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A Completion for Distributive Lattices

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Abstract: *A completion for a class of lattices is constructed and it is observed that a congruence relation on a given lattice can be extended to its completion.*
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I. Introduction

Distributive lattices do have some special properties to characterize ideals in them. Corollary 4 in section 3 of chapter 2 in [1] implies that any ideal in a distributive lattice is a congruence class, and a study of congruence relations in distributive lattices leads to fruitful results(cf:[2]). So, a completion for a class of distributive lattices is constructed through ideals in this article, and an extension of a congruence relation to completion is also discussed. Let us use the following definitions for this purpose.

Definition 1.1 Let us say that a sublattice $(L_1, \vee, \wedge)$ of a lattice $(L_2, \vee, \wedge)$ is dense in L_2, if any element x in L_2 is either a supremum of a collection of elements in L_1 or an infimum of a collection of elements in L_1.

Definition 1.2 If $(L_1,\vee, \wedge)$ is dense in $(L_2, \vee, \wedge)$, then let us say that $(L_2, \vee,\wedge)$ is a completion of $(L_1,\vee, \wedge)$, if L_2 is a complete lattice.
An extension of a congruence relation to completion is also to be discussed, and the following definition is applicable for this purpose.

Definition 1.3 Let $(L_2, \vee,\wedge)$ be a completion of $(L_1, \vee,\wedge)$. A congruence relation θ' on L_2 is said to be a complete extension of a congruence relation θ on L_1, if the restriction of θ' to L_1 is θ , and if $\vee_{i\in I} x_i \equiv \vee_{i\in I} y_i$ (mod θ ') in L_2, whenever $x_i \equiv y_i$ (mod θ), $\forall i \in I$, in L_1 and the suprema exist in L_2,and if $\wedge_{i\in I} x_i \equiv \wedge_{i\in I} y_i$ (mod θ ') in L_2, whenever, $x_i \equiv y_i$ (mod θ), $\forall i \in I$, in L_1 and the infima exist in L_2, for collections $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ in L_1.

Definition 1.4 A congruence relation on a lattice $(L, \vee,\wedge)$ is complete,if $(x_i)_{i\in I} \subseteq L$, $(y_i)_{i\in I} \subseteq L$ and $x_i \equiv y_i$ (mod θ) , $\forall i \in I$ imply $\vee_{i\in I} x_i \equiv \vee_{i\in I} y_i$ (mod θ) and $\wedge_{i\in I} x_i \equiv \wedge_{i\in I} y_i$ (mod θ), whenever these suprema and infima exist.

II. Lattice Completion

Let us recall that if L is the cartesian product of a collection of lattices $((L_i, \vee_i,\wedge_i))_{i\in I}$, then 'join' and 'meet' operations can be defined pointwisely on L by $(x_i)_{i\in I} \vee (y_i)_{i\in I} = (x_i \vee_i y_i)_{i\in I}$ and $(x_i)_{i\in I} \wedge (y_i)_{i\in I} = (x_i \wedge_i y_i)_{i\in I}$. These operations are to be considered in this article for sublattices of a cartesian product lattice.

Definition 2.1 Let $(L,\vee,\wedge)$ be a lattice. To each $a\in L$, let (a] denote the (ideal) sublattice $(a] = \{x \in L : x\leq a\}$. Let us now define the inverse limit L^* as a subset of the cartesian product lattice of the collection of lattices $((a])_{a\in L}$ by $L^* = \{(x_a)_{a\in L} : x_a \in (a];\ \forall a\in L, \text{ and } x_a = x_b \wedge a, \text{ whenever } a\leq b \text{ in } L\}$.

Lemma 2.2 Suppose $(x_a)_{a\in L} \in L^*$. Then $x_a = \vee_{c\in L} (x_c \wedge a)$, $\forall a\in L$.
Proof: Given $a,c \in L$, there exists $b\in L$ such that $b \geq a$, $b\geq c$, $x_c = x_b \wedge c$, and such that $x_c \wedge a = (x_b \wedge c) \wedge a = (x_b \wedge a) \wedge c = x_a \wedge c\leq x_a = x_a\wedge a$. This proves that $x_a=\vee_{i\in I}(x_c \wedge a)$.

Lemma 2.3 Let L and L^* be as in the definition 2.1. Then, to each $x \in L$, we have $(x \wedge a)_{a\in L} \in L^*$.
Proof: Fix $x\in L$. To each $a\in L$, we have $x \wedge a \in (a]$. If $a \leq b$ in L, then $(x \wedge b) \wedge a = x \wedge (b \wedge a) = x \wedge a$. So, $(x \wedge a)_{a\in L} \in L$.

Lemma 2.4 Suppose L given definition 2.1 is distributive so that $(x\vee y)\wedge z = (x\wedge z) \vee (y \wedge z)$; $\forall x,y,z \in L$. Then the inverse limit L^* given in definition 2.1 is a sublattice of the product lattice of the collection $((a])_{a\in L}$.
Proof: Let $(x_a)_{a\in L}$, $(y_a)_{a\in L} \in L$. Then $(x_a)_{a\in L} \wedge (y_a)_{a\in L} = (x_a \wedge y_a)_{a\in L}$ and $(x_a)_{a\in L} \vee (y_a)_{a\in L} = (x_a \vee y_a)_{a\in L}$. Also, if $a\leq b$ in L, then $(x_b\wedge y_b)\wedge a = (x_b \wedge a) \wedge (y_b\wedge a) = x_a\wedge y_a$; and $(x_b\vee y_b) \wedge a = (x_b \wedge a) \vee (y_b \wedge a) = x_a \vee y_a$. So, $(x_a)_{a\in L} \wedge (y_a)_{a\in L} \in L$ and $(x_a)_{a\in L} \vee (y_a)_{a\in L} \in L$ so that L^* is a lattice.

Lemma 2.5 Suppose (a] is a complete lattice for each $a \in L$, and suppose
$z\wedge (\vee_{i\in I}x_i) = \vee_{i\in I}(z\wedge x_i)$, whenever $x_i \in (a]$; $\forall i\in I$, for any fixed $a \in L$, and
$z \in (a]$. Then L^* given in the definition 2.1 is a complete lattice.
Proof: If x, y, z $\in$ L, then there is a $c \in L$ such that $x \leq c$, $y\leq c$, $z \leq c$ so that x, y, z$\in$ (c] and hence $(x\vee y) \wedge z = (x \wedge z) \vee (y\wedge z)$. So, by the previous lemma, L^* is a lattice.
Let $((x_a^{(i)})_{a\in L})_{i\in I}$ be a collection of elements in L^*. If $a\leq b$ in L, then $(\wedge_{i\in I} x_b^{(i)})\wedge a= \wedge_{i\in I}(x_b^{(i)}\wedge a)= \wedge_{i\in I}x_a(i) \in(a]$, $x_b^{(i)} \wedge a\in(a]$, and $x_b^{(i)} \in(b]$, $\forall i\in I$, and when (a] and (b] are complete. Thus $\wedge_{i\in I}(x_a^{(i)})_{a\in L}$ exists in L^*. If $a\leq b$ in L, then $(\vee_{i\in I} x_b^{(i)})\wedge a=\vee_{i\in I}(x_b^{(i)}\wedge a)=\vee_{i\in I}x_a^{(i).}$ This proves that L* is a complete lattice.

Lemma 2.6 Suppose L satisfies the hypothesis of lemma 2.4. If $T : L \to L^*$ is defined by $T (x) = (x \wedge a)_{a\in L}$, $\forall x \in L$, then T is an injective lattice homomorphism.
Proof: By lemma 2.3, $T (x) \in L^*$, $\forall x\in L$. If T (x) = T (y), then $x\wedge a = y\wedge a$, $\forall a \in L$, and hence $x = x\wedge x = y\wedge x = x\wedge y = y \wedge y = y$. This proves that T is 1-1. Let x, y $\in$ L. Then $T (x\wedge y) = ((x\wedge y)\wedge a)_{a\in L} = ((x\wedge a)\wedge(y\wedge a))_{a\in L} = (x\wedge a)_{a\in L} \wedge (y \wedge a)_{a\in L} = T (x) \wedge T (y)$, and $T (x\vee y) = ((x \vee y) \wedge a)_{a\in L} = ((x \wedge a)\vee (y \wedge a))_{a\in L} = (x \wedge a)_{a\in L}\vee (y \wedge a)_{a\in L} = T (x) \vee T (y)$. This proves the lemma.
So, if L is a distributive lattice, then we can identify L as the sublattice T (L) of L^*.

Lemma 2.7 If L and T are as in lemma 2.6, then T (L) is dense in L^* in the sense of definition 1.1.
Proof: Let $(x_a)_{a\in L} \in L$. To each fixed $c\in L$, define $x(c) \in (c]$ L by $x(c) = x_c$. To each $a \in L$, let π_a denote the coordinate projection of the product lattice defined by $((c])_{c\in L}$ onto (a]. Then $\pi_a(T (x(c))) = x_c \wedge a, \forall c \in L$ and $\forall a \in L$. Therefore, by lemma 2.2, we have $x_a=\vee_{c\in L}(x_c\wedge a)= \vee_{c\in L}\pi_a(T(x(c))=\pi_a(\vee_{c\in L}(T(x(c))$, $\forall$ $a\in L$. This proves that $(x_a)_{a\in L}= \vee_{c\in L} T(x(c))$, when $T(x(c)) \in T(L)$. This proves the lemma.

Theorem 2.8 Suppose L satisfies the hypostheses of lemma 2.5. Then L^* is a completion of L in the sense of definition 1.2.
Proof: Let us identify L with T (L), and then T (L) is dense in L^* and L is complete. So, L^* can be considered as a completion of L.

Remark 2.9 Since L is distributive, by the proof of the lemma 2.5, L^* of the previous theorem 2.8 is also distributive.

III. Extension of a congruence relation

Let θ be a congruence relation on a lattice L. When x and y are related by in L, let us write $x\equiv y(\text{mod } \theta)$. Define a relation θ^* on the product lattice of the collection $((a])_{a\in L}$ by $(x_a)_{a\in L}\equiv (y_a)_{a\in L} (\text{mod } \theta^*)$ if and only if $x_a\equiv y_a (\text{mod } \theta)$, $\forall a \in L$. Since θ is a congruence relation, θ^* is also a congruence relation. Let us use the same notation for its restriction to L^* and for its restriction to T (L) for T of lemma 2.6, when L is distributive. Let us assume in this section that L is distributive.

Lemma 3.1 $x \equiv y (\text{mod } \theta)$ in L if and only if $T (x) \equiv T (y) (\text{mod } \theta^*)$.
Proof: If $x\equiv y (\text{mod } \theta)$, then $x\wedge a\equiv y \wedge a (\text{mod } \theta)$, $\forall a \in L$, and hence
$T (x)\equiv T (y)(\text{mod } \theta^*)$.
Suppose $T (x) \equiv T (y) (\text{mod } \theta^*)$, for some x, y $\in$ L so that $x \wedge a\equiv y \wedge a$
$(\text{mod } \theta)$, $\forall a \in L$. Then
$x \wedge x \equiv y\wedge x(\text{mod } \theta)$
$\equiv x \wedge y (\text{mod } \theta)$
$\equiv y\wedge y(\text{mod } \theta)$ so that $x \equiv y (\text{mod } \theta)$. This proves the lemma.
By this lemma, θ^*on L can be considered as an extension of on L. With this identification, we can state the following theorem 3.2.

Theorem 3.2 Let θ' be a congruence relation on L^* which is a complete extension of θ to L^* . Then $\theta^*\leq \theta'$ on L^*.
Proof: Suppose $(x_a)_{a\in L} \in L^*$, $(y_a)_{a\in L} \in L^*$ and $(x_a)_{a\in L} \equiv (y_a)_{a\in L} (\text{mod } \theta^*))$. Then $x_a\equiv y_a(\text{mod } \theta)$, $\forall a\in L$. Hence

$x(c) \equiv y(c)$ (mod θ), when $x(c) = x_c$ and $y(c) = y_c$, $\forall c \in L$. Then by our identification of L with T(L), we have $T(x(c)) \equiv T(y(c))$ (mod θ'), $\forall c \in L$. So, $\vee_{c\in L} T(x(c)) \equiv \vee_{c\in L} T(y(c))$ (mod θ'), or $(x_a)_{a\in L} \equiv (y_a)_{a\in L}$ (mod θ') in view of the proof of the lemma 2.7.
$\theta^* \leq \theta'$ on L^*.

Theorem 3.3 Suppose (a] is complete and θ is complete on (a], $\forall a \in L$. Then θ^* is also complete on L^*.
Proof: Suppose $((x_a^{(i)})_{a\in L})_{i\in I}$, $((y_a^{(i)})_{a\in L})_{i\in I}$ are subcollections of L^* such that $(x_a^{(i)})_{a\in L} \equiv (y_a^{(i)})_{a\in L}$ (mod θ^*), $\forall i \in I$. Then $\vee_{i\in I} x_a^{(i)} \equiv \vee_{i\in I} y_a^{(i)}$ (mod θ) and $\wedge_{i\in I} x_a^{(i)} \equiv \wedge_{i\in I} y_a^{(i)}$ (mod θ), because the suprema and the infima exist in the complete lattice (a], $\forall a \in L$.So, $\vee_{i\in I}(x_a^{(i)})_{a\in L} \equiv \vee_{i\in I}(y_a^{(i)})_{a\in L}$ (mod θ^*) and $\wedge_{i\in I}(x_a^{(i)})_{a\in L} \equiv \wedge_{i\in I}(y_a^{(i)})_{a\in L}$ (mod θ^*). Thus θ^* is also complete on L^*.

Remark 3.4 If the conditions of lemma 2.5 and theorem 3.3 are satisfied, then L^* is a completion of L and θ^*on L is a complete extension of θ on L.

References

[1]. G.Gratzer, General lattice theory, Academic press, New York, 1978.
[2]. F.Wehrung, A solution to Dilworth's congruence lattice problem, Advances in Mathematics, 216(2007)610-625