IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 2 Ver. III (Mar - Apr. 2015), PP 33-37
www.iosrjournals.org

Stability Analysis in Dynamics of Plant – Herbivore System With Mating Induced Allee Effect

Vijayalakshmi.S[1], Sahayajothi.S[2], Gunasekaran.M[3,*]
[1,2]*Department of Mathematics, D.R.B.C.C.C. Hindu College, Chennai-600072.*
[3]*PG Department of Mathematics, Sri Subramaniya Swami Government Arts College, Tiruttani-631209.*

Abstract: *Insect herbivores are hypothesized to be major factors affecting the ecology and evolution of plants. Earlier, plant-herbivore models were phrased in terms of total vegetation biomass and total herbivore population. Allee effect is an important dynamics phenomenon believed to be manifested in several population process, notably extinction and invasion. In this paper, a discrete-time plant-herbivore model with mating induced Allee effect is investigated. We obtain asymptotically stable conditions of the equilibrium points which are subject to the Allee effect. The Allee effect which occurs on plant population is discussed by stability and numerical analysis. This study suggests that Allee effect has stabilizing effect on Plant-Herbivore system.*
Keywords: *Plant-herbivore system, Mating, Allee effect, Stability analysis, Equilibrium points.*

I. Introduction

Interactions between plants and herbivores have been studied by ecologists for many decades. Some ecological models, which are simple in mathematical expression, have been designed to study population temporal dynamics. In particular, the pioneering work in the field was initiated by May [18, 19]. Allee effect is a crucial phenomenon that has drawn considerable attention from ecologists [1, 3, 4, 6, 9, 15, and 20]. Various mechanisms related to Allee effect have been discussed as singular entities. It describes a positive interaction among individuals at low population sizes or densities, and these interactions may be critical for survival and reproductions. The population goes extinction below a threshold and increases above the threshold population density. Allee effect is therefore important in conservation of endangered and exploited species [13, 14, 16, and 21]. The Allee effect can be caused by any number of causes, for example, mating between the individuals of a species at low population densities. Other causes can be largely reduced defense against herbivores, special trends of social dysfunction, etc. Some investigations have shown that Allee effect may be a destabilizing factor in an ecological system [25]. But some papers have addressed the Allee effect with focus on the stabilizing effect [12, 24, and 31]. Most studies on plants and animals have shown that Allee effect is common in nature [1, 2, 10, 22, and 23]. Darwin was the first of a long series of evolutionary botanists interested in mating system evolution [7, 8]. At the heart of this approach was the central role of floral biology and pollination process.
The usual frame work for discrete-generation plant-herbivore model [17, 30] has the form

$$\begin{aligned} P_{t+1} &= P_t\, f_1(P_t) f_2(a, H_t) \\ H_{t+1} &= g_1(P_t)\, g_2(H_t) \end{aligned} \qquad (1)$$

Where
P_t - The density of edible plant biomass in generation t.
H_t - The population density of herbivore at time t.
$f_2(a, H_t)$ - The effect of the herbivore on the plant population growth rate with $f_2(a, 0) = 1$
a - The amount of damage caused by herbivore.
$g_1(P_t)$ - The function of plant density.
$g_2(H_t)$ - The non- linear function of herbivore density.

Many consumer-resource models assume a non-linear relationship between resource population size and attack rate [30]. For plants and insect herbivores, we similarly expect a non-linear functional relationship, due to herbivore foraging time and satiation. The relationship is expressed in terms of plant biomass units rather than population size, because herbivores are unlikely to kill entire plants.

The growth function $F(P_t)$ determines the amount of new leaves available for consumption for the herbivore in generation *t*. We assume that the herbivores search for plants randomly. The area consumed is measured by the parameter *a*, i.e., *a* is a constant that correlates to the total amount of the biomass that an herbivore consumes. The herbivore has a one year life cycle, the larger *a*, the faster the feeding rate. After attacks by herbivores, the biomass in the plant population is reduced to the following model which is reformulated from (1).

$$\begin{aligned} P_{t+1} &= F(P_t) e^{-aH_t} \\ H_{t+1} &= P_t[1 - e^{-aH_t}] \end{aligned} \qquad (\mathbf{2})$$

Where

$F(P_t)$ - The amount of new leaves available for consumption for the herbivore in generation t.

e^{-aH_t} - Probability of the plant escapes from the herbivore.

$[1-e^{-aH_t}]$ - Probability of the plant attacked by the herbivore.

Noting that the system (2) represents the dynamics of a system where the plant is attacked before it has a chance to grow.

II. The model

Now we consider the system $(2)$when the plant population is subject to Allee effect [17, 26, 27, 28, 29, and 30] as follows:

$$P_{t+1} = r\left[\frac{b\,P_t}{1+b\,P_t}\right]e^{-aH_t}$$
$$H_{t+1} = P_t[1-e^{-aH_t}] \qquad (3)$$

Where

$\left[\frac{b\,P_t}{1+b\,P_t}\right]$ - The term for mate-finding Allee effect.

b - The Allee effect constant.

r - The intrinsic growth rate of the plant.

Noting that the consequence of Allee effect is that to get rid of the hardness to find a mate (induced self- fertilization or pollinating agent) at low population density.

2.1. Fixed point and Local stability

We now study the existence of fixed points of the system (3), particularly we are interested in the non-negative interior fixed point and we list all possible fixed points.

i) $E_0 = (0,\ 0)$ is trivial or extinction fixed point.

ii) $E_1 = \left(\frac{rb-1}{b}, 0\right)$ is the axial or exclusion fixed point in the absence of the herbivore (H=0).

iii) $E_2 = (P^*, H^*)$ is the interior fixed point , where $P^* = \frac{X\log X}{a(X-1)}$ and $H^* = \frac{\log X}{a}$ (4)

where $X = \frac{rb}{1+bP}$.

III. The Dynamical behavior of the model

In this section, we investigate the local behavior of the model(3)around each fixed point. The local stability analysis of the model (3) can be studied by computing the variation matrix corresponding to each fixed point.

The variation matrix of the model at the state variable is given by

$$P_{t+1} = F_1(P, H)$$
$$H_{t+1} = F_2(P, H)$$

for which the Jacobian matrix is given by

$$J = J(P,H) = \begin{pmatrix} \frac{rb\,e^{-aH}}{(1+b\,P)^2} & \frac{-arbP\,e^{-aH}}{1+b\,P} \\ 1-e^{-aH} & Pae^{-aH} \end{pmatrix} \qquad (5)$$

The characteristic equation of Jacobian matrix can be written as $\lambda^2 - \lambda\, Tr(J) + Det(J) = 0$ where *Tr* is the trace and *Det* is the determinant of the Jacobian matrix *J (P, H)* which are defined as

$Tr = \frac{rb\,e^{-aH}}{(1+b\,P)^2} + Pae^{-aH}$ and $Det = \frac{rabPe^{-aH}[1+bP-bPe^{-aH}]}{(1+bP)^2}$

Hence the model (3) is a Dissipative dynamical system if $\left|\frac{rabPe^{-aH}[1+bP-bPe^{-aH}]}{(1+bP)^2}\right| < 1$ and Conservative dynamical one if and only if $\left|\frac{rabPe^{-aH}[1+bP-bPe^{-aH}]}{(1+bP)^2}\right| = 0$. In order to study the stability of the fixed point model, we first give the following theorem and lemma.

Theorem:

If $P(\lambda) = \lambda^3 + B\,\lambda^2 + C\,\lambda + D = 0$ is the characteristic equation of the matrix then the following statements are true:

a) If every root of the equation has absolute value less than one, then the fixed point of the system is locally asymptotically stable and the fixed point is called a sink.

b) If at least one of the roots of the equation has absolute value greater than one, then the fixed point of the system is unstable and the fixed point is called a saddle.

c) If every root of the equation has absolute value greater than one, then the fixed point of the system is a source.
d) The fixed point of the system is called hyperbolic if no root of the equation has absolute value equal to one. If there exists a root of equation with absolute value equal to one, then the fixed point is called Non-hyperbolic [11].

Preposition: 1
The fixed point $\boldsymbol{E_0}$ is locally asymptotically stable if $r < \frac{1}{b}$, otherwise unstable.
Proof:
In order to prove the result, we estimate the Eigen values of the Jacobian matrix J at $\boldsymbol{E_0}$ is given by

$$J(E_0) = \begin{pmatrix} rb & 0 \\ 0 & 0 \end{pmatrix}$$

The Eigen values of J at $\boldsymbol{E_0}$ are $\boldsymbol{\lambda_1}$ rb and $\boldsymbol{\lambda_2} = 0$. Thus $\boldsymbol{E_0}$ is stable if $r < \frac{1}{b}$. If $r > \frac{1}{b}$ then the fixed point $\boldsymbol{E_0}$ is unstable.
Preposition: 2
The fixed point $\boldsymbol{E_1}$ is stable if it satisfies the condition $\frac{1}{b}$ $r < \frac{a+b}{a\,b}$.
Proof:
In order to prove the result, we estimate the Eigen values of the Jacobian matrix J at $\boldsymbol{E_1}$ is given by

$$J(E_1) = \begin{pmatrix} \frac{1}{rb} & \frac{a\,(1-rb)}{b} \\ 0 & \frac{a\,(rb\,-1)}{b} \end{pmatrix}$$

The eigen values of J at $\boldsymbol{E_1}$ are $\boldsymbol{\lambda_1} = \frac{1}{rb}$ and $\boldsymbol{\lambda_2} = \frac{a\,(rb-1)}{b}$. The point $\boldsymbol{E_1}$ is locally asymptotically stable if $\frac{1}{rb} < 1$ and $\frac{a\,(rb-1)}{b} < 1$, which implies that $r > \frac{1}{b}$ and r $\frac{a+b}{a\,b}$. Thus $\boldsymbol{E_1}$ is stable if $\frac{1}{b}$ $r < \frac{a+b}{a\,b}$.

Lemma:
If the Eigen values of the Jacobian matrix of the fixed point are inside the unit circle of the complex plane, the fixed point $\boldsymbol{E_2}$ is locally stable. Using Jury's condition [5], the necessary and sufficient conditions for local stability of interior fixed points for $|\lambda_{1,2}|<1$, are
(i) $1+Tr(J)+Det(J)>0$
(ii) 1 $Tr(J)+Det(J)>0$
(iii) $|\boldsymbol{Det(J)}|<1$

3.1. Local Stability and Dynamical behavior around interior fixed point $\boldsymbol{E_2}$
We now investigate the local stability of interior fixed point $\boldsymbol{E_2}$. The Jacobian matrix (5) at $\boldsymbol{E_2}$ has the form

$$J(E_2) = \begin{pmatrix} \frac{ra^2b(X-1)^2}{X[a(X-1)+bXlogX]^2} & \frac{-rablogX}{[a(X-1)+bXlogX]} \\ \frac{X-1}{X} & \frac{logX}{X-1} \end{pmatrix} \quad (6)$$

Its characteristic equation is $\lambda^2 - \lambda\, Tr(J(E_2)) + Det(J(E_2)) = 0.$ (7)
where $Tr(J(E_2)) = \frac{ra^2b(X-1)^2}{X[a(X-1)+bXlogX]^2} + \frac{logX}{X-1} = \boldsymbol{B_1}$ and $Det(J(E_2)) = \frac{(X-1)rab\,logX\,[a+blogX]}{[a(X-1)+bX\,logX]^2} = \boldsymbol{B_2}$.
By Lemma, using formulas of *Tr* and *Det*, the interior fixed point is locally stable if we find that the inequality (i) is equivalent to $1+\boldsymbol{B_1}+\boldsymbol{B_2}>0$ which implies that $\boldsymbol{B_1}+\boldsymbol{B_2}>$ 1, the inequality (ii) is equivalent to $\boldsymbol{B_1}-\boldsymbol{B_2}<1$, the inequality(iii) is equivalent to $|\boldsymbol{B_2}|<1$. For the interior fixed point $\boldsymbol{E_2}$ the roots of eq. (7) are $\lambda_{1,2} = \frac{B_1 \pm \sqrt{{B_1}^2 - 4\,B_2}}{2}$. Both eigen values are locally asymptotically stable if $\boldsymbol{B_1} + \sqrt{{B_1}^2 - 4\,B_2} < 2$ and $\boldsymbol{B_1}$ $\sqrt{{B_1}^2 - 4\,B_2} < 2$.

IV. Numerical Analysis

In this section, we numerically describe our analytical results obtained in the former sections by using MATLAB programming. We present the time series graphs for the plant-herbivore system with Allee effect. In particular, we illustrate the stabilizing effects of Allee function on plant population.

Fig.1, Fig.2 and Fig.3 (See Appendix) shows the time series graphs for Herbivore, Plant and

Plant-Herbivore populations respectively, when a=0.45, b=15.4, P_0=0.2 and H_0=0.25. These diagrams are consistent with the analytical results obtained in the former sections.

V. Discussion and Conclusion

Ecological facts lead us to the importance of an Allee effect on population dynamics. In this paper we studied such an effect on plant population. By combining mathematical and numerical analysis, we have shown the impact of Allee effect on plant population on the stability of the positive equilibrium point for a discrete-time plant-herbivore system.

As it is expressed by many researchers, the impact of the Allee effect on the stability population models can show different dynamics which strictly depend on the assumptions of the corresponding model. Hence, it is not surprising that while Allee effect is a destabilizing force for some models, for some others it might be a stabilizing force. As we analyze in this study for our plant-herbivore model, Allee effect has a stabilizing force. Furthermore, we conclude from numerical analysis that, for some fixed parameter values, Allee effect changes the stability of the positive equilibrium points from unstable to stable position.

In this paper, we have studied the Allee effect on the plant population. However, it may be very complicated structure when the herbivore population (or, both plant and herbivore population) is subject to an Allee effect in the system. In the future studies, it would be very interesting to improve such structures.

References

[1]. Allee, W.C., 1931.Animal Aggregations: A study in General Sociology. University of Chicago Press, Chicago.
[2]. Amarasekare, P., 1998. Interactions between local dynamics and dispersal: insights from single species models. Theor. Popul. Biol. 53, 44-59.
[3]. Andrew M. Kramer, Brain Dennis, Andrew M. Liebhold, John M.Drake., 2009. The evidence for Allee effects.Popul.Ecol.51:341-354.
[4]. Birkhead, T.R., 1977.The effect of habitat and density on breeding success in the common guillemot (Uria aagle).J.Anim.Ecol.46, 751-764.
[5]. CheryI.J.Briggs and Martha.F.Hoopes.2004. Stabilising effects in spatial parasitoid-host and predator-prey models: a review, Theor.Popul.Biol.65, 299-315.
[6]. Courchamp, F, L., Berec and J.Gascoigne. 2008. Allee effects in Ecology and conservation, Oxford University press, oxford.
[7]. Darwin, C.R., 1876.The effects of cross and self fertilization in the vegetable kingdom.London, Murray.
[8]. Darwin, C.R., 1876.The different forms of flower on plants of the same species.London.Murray.
[9]. Dennis.B., 1989. Allee effect population growth, critical density, and the chance of extinction.Nat.Res.Model.3, 481-538.
[10]. Dobson. A.P., Lyle, A.M., 1989.The population dynamics and conservation of primate populations.Conserv.Biol.4, 362-380.
[11]. Elaydi.S.N.1996. An Introduction to Difference Equations, Springer- Verlag Publishers.
[12]. Fowler, M.S., Ruxto, G.D., 2002. Population dynamic consequences of Allee effect. J.Theor. Biol.215, 39-46.
[13]. Gascoigne, J., Lipcius, R.N., 2004. Allee effects in marine systems. Mar.Ecol.Prog.Ser.269, 49-59.
[14]. Gyllenberg, M., Hemminki, J., Tammaru, T., 1999. Allee effects can both conserve and create spatial heterogeneity in population densities. Theor. Popul.Biol.56, 231-242.
[15]. Lande, R., Engen, S., Sather, B.E., 2003. Stochastic population Dynamics in Ecology and Conservation. Oxford University Press, Oxford.
[16]. Leah Edelstein-Keshet.1986. Mathematical theory for plant-herbivore systems. Journal of Mathematical Biology, 24:25-58.
[17]. Livadiotis, G., Assas, L., Dennis, B., Elaydi, S., and Kwessi, E., 2014. A discrete-time host-parasitoid model with an Allee effect. Journal of Biological Dynamics.Vol.9, No.1, 34-51.
[18]. May, R.M., 1974. Biological population with non-overlapping generations: stable points, stable cycles and chaos. Science 186,645-647.
[19]. May, R.M., 1976. Simple mathematical models with very complicated dynamics. Nature 261,459-467.
[20]. McCarthy, M.A., Lindenmayer, D.B., Dreschler, M., 1997. Extinction debts and risks faced by abundant species.Conserv.Biol.11, 221-226.
[21]. Saether, B.E., Ringsby, and T.H., Roskaft, E., 1996. Life history variation, population processes and priorities in species conservation: towards a reunion of research paradigms.Oikos.77, 217-226.
[22]. Stephens, P.A., Sutherland, W.J., Freckleton, R.P., 1999. What is the Allee effect? Oikos 87, 185-190.
[23]. Stephens, P.A., Sutherland, W.J., 1999. Consequences of the Allee effect for behavior.
[24]. Scheuring, I., 1999. Allee effect increases the dynamics stability of populations. J. Theor. Biol.199, 407-414.
[25]. Wang, G., Liang, X.G., Wang, F.Z., 1999. The competitive dynamics of populations subject to Allee effect. Ecol. Model.124, 183-192.
[26]. Wan-Xiong Wang, Yan-Bo Zhang, Chang-zhong Liu, 2011. Analysis of a discrete-time predator-prey system with Allee effect. Ecological complexity 8, 81-85.
[27]. Y. Kang, Armbruster and Y.Kuang.2008. Dynamics of a plant-herbivore model.J.Biol.Dynamics, Volume 2, Issue 2, 89-101.
[28]. Y. Kang and Yakubu, A.A. 2011.Weak Allee effects and species coexistence, Nonlinear Analysis. Real World App.12.pp:3329-3345.
[29]. Y. Kang and Yakubu, A.A. 2008.Weak Allee effects and species coexistence, Nonlinear Analysis. Real World App.12.pp:89-101.
[30]. Yun Kang. 2011. Noise and seasonal effects on the dynamics of plant-herbivore models with monotonic plant growth functions. International Journal of Biomathematics, World Scientific Publishing Company.
[31]. Zhou, S.R., Liu, Y.F., Wang, G., 2005. The stability of predator-prey system subject to the Allee effects. Theor. Popul. Biol. 67, 23-31.

Dr M Gunasekaran received B.Sc.,M.Sc., and M.Phil., in Mathematics from Bharathidasan University, Tiruchirapalli, Tamil Nadu in 1989, 1991 and 1992 respectively. He completed his Ph.D degree in Mathematics-Zoology from University of Madras, Chennai. He is a Professor of Mathematics with 23 years of teaching experience and now he is working in Sri SubramaniyaSwamy Government Arts College, Tiruttani. His areas of research are Drug designing, Mathematical models in Prey-Predator system, Plant-Herbivore system and Queuing system. He has published 15 international journals, presented 7 research papers in various National and International Conferences. He has supervised 15 M.Phil., scholars in the area of various Queuing models. He has also given invited talks in National and International conferences in his specialized areas.

Appendix:

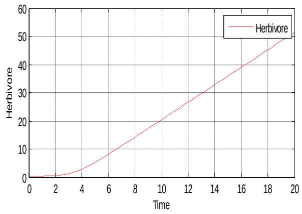

Fig.1 Time series graph for Herbivore population (a=0.45; b=15.4;p_0=0.2,H_0=0.25)

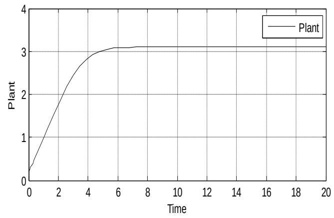

Fig.2 Time series graph for Plant population (a=0.45; b=15.4;p_0=0.2,H_0=0.25)

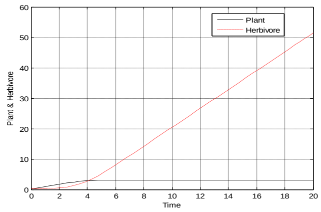

Fig.3 Time Series graph of Plant-Herbivore system (a=0.45; b=15.4; Po=0.2, Ho=0.25)