, \varepsilon)$ using (28), in which *a*, *b*, *c*,*d* and *h* are obtained from (27) and u_1 is calculated from equation (25). The result obtained from

(28) for various values of t, and the corresponding numerical solution obtained by a fourth order *Runge-Kutta* method is presented in the following**Fig.1**,**Fig.2**,**Fig.3** and **Fig.4**respectively.

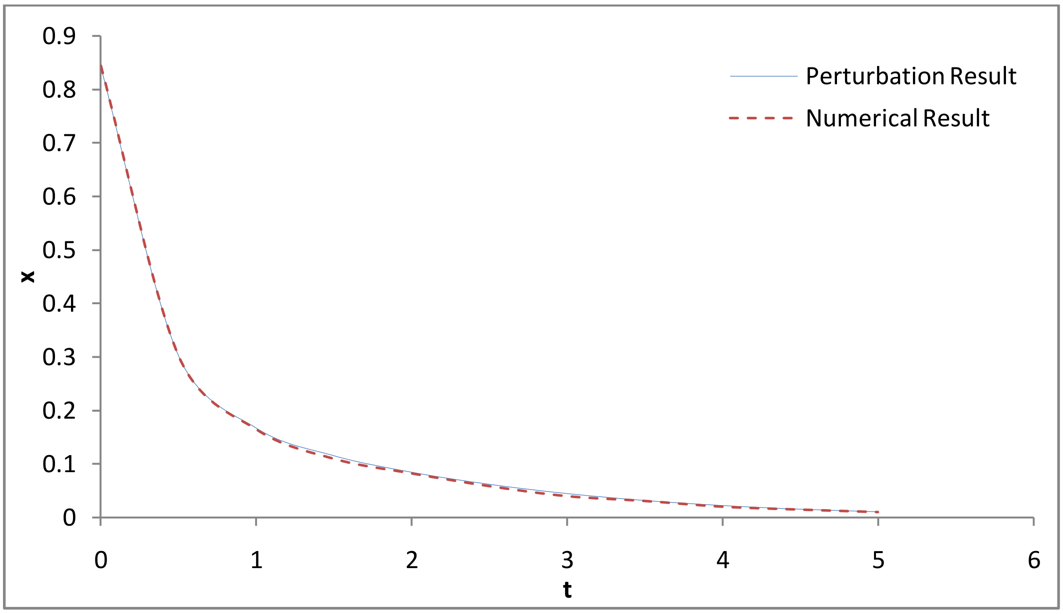

Figure 1: Comparison between perturbation and numerical results for $\mu = 3.6, \lambda = 0.95$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.45,\ b_0 = 0.25,\ c_0 = 0.15,\ d_0 = 0.30,\ h_0 = 0.25$.

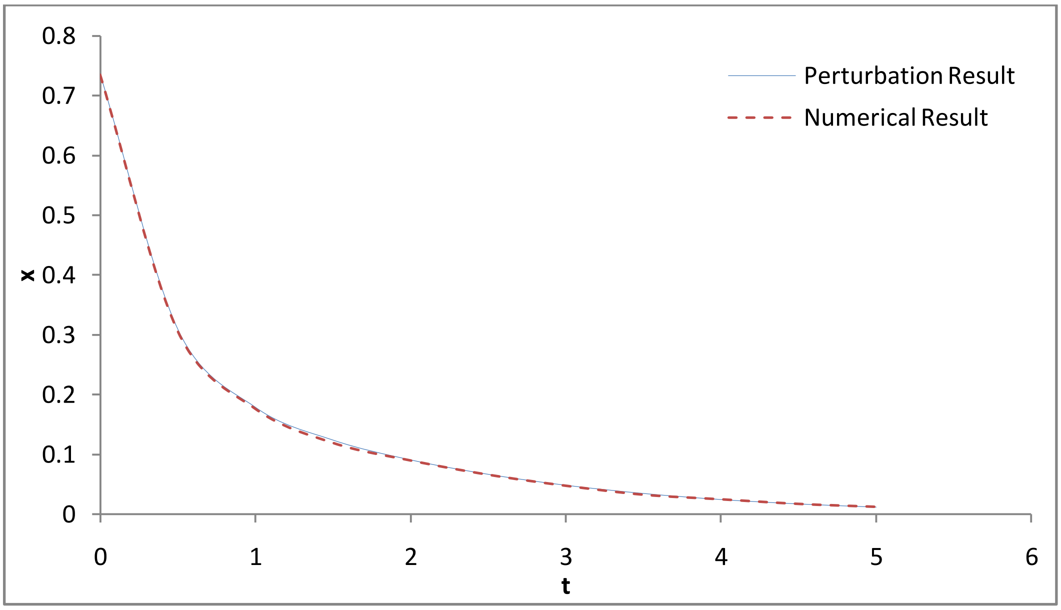

Figure 2: Comparison between perturbation and numerical results for $\mu = 3.5,\ \lambda = 0.9$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.45,\ b_0 = 0.35,\ c_0 = 0.10,\ d_0 = 0.25,\ h_0 = 0.20$.

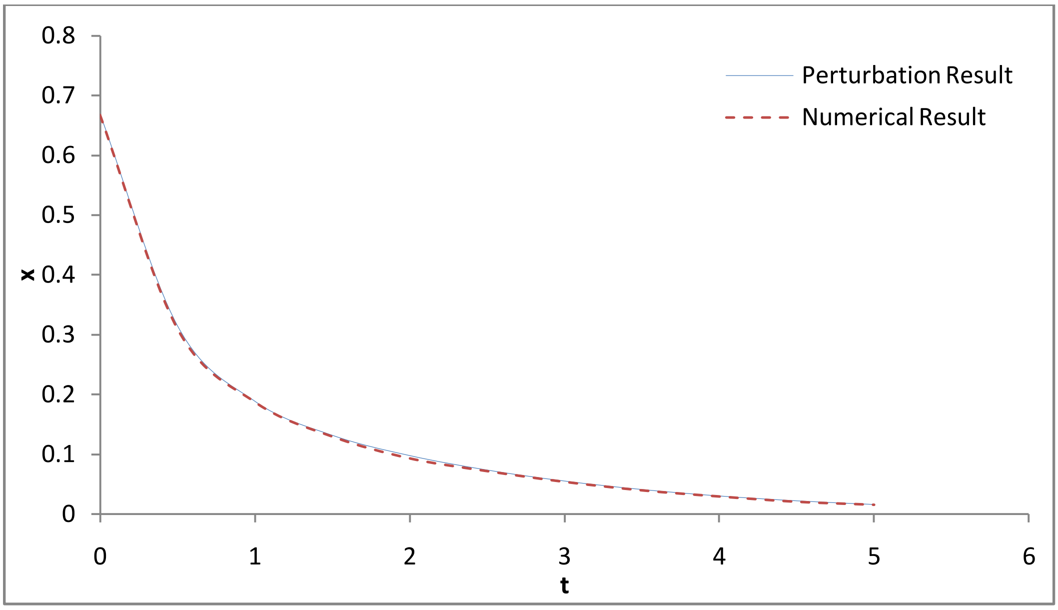

Figure 3: Comparison between perturbation and numerical results for $\mu = 3.2,\ \lambda = 0.8$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.40,\ b_0 = 0.30,\ c_0 = 0.10,\ d_0 = 0.25,\ h_0 = 0.15.$

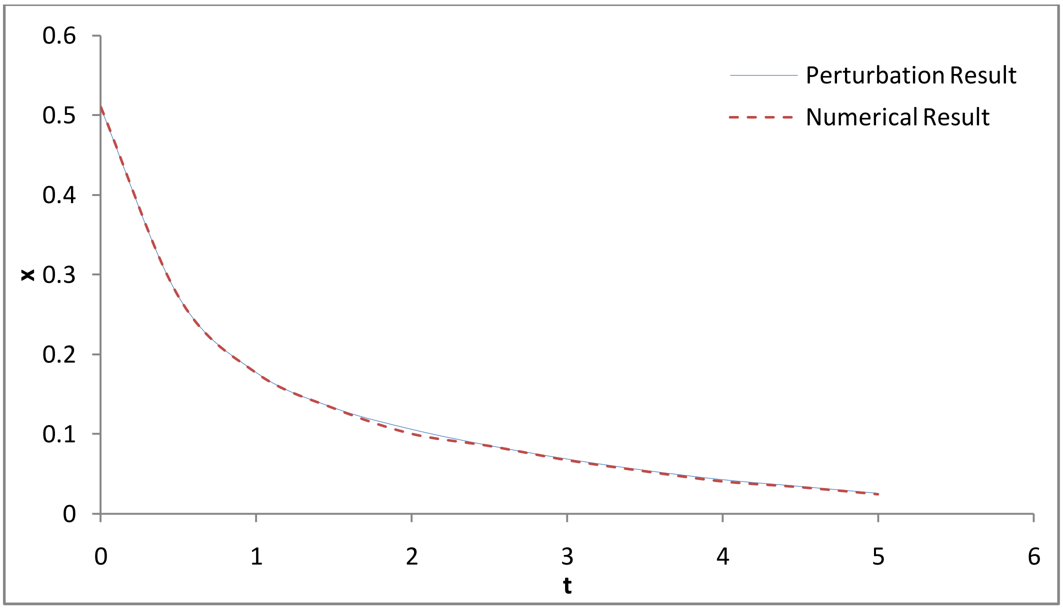

Figure 4: Comparison between perturbation and numerical results for $\mu = 3,\ \lambda = 0.7$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.35,\ b_0 = 0.35,\ c_0 = 0.10,\ d_0 = 0.15,\ h_0 = 0.15.$

For various values of *t*, the corresponding perturbation andnumerical results are shown in the following **Table 1**, **Table 2**, **Table 3**and**Table 4** respectively.

Table 1:*Comparison between perturbation and numeicalresults*

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	0.843764	0.843764
0.5	0.302987	0.301971
1	0.167044	0.164832
1.5	0.115136	0.109746
2	0.084293	0.082527
2.5	0.061659	0.058746
3	0.044406	0.040024
3.5	0.031458	0.031003
4	0.02197	0.020325
4.5	0.015163	0.014913
5	0.010364	0.010267

Table 2:*Comparison between perturbation and numerical results*

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	0.734501	0.734501
0.5	0.308836	0.305815
1	0.178852	0.175932
1.5	0.124124	0.118942
2	0.091032	0.089527
2.5	0.066961	0.065746
3	0.048701	0.047325
3.5	0.034957	0.032327
4	0.024796	0.024623
4.5	0.017413	0.016913
5	0.012125	0.012027

Table 3: Comparison between perturbation and numerical results

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	0.667447	0.667447
0.5	0.314167	0.310231
1	0.188513	0.187692
1.5	0.131734	0.129983
2	0.097887	0.093529
2.5	0.073761	0.072276
3	0.055358	0.054323
3.5	0.041241	0.040123
4	0.030437	0.029825
4.5	0.022284	0.021016
5	0.016202	0.015975

Table 4: Comparison between perturbation and numerical results

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	0.509836	0.509836
0.5	0.272878	0.271985
1	0.177268	0.176334
1.5	0.132883	0.131949
2	0.105916	0.099987
2.5	0.085495	0.084784
3	0.068591	0.067021
3.5	0.054424	0.053042
4	0.042705	0.040325
4.5	0.034179	0.033113
5	0.02556	0.024167

V. Conclusion

In this study, we have carried out the modification of the KBM method and successfully applied the modified method to the fifth order more critically damped nonlinear systems. Based on the modified KBM method transient responses of nonlinear differential systems have been investigated. For fifth order more critically damped systems the solutions are looked for such circumstances where in the triply eigenvalues are small and the two eigenvalues are large. For different sets of initial conditions of the modified KBM method, the results provide solutions which show well agreement with the numerical solutions

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