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Pythagorean Triplets—Views, Analysis and Classification

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Abstract : *In this paper on the most known and popular subject, our efforts are to establish many algebraic properties of Pythagoras Triplets and associate them with different branches of mathematics. Some new methods of generating Pythagoras triplets and number of primitive triplets to a given integer, if it exists, have been elaborated. In this paper and many following papers, classification of Pythagorean triplets into three families—Plato's, Pythagoras', and Fermat's, will play important role showing inter connectivity between the different branches –Number theory, matrices, abstract algebra, graph theory and some more.*

Keywords: *Primitive Triplets, Hypotenuse, pro-addition, pro-multiplication, Plato, Pythagoras and Fermat family.*

I. Introduction

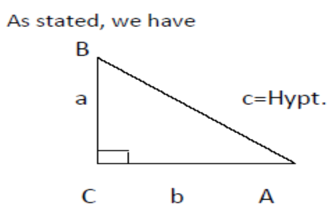
In this paper, our principal focus is on the most known and oldest topic –Pythagorean Triplets. Referring to old papers the work, to a little extent, may look repetitive but our efforts have constantly remained to either provide our own views or to look into it with different perspectives. We have attempted in different approaches to incorporate many aspects in this paper. Some new operations like, pro-addition, and pro-product for even and odd Pythagorean triplets of different classes have been introduced and also based on the results we have established inter-connectivity between different families. The concept of locating a triplet at a given distance from a given triplet is applicable without loss of generality. All or some of these concepts, to some extent, provide link to the other three papers to follow in Fermat family, Pell-sequence and Jha sequence.

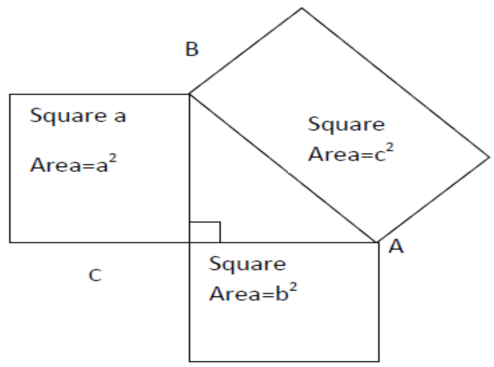
1.1. Historical facts:

From the historical facts dating thousands of years, we just pick-up what we are closely concerned about.

(a) Pythagoras (C.560 to C.480 B.C.) is often credited with identifying and divulging the natural property of distance relationship. It states that "For any right triangle, the square of/on the hypotenuse 'c' equals the sum of squares of/on the two (shorter) "legs" –lengths 'a' and 'b' which is written as $a^2 + b^2 = c^2$."

(Evidences claim that the same property was identified and mentioned in the book ' Baudhayana Sulbha Sutra' written by Baudhayana in the century earlier around C.800 B.C.)





The figure evidently expresses what does it mean by $a^2 + b^2 = c^2$.

In context of natural numbers, we write that three different integers a, b, and c satisfying the above relation are called to frame Pythagorean configuration. In the detailed work, as a part of sequential papers we will relax some conditions to construct 'Pythagorean spectrum'.

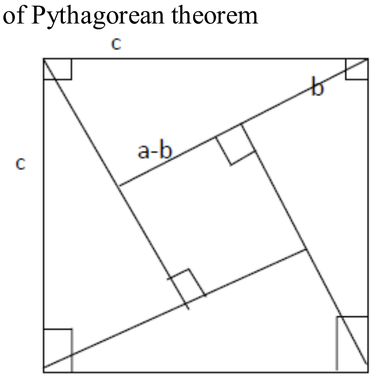
At this point of time it is highly justifiable that Euclid (C. 300 B.C.) proved that Pythagoras theorem is reversible which means that converse of the theorem is also true. (It states that the triangle with the sides a, b, and c which satisfies the relation $a^2 + b^2 = c^2$ (1)
is necessarily a right angle triangle with angle $C = 90^{\circ}$).

(b) Some Alternatives:
(1) For the right triangle ABC (angle $C = 90^{\circ}$) Pythagoras statement is equivalent to
$\sin^2 A + \sin^2 B = 1$ **or** $\sin^2 A + \sin^2 B + \sin^2 C = 2$; (2)
which is equivalent to $a^2 + b^2 + c^2 = 2d^2$ (3)
where 'd ' is the diameter of the circum-circle of the triangle ABC and for the angle $C = 90^{\circ}$,
we have c = d. Some mathematicians have inclination to state the same thing as
$\cos^2 A + \cos^2 B + \cos^2 C = 1$ (4)

(2) Prof. Dijkstra contributed an outstanding generalization of Pythagoras theorem which is as follows.
In a triangle ABC, let a, b, and c be the sides opposite to corresponding vertices then
sign(x) = -1 if x < 0 , sign(0) = 0, and sign(x) =1 if x > 0
As a result for angle $C = 90^{\circ}$, we have $a^2 + b^2 = c^2$

(3) About Proofs of the Theorem:
In the context of different proofs, we have a book – the Pythagoras propositions by Prof. Elisha Scott Loomis. There is a collection of 367 proofs Pythagorean Theorem. The author asserts that the number of algebraic proofs is limitless as the number of geometric proofs is.
The most popular among all is the first of Euclid's two proofs (I-47). The popular configuration is known as 'Brides Chair'; which is the most popular one.
The smallest of all the proofs is given by an Indian mathematician Bhaskara (1114.ca1185). It shows the dissection of the square figure of side 'c' and compares the sum of areas of all right triangles within the square. The proof is of one word -'behold'.

Shri Bhaskara's Proof of Pythagorean theorem



Behold

[The square figure of side 'c' has sub-division into four congruent right triangles of sides 'a' and 'b' and a square of side 'b-a'.]

4. All about Triplets:

For the sake of convenience we introduce notations as follows. The set of natural numbers N has been bifurcated as two infinite sets N_1 – a set of all odd integers and N_2 –a set of all even integers.
Clearly $N = N_1 \cup N_2$ and $N_1 \cap N_2 = \phi$.
In connection to proceeding of this paper, many times we refer to 'Primitive Triplet'. A triplet (**a, b, c**), with a, b, and c some natural numbers with the properties

(1) G.C.D. of **a** and **b = 1 ,** symbolically (**a, b) = 1**
(2) a < b and (3) $a^2 + b^2 = c^2$

These properties help differentiate a primitive triplet from a triplet.
There are many ways to construct Pythagorean triplets. Many mathematician and experts in other branches have found and divulged different ways to express their thought about triplets .We suggest one more and honestly

claim that it involves some further details and works in many different situations that we will handle from time to time.

(4.1) For odd integers:
Let $a \in N_1$- {1} be a given positive integer, then for some $i \in N_1$, we have integers

$$a \ , \ b = \frac{a^2 - i^2}{2i} \ , h = \frac{a^2 + i^2}{2i} \qquad (5)$$

which satisfies $a^2 + b^2 = h^2$, form a Pythagorean triplet. Note that $h = b + i$
We call a triplet an odd triplet if the smallest of all three integers i.e. **a is an odd integer**.

This calls for $\frac{a^2 + i^2}{2i} > \frac{a^2 - i^2}{2i} > a$
From this we can get condition on i.
From the inequality, we have $\frac{a^2 - i^2}{2i} > a$; which means that $(a - i)^2 > 2(i)^2$

This implies that $a - i > (\sqrt{2})i \ and \ a - i < -(\sqrt{2}) \ i \ and \ a > (\sqrt{2} + 1) \ i$
is the only possible criterion.
We have $i < a\left(\sqrt{2} - 1\right)$ (6)
This helps consider feasible values for **i** and possibly predict additional primitive triplets if at all they exist.
At this point it is justifiable to share two comments;

(A) For odd integers i, $b = \frac{a^2 - i^2}{2i}$ is always an even integer (as **a** is an odd integer other than 1).

(B) In the search of additional triplets to the one given by result (5) for $i = 1$,
the condition (6) implies that $i < a(\sqrt{2} - 1)$ must be an odd integer in order to satisfy evenness of $\frac{a^2 - i^2}{2i}$.
e.g.; for $a = 3$, the integer $b = \frac{a^2 - i^2}{2i} \ for \ i = 1, is$ 4 and the next the highest in the sequence, we call it an hypotenuse, $h = \frac{a^2 + i^2}{2i} = 5.$
i.e. $3^2 + 4^2 = 5^2$ is an odd Pythagorean triplet.
At this point it is timely to let the readers know that there are infinitely odd integers which possess more than one – may be two, three and even more, Pythagorean primitive * triplets.

(4.2) For Even integers:
There are some more points of clarifications when the first integer of the primitive triplet i.e. **a** is an even integer. We will take up all such diversifications before they hit us on the way.
Let **a** $\in N_2$ – {2}
This means that for some **a,** an even integer, if we find $i \in N_2$
so that the triplet of positive integers $a \ , \ b = \frac{a^2 - i^2}{2i} \ , h = \frac{a^2 + i^2}{2i}$
satisfy Pythagorean condition i.e. $a^2 + b^2 = h^2$. Note that $h = b + i$.

If **a** is even then we call **a, b**, and **h** to form an even triplet
e.g; for $a = 8$ and $i = 2$, we have $b = \frac{a^2 - i^2}{2i}$
$\therefore \ b = \frac{8^2 - 2^2}{2(2)} = 15 \ \ and \ h = b + 0 = 15 + 2 = 17$
A first even primitive* Pythagorean triplet is 8, 15, and 17.

(4.3) Primitive Triplets and Some Facts:
As discussed above, we have some views that help develop detail regarding Pythagorean triplets. In the primitive triplets with a , $b = \frac{a^2 - i^2}{2i}$, $h = \frac{a^2 + i^2}{2i}$ with $i \in N_1$ or N_2 as the case may be for **a** $\in N_1$ or N_2. We have to our notes some known and obvious results.

(A) From the properties inherent in the primitive Pythagorean triplets we deduce that if **a** is odd or even integer so is **b** even or odd integer and as a result the hypotenuse **h** is always an odd integer. Also as shown above, for the class of primitive triplets, **(a, b) =** 1 with a < b.

(B) In most of the cases, at least one of the integers **a, b,** and **h** is prime or divisible by 5.

(C) A member integer of the set {1, 2, 4, 6} U {2(2n +3)| n ϵ N} cannot possess primitive Pythagorean triplet. For the member of {1, 2, 4, 6}; it can be easily verified. In the case of an integer of the form 2(2n +3) ; n ϵ N we prove it as follows.

Proof: If **a** ϵ {2(2n +3)| n ϵ N} then by its nature, '**a**' is an even integer. For the primitive Pythagorean triplet to exist, b = $\frac{a^2 - i^2}{2i}$ must be an **odd** integer. Also, for **a** even we have i ϵ N_2 and $i < a(\sqrt{2} - 1)$

We have b = $\frac{(2(2n+3))^2 - 2^2}{2\,(2)}$ =$(2n + 3)^2 - 1$, which is an even integer.

This contradicts with conditions (for **a** an even integer**, b** is an odd integer with **a < b)** of primitive triplet. Hence we conclude that integers of the form {2(2n +3)| n ϵ N} cannot possess primitive triplets.

(D) Different forms of Triplets:

In the above section we have seen the first form of a primitive triplet where the given positive integer **a** is in the first position; i.e. **a** is the smallest of all sides. The different forms are as under.

(1) The first type, for a given **a,** as we have already seen is

$a\ ,\ b = \frac{a^2 - i^2}{2i},\ h = \frac{a^2 + i^2}{2i}$, with (a, b) =1 for any a ϵ N = N_1 U N_2 and i can be either even or odd positive integer as the integer **a** is considered.

This will distinctly derive all possible triplets including primitive triplets also. It is important to note that hypotenuse $\boldsymbol{h} = \boldsymbol{b} + \boldsymbol{i}$ in all such cases.

If a given integer **a** does not fit in this format then it cannot be a part of Pythagorean triplet.

For **a** = 1 an odd integer and i =1, we get **b** = 0 and this helps us conclude that there cannot be a Pythagorean triplet in which **a** = 1 can be the shorter leg.

For **a =** 2, an even integer and i = 2, we get **b =** 0 and this helps us conclude that there cannot be a Pythagorean triplet in which **a** = 2 can be the shorter leg.

In addition to this, we leave to reader's discretion to compare this result with that of Euclid's formula where in we use any one of the results

a = 2mn, b = m^2- n^2, and h = m^2 + n^2 with co-primes m, and n so that m > n.

In this case, the triplet begins with an even integer. As a prime condition for **a** < **b**, we should have

2mn < m^2- n^2 ⇒ $(m + n)^2 < 2m^2$ which does not hold true for m = 2 , and n = 1

a = m^2- n^2, b = 2mn, and h = m^2 + n^2 with co-primes m, and n so that m > n.

In this case we should keep m an odd integer and n an even integer and that can allow **a** to be an odd integer and the term **b = 2mn** > **a** with h = m^2 + n^2 will make the triplet an odd triplet.

(2) Now we discuss a case in which **a** takes up its position in the middle position, i.e. the value of **a** as the larger one of the two legs. In this case the triplet takes up the form as follows.

$\sqrt{i(2\boldsymbol{a} + i)},\ \boldsymbol{a}\ ,\ \boldsymbol{a} + \boldsymbol{i}$; and $\sqrt{i(2\boldsymbol{a} + i)} < \boldsymbol{a} < \boldsymbol{a} + \boldsymbol{i}$

Importance of this result is to find whether to a given integral value of a leg of a right triangle, the leg smaller to that exists. It is note-worthy point that if **a** is even then the value of **i** is an odd integer and vice-versa. Also, the hypotenuse **h = a + i** establishes the same relation that we saw in the previous section.

We consider two examples.

Let **a** = 4 then for i= 1, $\sqrt{i(2\boldsymbol{a} + i)}$ = $\sqrt{1(2(\boldsymbol{4}) + 1)}$ = 3 and the hypotenuse = $\boldsymbol{a} + \boldsymbol{i}$ = 4+1 =5

That is for **a = 4** ; a given value of a leg of a right triangle, there exists a smaller leg with measure = 3 and the hypotenuse is =5

Now, we consider **a = 5**, corresponding to this value of **a** there is no integer value of i which can make $\sqrt{i(2\boldsymbol{a} + i)}$ an integer less than $\boldsymbol{a} = \mathbf{5}$**.**

We can derive the upper limit of **i** for a given value of $\boldsymbol{a}$**;**

this will make search for even or odd integral value of **i** (depending on odd or even value of **a**)

We have, $\sqrt{i(2a + i)} < a$

⇒ $i(2a + i) < a^2$

⇒ $0 < a^2 - 2ia - i^2$

$\Rightarrow\ a^2 - 2ia + i^2 > 2i^2$
$\Rightarrow (a-i)^2 > 2i^2$

The concerned part of this inequality at this stage is
$a - i > \sqrt{2}i$
$\therefore a > (1+\sqrt{2})i$
For given $a \in N, i < a(\sqrt{2}-1)$
As $\sqrt{2}-1 < 1$, it implies that $i \in N$ be such that $i < a$
[In fact approximately $i < \frac{a}{2}$]

(3) In this section we consider the case when the given positive integer **a** plays the role of a hypotenuse of a right triangle.

We consider the triplet $\boldsymbol{a-i}, \sqrt{\boldsymbol{i(2a-i)}},\ \boldsymbol{a}$ where **a** is a given integer
In this case, if **a** is an odd integer then **i** is necessarily an even integer and such values of **a** and **i** produces an odd primitive Pythagorean triplet.
In case, if **a** is an even integer then the given triplet is not a primitive one. For even value of **a** we have to take even integer value of i.
In this case also, we can find the range of **i** depending on given value of **a.**

As $\boldsymbol{a-i}$ **ϵ N,** it follows that **a** > **i.**
Also, the inequality $\boldsymbol{a-i} < \sqrt{\boldsymbol{i(2a-i)}}$, and $\sqrt{\boldsymbol{i(2a-i)}} < \boldsymbol{a}$
$\therefore (a-i)^2 < i(2a-i)$
$\Rightarrow (a-2i)^2 < 2i^2$
$\Rightarrow a - 2i < \sqrt{2}i$
$\therefore a < \sqrt{2}(\sqrt{2}+1)i$ or for given $a \in N$
$i > \frac{a}{2}(\sqrt{2}-1)(\sqrt{2})$
$i > \frac{a}{2}(2-\sqrt{2})$
$\therefore i > \left[\frac{a}{2}(2-\sqrt{2})\right]$
[$i > 0.3\ a$ Approximation for $\sqrt{2} \cong 1.4$]
Combining the above results we have a > i and $i > 0.3\ a$
All the three positions for a given integer value of **a** help take decision regarding pattern of Pythagorean triplets of all types.

(4.4) Multiple Primitive Triplets:
There are many positive integers which possess more than one primitive triplet. We have classified them in two classes.

(A) Positive integers of the form 4(2n +3) with n ϵ N possess more than one primitive triplets.
Let **a** = 4(2n +3) **---** an even positive integer which is a member of the infinite set
{20, 28, 36,……. }
We have $\mathbf{a^2 = 16(2n+3)^2}$**,** For **b=** $\frac{a^2-i^2}{2i}$ **;** as discussed earlier for a primitive triplet, we have
i = an **even** positive integer and it **b** must be an **odd** positive integer**.** As a consequence,
$\mathbf{h = b + i} = \frac{\boldsymbol{a^2+i^2}}{\boldsymbol{2i}}$**.**

As $\mathbf{a^2}$ is an odd multiple of 16, and for **b** to be an **odd** integer, i = 2, 8 etc. will fit better.
On i =2 and on taking i = 8, we have two primitive triplets for one value of **a** of the form 4(2n +3).
e.g; for **a** = 20,

For i = 2, b = $\frac{a^2-i^2}{2i} = = \frac{20^2-2^2}{2(2)}$ = 99 and **h = b + i** gives **h** = 101
A triplet 20, 99, and 101 satisfying $\mathbf{a^2 + b^2 = h^2}$

For i = 8, b = $\frac{a^2 - i^2}{2i}$ = = $\frac{20^2 - 8^2}{2(8)}$ = 21 and **h = b + i** gives **h** = 29
A triplet 20, 21, and 29 satisfying $\mathbf{a^2 + b^2 = h^2}$.
We tabulate some results.

Primitive Triplets

Numbers of the Form 4(2n +3) = **a**	Numbers of Triplets	Value of i
20,28, 36,44, 52,68,76 92,100,108,116,124…….	2	2,8
60,84,132,140,156…..	3	(2,8,18), (2,8,50)…
204,228,276,…….	4	(2,8,18,72),(2,8,50,200)…..
780,924,……	6	2,8,18,72,98,242

Observation: We have, through 20 up to 1012, 125 numbers of the form 4(2n +3). Based up on number of triplets, they are classified as follows.

Number of triplets:	2	3	4	6	7*
Numbers:	62	32	27	3	1

We have two immediate questions which need special attention.
* The integer 660 has 7 primitive triplets. The first ten integer multiple of 660 exhibits different nature in generating triplets. The integer 2640 (= 660 x 4) and 3960 (=660x6) have 6 primitive triplets ; 4620 (= 660 x 7) has **13** primitive triplets and the others have 7 primitive triplets.
(1) Are there any integers of the form 4(2n+3) which, for some n ϵ N, possess 5 primitive triplets?
(2) Does the number of numbers of triplets keep on increasing as the number 4(2n+3) increase?

(B) Positive integers of the form 3(2n +9) with n ϵ N possess one or more than one primitive triplets.
Let **a** = 3(2n +9) **---** an odd positive integer which is a member of the infinite set {33, 39, 45,……}
We have $\mathbf{a^2 = 9(2n + 9)^2}$**,** For **b=** $\frac{a^2 - i^2}{2i}$ **;** as discussed earlier for a primitive triplet, we have
i = an **odd** positive integer and it **b** must be an **even** positive integer**.** As a consequence,
$\mathbf{h = b + i = \frac{a^2 + i^2}{2i}}$**.**
As $\mathbf{a^2}$ is an odd multiple of 9, and for **b** to be an **even** integer, i = 1, 9 etc. will fit better.
For i = 1, b = $\frac{a^2 - i^2}{2i}$ = = $\frac{33^2 - 1^2}{2(1)}$ = 544 and **h = b + i** gives **h** = 545
A triplet 33, 544, and 545 satisfying $\mathbf{a^2 + b^2 = h^2}$
For i = 9, b = $\frac{a^2 - i^2}{2i}$ = = $\frac{33^2 - 9^2}{2(9)}$ = 56 and **h = b + i** gives **h** = 65

A triplet 33, 56, and 65 satisfying $\mathbf{a^2 + b^2 = h^2}$.

As a special case, when n = 3k, with k ϵ {1,2,3,4,5,7}U{8,12, 18,24,27}U{33,36,39,42…}
then, **a** = 3(2n +9) has only one primitive triplet.
We tabulate some results.

Primitive Triplets

Numbers of the Form 3(2n +9) = **a**	Number of Triplets	Value of i
**45,63,81.99,117,153,171, 243,351,459,513,621,675 729,783, 837, 891, 945, 999,…	1	1
33,39,51,57,…,135,….567,...	2	(1,9),((1,25), (1,49)…
105,165,195,231,…….	3	(1,9,25),(1,9,49)…..
315,429,……	4	(1,25,49,81),(1,9,121,169)

****** It apparently looks that there is some symmetry but it is not easy to generalize as the difference between consecutive numbers keeps on increasing.

Observations: We have, through 33 up to 885, 143 numbers of the form 3(2n +9). Based up on number of triplets, they are classified as follows.

Number of triplets:	1	2	3	4
Numbers:	16	88	25	14

We have three immediate questions which need special attention.

(1) Are there any integers of the form 3(2n+9) which, for some n ϵ N, possess 5 primitive triplets?
(2) Does the number of numbers of triplets keep on increasing as the number 3(2n+9) increase?
(3) What is the general pattern of the numbers of the form 3(2n+9) that possess exactly one primitive triplet?

(5) Different Methods of Generating Pythagorean Triplets:

In this section, we will show different patterns of expressing triplets and meanings associated with them. Methods like Euclid's method, Fibonacci method, Progression of whole and fractional method, Dickson's method and many other help generate Pythagorean triplet and verify Pythagorean property for three given positive integers.

(5.1) Master Method:

We mention an interesting method for generating Pythagorean triplets using a pattern like Fibonacci sequence.
Let **a** and **b** be any two positive integers, generally **a ≤ b.**
Let **a + b = c** and **b + c = d**
Pythagorean triplet using these positive integers is **(ad, 2bc, a^2 +2bc)**
e.g; let **a = 2** and **b = 5** then **c =7** and **d = 12** using these numbers, the triplet is (24, 70, 74) which is equivalent to 2(12, 35, 37).
This can be achieved by letting **a = b= 2**; as a result **c = 4** and **d = 6** in the above result **(ad, 2bc, a^2 +2bc)**
We appreciate and claim total uniqueness of this method of generating Pythagorean triplets.

Special note:

Ternary tree approach is an extension to this approach which helps generate all Pythagorean triplets. We make a matrix $\begin{bmatrix} q & p \\ r & s \end{bmatrix}$ with two given positive integer p and q.
We define r = p + q and s = q + r.
The Pythagorean triplet is **a = p s, b = 2qr,** and **h= p r + q s** with $a^2 + b^2 = h^2$
There is no guarantee that it will generate primitive triplets.
e.g; Let us consider p = 2 and q =5 then the above matrix $\begin{bmatrix} 5 & 2 \\ 7 & 12 \end{bmatrix}$
With this values in the above relation, **a = 24, b = 70,** and **h = 14 + 60 = 74**
Primitive triplet derived is **a = 12, b = 35,** and **h = 37**
We can obtain more matrices using the elements of the above matrix. This process will help generate many triplets.

(5.2) Michael Stifel and Jacques Ozanam's Approach:

Let **a** ϵ N. We can construct a sequence of integers and fractions in the form
$\mathbf{a} + \frac{a}{2a+1} = \mathbf{a}\,\frac{a}{2a+1}$ for a given integer **a.** The integral value of the denominator and that of the numerator considered in order are the measures of two consecutive legs of a right triangle and the hypotenuse **h** is one (1) more than the second leg.
e.g; For a =1, we have $\mathbf{a} + \frac{a}{2a+1} = \mathbf{1} + \frac{1}{2(1)+1} = \mathbf{1}\,\frac{1}{3} = \frac{4}{3}$
Denominator = **a** = 3 (The smaller leg), Numerator =**b** = 4 (The second leg)
The hypotenuse is = **h = b+1 = 4 +1 =5**
As a result, for different integer values of **a,** we have a sequence as follows.
$1\frac{1}{3}, 2\frac{2}{5}, 3\frac{3}{7}$ ……which is equivalent to $\frac{4}{3}, \frac{12}{5}, \frac{24}{7}$,… [Triplets are as (3, 4, 5), (5,12, 13) etc.]
This mechanism of generating Pythagorean triplets is attributed to the German monk and mathematician Michael Stifel (1544). There two important features in this pattern;

(a) The pattern generates all odd primitive triplets. There is no reference of even primitive triplets.
(b) The two legs in ascending order are given by the denominator and the numerator respectively and the hypotenuse is just one (1) more than the numerator value.

To include the even numbers, as a supplement to the above result, Jacques Ozanam (1694) gave supplementary result which is as follows.
For each n ϵ N, the pattern $\mathbf{n} + \frac{4n+3}{4n+4}$ generates even primitive triplets. The shorter leg equals the number in the denominator and the next leg being the numerator of the simplified fraction;

i.e. n.(4n +4) + (4n +3) = $4n^2 + 8n + 3$. The hypotenuse, in this case is just two (2) more than that of the numerator; i.e. h = $4n^2 + 8n + 5$

e.g. for n =1, the triplet is $\mathbf{1+\frac{4(1)+3}{4(1)+4} = \frac{15}{8}}$, the triplet being 8,15, 17

for n =2, the triplet is $\mathbf{2+\frac{4(2)+3}{4(2)+4} = \frac{35}{12}}$, the triplet being 12,35, 37

There are many methods of generating Pythagorean triplets

(5.3) Combination Method:

In this method the prime feature is that it begins with a known triplet and using generalized Fibonacci sequence finds the next triplet, using some formula, finds the next triplet following the same pattern.

To clarify these points let us consider $(a_n , b_n , h_n) = (3 ,4 ,5)$ for n = 3 which a Pythagorean triplet.

We now consider Fibonacci sequence (F_n) with $F_1 = 0$ and $F_2 =1$ and continue with the known relation $F_{n+2} = F_{n+1} + F_n$ for all n ϵ N. Some terms of the sequence are 0, 1, 1, 2, 3,5,8,13,21,34….

The recurrence relation $(a_{n+1} , b_{n+1} , h_{n+1}) = (F_{2n+1} - a_n , a_n + b_n + h_n , F_{2(n+1)})$. This method links a given triplet to its next one using Fibonacci sequence. This method fails to generate all triplets. In addition to this, it generates, in many cases, non-primitive triplets also.

(6) A Group of Different Families:

The study and the pattern of triplets help classify Pythagorean triplets in different classes and each class is named after great mathematicians who shared remarkable contribution in the area. We consider the triplet in the form **(a, b, h)**; where all a, b, and h are positive integers; **a** stands for the shorter leg of the right triangle in consideration, the second leg is **b** and the hypotenuse is **h**. We tend to consider a < b with (a, b) = 1 and satisfying the relation $\mathbf{a^2 + b^2 = h^2}$.

All the classes, as we believe are sub classes to one class--- A Pythagorean Family.

Triplet members of Pythagorean family may be considered with above constraints or triplets which result as an integer multiple of some primitive triplet.

If **(a, b, h)** is a primitive triplets then for some k ϵ N, k**(a, b, h) = (ka, kb, kh)** is also a Pythagorean triplet.

Let us denote the universal set of triples of the form **(a, b, h)** by the symbol P**.**

P = {(a, b, h)| a, b, and h ϵ N, and $a^2 + b^2 = h^2$ } (7)

(a) Plato Family P_1 :

We define, the infinite set known as Plato Family of Pythagorean Triplets

P_1= {(a, b, h) | a, b, and h ϵ N, a < b, (a, b) = 1, h-b =1} (8)

Triplets like, (3, 4, 5) , (7, 24, 25) ϵ P_1. Triplets are odd primitive triplets

This class is referred as **Plato** family.

(b) Pythagorean Sub-family P_2:

We define, the infinite known as Pythagorean set P_2

P_2={(a, b, h)|a, b, h ϵ N, a < b, (a, b)= 1 and h-b=2} (9)

Triplets like, (8, 15, 17), (12, 35, 37) ϵ P_2 . Triplets are even primitive triplets

This class is referred as **Pythagorean Sub-family**.

(c) Fermat Family P_3:

We define, the infinite set known as Plato Family of Pythagorean Triplets

P_3= {(a, b, h) | a, b, and h ϵ N, a < b, (a, b) = 1, b-a =1} (10)

Triplets like, (3, 4, 5) , (20,21, 29) ϵ P_3. Triplets in this class are primitive triplets

This class is referred as **Fermat** family.

At this stage it is note- worthy that the set

P' = P – {P_1 U P_2 U P_3} is an infinite set . Triplets like (20, 45, 53), (39, 80, 89) are members of the infinite set P'.

(7) Inter- Connectivity of P_1 *and* P_2 and some Properties of P_2:

We define certain special operations on member triples of Plato family P_1 and the set P_2.

These operations parallel to regular addition and multiplication and they either establish closure property on the set or establish inter-connectivity between the sets P_1 and P_2.

(A) Pro-addition in (╫)P_1 :

Let $T_1 = (a_1, b_1, h_1)$ and $T_2 = (a_2, b_2, h_2)$ be the member of P_1. Let $a_2 \geq a_1$.

We define a special operation, called **'pro-addition'**, denoted as ╫, on members of P_1.

For $T_1 , T_2 \epsilon P_1$

$T_1 \mathbin{+\!\!+} T_2 = (a_1 + a_2,\ b_1 + b_2 - d^2,\ h_1 + h_2 - d^2)$ (11) ;

where $d = \left(\frac{a_2 - a_1}{2}\right)$.

We note that $T_1 \mathbin{+\!\!+} T_2 \in P_2$; it is an even triplet.

If $T_1 = T_2$ then,

$T_1 \mathbin{+\!\!+} T_2 = T_1 \mathbin{+\!\!+} T_1 = 2T_1 = (2a_1, 2b_1, 2h_1)$ and $2\,T_1 \in P_2$.

e.g. For $T_1 = (3, 4, 5)$ and $T_2 = (5,12,13)$

Then their pro-addition $T_1 \mathbin{+\!\!+} T_2 = \left(3 + 5, 4 + 12 - \left(\frac{5-3}{2}\right)^2, 5 + 13 - \left(\frac{5-3}{2}\right)^2\right)$ $= (8,15,17)$ and $T_1 \mathbin{+\!\!+} T_2 \in P_2$.

(B) Pro-addition in P_2:

Let $T_1 = (a_1, b_1, h_1)\ and\ T_2 = (a_2, b_2, h_2)$ be the two triplets of class $P_2.with\ a_2 \geq a_1$.
then their **'pro-addition'**, denoted $\boldsymbol{T_1 \mathbin{+\!\!+} T_2}$, is defined as follows

$$T_1 \mathbin{+\!\!+} T_2 = \left(a_1 + a_2, \left(a_1 + \frac{k}{2}\right)^2 - 1, \left(a_1 + \frac{k}{2}\right)^2 + 1\right) \qquad (12)$$

Where $|a_2 - a_1| = k$; Obviously $T_1 \mathbin{+\!\!+} T_2 \in P_2$.

If $T_1 = T_2$ then $T_1 \mathbin{+\!\!+} T_1 = 2\,T_1$ which is a non-primitive triplet.

e.g; $T_1 = (8 , 15 , 17) \in P_2$ and $T_2 = (24, 143, 145) \in P_2$

We have $k = 124 - 81 = 16$

Then $T_1 \mathbin{+\!\!+} T_2 = (32, (8 + 8)^2 - 1 , (8 + 8)^2 + 1) = (32, 255, 257) \in P_2$

(C) Pro-addition of Members of P_1 and P_2:

Let $T_1 = (a_1, b_1, h_1) \in P_1\ and\ T_2 = (a_2, b_2, h_2) \in P_2$ be the two triplets, then their **'pro-addition'**, denoted $\boldsymbol{T_1 \mathbin{+\!\!+} T_2}$, is defined as follows.

$$T_1 \mathbin{+\!\!+} T_2 = (a_1 + a_2\ ,\ 3(b_1 + b_2 - 2) + 1\ ,\ 3(b_1 + b_2 - 2) + 2) \qquad (13)$$

We note that $T_1 \mathbin{+\!\!+} T_2 \in P1$

E.G. Let $T_1 = (5,12,13)$ and $T_2 = (16, 63, 65)$ then their pro-addition is

$T_1 \mathbin{+\!\!+} T_2 = (5 + 16\ ,\ 3(12 + 63 - 2) + 1\ ,\ 3(12 + 63 - 2) + 2) = (21,220, 221) \in P1$

(D) Pro- Product in P_1 :

Let $T_1 = (a_1, b_1, h_1)$ and $T_2 = (a_2, b_2, h_2)$ be the member of P_1. Let $a_2 \geq a_1$.

We define a special operation, called **'pro-product'**, denoted as $\boldsymbol{T_1}$ ٩ $\boldsymbol{T_2}$.

T_1 ٩ $T_2 = (a_1a_2\ , b_1+b_2+2b_1b_2\ , b_1+b_2+2b_1b_2 +1)$ (14)

The operation is a binary operation on the set P_1.

$\boldsymbol{T_1}$ ٩ $\boldsymbol{T_2}$ = $\boldsymbol{T_2}$ ٩ $\boldsymbol{T_1}$ and both belong to P_1.

In the same way on taking $T_2 = T_1$, we write the pro-product as T_1 ٩ T_1 by the notation T_1^2 . It is obtained by putting b_1 for b_2. $T_1^2 = (a_1^2 , 2b_1 (b_1 + 1), 2b_1 (b_1 + 1) + 1)$.

In the same way, we can define higher powers of a given triplet and can find symmetry.

E.G. $T_1 = (5, 12, 13)$ and $T_2 = (11, 60, 61)$ then their pro-product denoted as $T_1$٩ T_2 is as follows.

$T_1$٩ T_2 = (5 x 11, 2x (12x60)+12+60, 2x(12x60)+12+60 +1) = (55, 1512, 1513) $\in P_1$

(E) Pro-Product in P_2:

Let $T_1 = (a_1, b_1, h_1)$ and $T_2 = (a_2, b_2, h_2)$ be the member of P_2. Let $a_2 \geq a_1$.

We define a special operation, called **'pro-product'**, denoted as $\boldsymbol{T_1}$ ٩ $\boldsymbol{T_2}$, is defined as follows.

$\mathbf{T_1}$ ٩ $\mathbf{T_2} = (a_1a_2\ , 4(b_1+1)(b_2+1)-1\ , 4(b_1+1)(b_2+1)+1)$ (15)

Note that $\boldsymbol{T_1}$ ٩ $\boldsymbol{T_2}$ = $\boldsymbol{T_2}$ ٩ $\boldsymbol{T_1}$ and both belong to P_1.

In the same way on taking $T_2 = T_1$, we can write the pro-product as T_1 ٩ T_1 as T_1^2.

We write $T_1^2 = (a_1^2 , 4(b_1 + 1)^2 - 1, 4(b_1 + 1)^2 + 1)$.

In the same way we define $T_1^3 = (a_1^3 , 4^2(b_1 + 1)^3 - 1 , 4^2(b_1 + 1)^3 + 1)$

This helps define higher powers of a given triplet.

Let $T_1 = (8 , 15, 17)$ and $T_2 = (20 , 99, 101)$ then their pro-product, using the above definition,

$T_1$٩ T_2 = (160, 4 x (15+1) x (99+1) -1, 4 x (15+1) x (99+1) +1) = (160, 6399, 6401) $\in P_2$

(F) Pro-Product of members of P_1 and P_2:

Let $T_1 = (a_1, b_1, h_1) \in P_1\ and\ T_2 = (a_2, b_2, h_2) \in P_2$ be the two primitive triplets, then their **'pro-product'**, denoted $\boldsymbol{T_1}$ ٩ $\boldsymbol{T_2}$, is defined as follows.

$T_1 ᕋ T_2 = (a_1 a_2\,,\ (2b_1+1)(b_2+1)-1\,,(2b_1+1)(b_2+1)+1)$ (16)

(**Notes:**
(1) T_1 is an odd triplet and hence a_1 is odd and b_1 is even. The result of operation of Pro-product of T_1 with T_2 and that of T_2 with T_1 is same but care should be taken to use the formula (15). Else we should design corresponding formula for the pro-product of T_2 of P_2 with that of T_1 of P_1.

(2) The above defined two special operations Pro-addition ╫ and Pro-product ᕋ
are defined to operate on set P_1 and P_2 as shown ; they cannot be interchanged with respect to the sets P_1 and P_2. They parallel to standard operations addition and multiplication only with respect to the first element of the triplets)

(G) Triplet at a Distance 'd' from a Given Triplet:
Let $T_1 = (a_1, b_1, h_1) \in P_2$ be an even triplet with $a_1 \geq 6$. A triplet at a distance'd', where **d** is an even integer**,** from (a_1, b_1, h_1) is a triplet denoted as $T_1@d$ with

$$T_1@d = T_2 = \left(a_1 + d, \left(\frac{d}{2}\right)\left(\frac{d}{2} + a_1\right) + b_1, \left(\frac{d}{2}\right)\left(\frac{d}{2} + a_1\right) + b_1 + 2\right) \qquad (17)$$

e.g. $T_1 = (8,15,17) \in P_2$ and $d = 10$
then $T_1@10 = (18, (5)(5+8)+15, (5)(5+8)+15+2) = (18, 80, 82)$.

With a little change, the corresponding result for an odd triplet, the result is as follows.
Let $T_1 = (a_1, b_1, h_1) \in P_1$ then the triplet at an even distance **d**, denoted as $T_1@d$, is given as follows.

$$T_1@d = T_2 = \left(a_1 + d, (d)\left(\frac{d}{2} + a_1\right) + b_1, (d)\left(\frac{d}{2} + a_1\right) + b_1 + 1\right) \qquad (18)$$

e.g. $T_1 = (9, 40, 41) \in P_2$ and $d = 10$
then the triplet T_2 at a distance 10 units from the triplet T_1 is given as follows.
$T_2 = T_1@10 = (19, (10)(5+9)+40, (10)(5+9)+40+1) = (19, 180, 181)$.

II. Conclusion

Our discussion on triplets and primitive triplets has clearly identified many features. Some positive integers have unique triplets while some have more than one triplets. Integer having 13 primitive triplets has been identified in our work and there are some integers they do not possess any primitive triplets. The classification of triplets into different families and basic algebraic operations have established some rigors in the sets of triplets and in addition to this, special operations like pro-addition and pro-product have established inter-connectivity between the member triplets of the family of triplets. The last feature of locating a triplet at a given distance from a given triplet helps write computer program to achieve generalization.

Future Projections

In close connection to this paper, our immediate dedication follow on to study of all properties of members of Fermat family and its connection with Pell-sequence, 'Jha' sequence, 'Jha-Pell' sequence and many other sequences in the same group. On accepting certain relaxation, we have established Primal –dual type properties of Pythagorean triplets and spectral Algebra of triplets.

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