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On pairwise δ_I - semi-homeomorphism

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Abstract: *In this paper, the notions of pairwise δ_I-semi-open functions, pairwise δ_I-semi-closed functions, pairwise δ_I-semi-homeomorphism and pairwise δ_I^*-semi-homeomorphism are introduced and investigated some characterizations of these functions in ideal bitopological spaces.*
Keywords and Phrases*: pairwise δ_I-semi-open functions, pairwise δ_I-semi-closed functions, pairwise δ_I-semi-homeomorphism, pairwise δ_I^*-semi-homeomorphism.*

I. Introduction

In 1968, the concept of δ-open sets in topological spaces are introduced by
N. V. Velicko [7]. In 2005, δ-open sets are introduced by S. Yuksel et al. [8] in ideal topological spaces. A triple (X, τ_1, τ_2) where X is a non-empty set and τ_1 and τ_2 are topologies on X is called a bitopological space. In 1963, Kelly [4] initiated the systematic study of such spaces. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [5] and Vaidyanathasamy [6]. An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal bitopological space is a bitopological space (X, τ_1, τ_2) with an ideal I on X and is denoted by (X,τ_1,τ_2,I). For a subset A of X and j = 1,2
$A^*_{\tau j}(I) = \{ x \in X : A \cap U \notin I$ for every $U \in \tau_j(X, x)\}$ is called the local function [5] of A with respect to I and τ_j. We simply write A_j^* instead of $A^*_{\tau j}(I)$ in case there is no chance for confusion. For every ideal bitopological space
(X,τ_1,τ_2,I), there exists a topology $\tau_j^*(I)$, finer than τ_j, generated by $\beta(I,\tau_j) =$
$\{U - I : U \in \tau_j$ and $I \in I\}$, but in general $\beta(I,\tau_j)$ is not always a topology [3]. Additionally, τ_j - $cl^*(A) = A \cup A_j^*$ is a Kuratowski closure operator [6] for
$\tau_j^*(I)$. In this paper, the notions of δ_I-semi-open functions, pairwise δ_I-semi-closed functions, pairwise δ_I-semi-homeomorphism and pairwise δ_I^*-semi-homeomorphism are introducted and study some of its properties in ideal bitopological spaces.

II. Preliminaries

Throughout this paper, (X, τ_1, τ_2, I), (Y, σ_1, σ_2, I) and (Z, γ_1, γ_2, I)
(or simply X, Y and Z) always mean ideal topological spaces on which no separation axioms are assumed. For a subset A of a space (X, τ_1, τ_2, I),
$\tau_i - \text{int}(A)$, $\tau_j - \text{cl}(A)$,), $\tau_i - \text{int}_\delta(A)$, $\tau_j - \text{cl}_\delta(A)$, $(i,j) - \text{sint}_{\delta_I}(A)$, and $(i,j) - \text{scl}_{\delta_I}(A)$ denote τ_i - interior, τ_j-closure, τ_i - δ -interior, τ_j-δ-closure , (i, j)-δ_I-semi-interior and (i, j)-δ_I-semi-closure of A respectively.
Definition 2.1. [1] A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j)-δ_I-semi-open if $A \subseteq \tau_j$- $cl^*(\tau_i - \text{int}_\delta(A))$.
Definition 2.2 [2] A function f : (X, τ_1, τ_2, I) → (Y, σ_1, σ_2) is said to be p-δ_I-semi-continuous $f^{-1}(V)$ is (i, j)-δ_I-semi-open in (X, τ_1, τ_2, I) for each σ_i -open set V of (Y, σ_1, σ_2).
Lemma 2.3 [1] For a subset $A \subseteq$ (X, τ_1, τ_2, I), the following hold:
1. $(i,j) - \text{scl}_{\delta_I}(X - A) = X - (i,j) - \text{sint}_{\delta_I}(A)$
2. $(i,j) - \text{sint}_{\delta_I}(X - A) = X - (i,j) - \text{scl}_{\delta_I}(A)$

III. Pairwise δ_I-semi-open and pairwise δ_I-semi-closed functions

Definition 3.1. A function f : (X, τ_1, τ_2, I) → (Y, σ_1, σ_2, I) is called pairwise -δ_I-semi-open (briefly p-δ_I-semi-open) if for each τ_i – open set U of X, f(U) is (i, j)-δ_I-semi-open in Y.

Definition 3.2. A function f : (X, τ_1, τ_2, I) → (Y, σ_1, σ_2, I) is called pairwise -δ_I-semi-closed (briefly p-δ_I-semi-closed) if for each τ_i – closed set U of X, f(U) is (i, j)-δ_I-semi-closed in Y.
Example 3.3. Let X = {a, b, c} with topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$,
$\tau_2 = \{\phi, \{a, c\}, X\}$ and let Y = {p, q, r} with topologies $\sigma_1 = \{\phi, \{q\}, \{r\}, \{q, r\}, Y\}$,

$\sigma_2=\{\phi,\{q\},\{r\},\{q,r\},\{p,r\},Y\}$ and an ideal $I=\{\phi,\{p\}\}$. Let $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$ be a function defined as $f(a)=q, f(b)=r$ and $f(c)=p$. Then f is $p-\delta_I$- semi - open and $p-\delta_I$- semi-closed.

Theorem 3.4. A function $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$ is p-δ_I-semi-open if and only if for each $x\in X$ and each neighborhood U of x, there exists $V\in(i,j)$-$\delta_I SO(Y)$ containing f (x) such that $V\subseteq f(U)$.

Proof: Suppose that f is p -δ_I-semi-open. For each $x\in X$ and each neighborhood U of x , there exists an open set, U_0 such that $x\in U_0\subseteq U$. Since f is p-δ_I-semi-open, $V=f(U_0)\in(i,j)$ -$\delta_I SO(Y)$ and $f(x)\in V\subseteq f(U)$.
Conversely, let U be an τ_i – open set of X. For each $x\in U$, there exists $V_x\in(i,j)$-$\delta_I SO(Y)$ such that $f(x)\in V_x\subseteq f(U)$.Therefore, we obtain $f(U)=\cup\{V_x|x\in U\}$ and hence f(U) is (i, j)-δ_I-semi-open in Y, by Theorem 3.16 of [1]. This implies that f is p -δ_I- semi – open.

Theorem 3.5. A bijective function $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$ is p-δ_I-semi-open if and only if for each subset $W\subseteq Y$ and each τ_i – closed set F of X containing $f^{-1}(W)$ there exists an (i, j)-δ_I-semi-closed set $H\subseteq Y$ containing W such that $f^{-1}(H)\subseteq F$.

Proof: **Necessity:** Suppose that f is a p-δ_I-semi-open function. Let W be any subset of Y and F is a τ_i- closed subset of X containing $f^{-1}(W)$. Then $X-F$ is τ_i –open and since f is p-δ_I-semi-open, $f(X-F)$ is (i, j)-δ_I-semi-open in Y. Hence $H=Y-f(X-F)$ is p-δ_I-semi-closed in Y. $f^{-1}(W)\subseteq F$ implies that $W\subseteq H$. Moreover, we obtain $f^{-1}(H)=f^{-1}(Y-f(X-F))=f^{-1}(Y)-f^{-1}(f(X-F))\subseteq X-(X-F)=F$. Hence $f^{-1}(H)\subseteq F$.
Sufficiency: Let U be any τ_i- open set of X and $W=Y-f(U)$. Then $f^{-1}(W)=f^{-1}(Y-f(U))=f^{-1}(Y)-f^{-1}(f(U))\subseteq X-U$ and $X-U$ is τ_i- closed. By hypothesis, there exists an (i, j)-δ_I-semi-closed set H of Y containing W such that $f^{-1}(H)\subseteq X-U$. Then we have $f^{-1}(H)\cap U=\phi$ and $H\cap f(U)=\phi$. Therefore we obtain $Y-f(U)\supseteq H\supseteq W=Y-f(U)$ and f(U) is (i, j)-δ_I-semi-open in Y. This shows that f is p -δ_I-semi-open.

Theorem 3.6. For a function $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$, the following are equivalent:
1. f is p-δ_I-semi-open.
2. For every subset A of X , $f[\tau_i-int(A)]\subseteq(i,j)-sint_{\delta_I}(f(A))$

Proof: (1) → (2): Let f be p-δ_I-semi-open. Since $\tau_i-int(A)$ is τ_i-open in X,
$f[\tau_i-int(A)]$ is (i, j)-δ_I-semi-open in Y. Then we have $(i,j)-sint_{\delta_I}(f[\tau_i-int(A)])=f[\tau_i-int(A)]\subseteq f(A)$. This implies $f[\tau_i-int(A)]\subseteq(i,j)-sint_{\delta_I}(f(A))$.
(2) → (1): Let A be an τ_i-open set in X. Then, $f(A)=f[\tau_i-int(A)]\subseteq(i,j)-sint_{\delta_I}(f(A))$ and so $(i,j)-sint_{\delta_I}(f(A))=f(A)$. Therefore f(A) is (i, j)-δ_I-semi-open in Y. Hence f is p-δ_I-semi-open.

Theorem 3.7. For a function $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$, the following are equivalent:
1. f is p-δ_I-semi-closed.
2. For every subset A of X , $(i,j)-scl_{\delta_I}(f(A))\subseteq f[\tau_i-cl(A)]$.

Proof: Obvious from the Theorem 3.6 and Lemma 2.3.

Theorem 3.8. For any bijective function $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$, the following are equivalent:
1. f^{-1} is p-δ_I-semi-continuous.
2. f is p-δ_I-semi-open.
3. f is p-δ_I-semi-closed.

Proof: (1) ⇒ (2): Let U be a τ_i-open set of X. Then, by assumption, $(f^{-1})^{-1}(U)=f(U)$ is p-δ_I-semi-open in Y. Hence f is p-δ_I-semi-open.
(2) ⇒ (3): Let F be a τ_i-closed set of X. Then F^c is τ_i-open set in X. Since
f is p-δ_I-semi-open, $f(F^c)$ is (i, j)-δ_I-semi-open in Y. Hence f is p-δ_I-semi-closed.
(3) ⇒ (1): Let F be a τ_i-closed set of X. Then by assumption f(F) is (i, j)-δ_I-semi-closed set in Y and we have $f(F)=(f^{-1})^{-1}(F)$. Hence f^{-1} is p-δ_I-semi-continuous.

IV. Pairwise δ_I-semi-homeomorphism and pairwise δ_I^*-semi-homeomorphism

Definition 4.1. A bijective function $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$ is called pairwise -δ_I-semi-homeomorphism (briefly (i, j)-δ_I-semi-homeomorphism) if both f and f^{-1} are is p-δ_I-semi-continuous.
The family of all p-δ_I-semi-homeomorphisms of an ideal bitopological space (X,τ_1,τ_2,I) onto itself is denoted by $(i,j)-S_{\delta_I}H(X,\tau_1,\tau_2,I)$.

Theorem 4.2. Let $f:(X,\tau_1,\tau_2,I)\rightarrow(Y,\sigma_1,\sigma_2,I)$ be a bijective and p-δ_I-semi-continuous function. Then the following are equivalent.

1. f is p-δ_I-semi-open.
2. f is p-δ_I-semi-homeomorphism.
3. f is p-δ_I-semi-closed.

Proof: (1) ⇒ (2): Let V be τ_i-open set of X. Then, by assumption, f(V) is (i, j)-δ_I-semi-open in Y. But $f(V) = (f^{-1})^{-1}(V)$ and so $(f^{-1})^{-1}(V)$ is (i, j)-δ_I-semi-open in Y . This shows that f^{-1} is p-δ_I-semi-continuous. Hence f is p-δ_I-semi-homeomorophism.
(2) ⇒ (3): Let F be a τ_i-closed set in X. Then by assumption f^{-1} is p-δ_I-semi-continuous and so $(f^{-1})^{-1}(F) = f(F)$ is (i, j)-δ_I-semi-closed in Y. Thus f is p-δ_I-semi-closed.
(3) ⇒ (1): Let V be a τ_i-open in X. Then V^c is τ_i-closed in X. By assumption $f(V^c)$ is (i, j)-δ_I-semi-closed in Y. But $f(V^c) = (f(V))^c$. This implies that $(f(V))^c$ is (i, j)-δ_I-semi-closed in Y and so f(V) is (i, j)-δ_I-semi-open in Y.
Hence f is p-δ_I-semi-open.

Definition 4.3. A bijective function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, I)$ is called pairwise - δ_I^*-semi-homeomorphism (briefly p- δ_I^*-semi-homeomorphism) iff both f and f^{-1} are is p-δ_I-semi-irresolute.

Remark 4.4. 1. The spaces (X, τ_1, τ_2, I) and $(Y, \sigma_1, \sigma_2, I)$ are p- δ_I^*-semi-homeomorphic if there exists a p- δ_I^*-semi- homeomorphism from (X, τ_1, τ_2, I) onto $(Y, \sigma_1, \sigma_2, I)$.
2. The family of all p- δ_I^*-semi- homeomorphisms of an ideal bitopological space (X, τ_1, τ_2, I) onto itself is denoted by p- $S^*_{\delta_I}H(X)$.

Theorem 4.5. If the bijective function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, I)$ is an p- δ_I^*-semi- homeomorphism, then $(i,j) - scl_{\delta_I}(f^{-1}(B)) = f^{-1}\left((i,j) - scl_{\delta_I}(B)\right)$ for every $B \subseteq Y$.

Proof. Since f is p- δ_I^*-semi- homeomorphism, then both f and f^{-1} are p-δ_I-semi-irresolute. Let B is a subset of Y. Since $(i,j) - scl_{\delta_I}(B)$ is (i, j)-δ_I-semi-closed in Y, $f^{-1}\left((i,j) - scl_{\delta_I}(B)\right)$ is (i, j)-δ_I-semi-closed in X. But $(i,j) - scl_{\delta_I}(f^{-1}(B))$ is the smallest (i, j)-δ_I-semi-closed set containing $f^{-1}(B)$. $(i,j) - scl_{\delta_I}(f^{-1}(B)) \subseteq f^{-1}\left((i,j) - scl_{\delta_I}(B)\right)$. Again, $(i,j) - scl_{\delta_I}(f^{-1}(B))$ is (i, j)-δ_I-semi-closed in X. Since f^{-1} is p-δ_I-semi-irresolute, $f((i,j) - scl_{\delta_I}(f^{-1}(B)))$ is (i, j)-δ_I-semi-closed in Y. Now
$B = f(f^{-1}(B)) \subseteq f((i,j) - scl_{\delta_I}(f^{-1}(B)))$.
$(i,j) - scl_{\delta_I}(B) \subseteq \left((i,j) - scl_{\delta_I}(f((i,j) - scl_{\delta_I}(f^{-1}(B))))\right) = f\left((i,j) - scl_{\delta_I}(f^{-1}(B))\right)$. This implies
$f^{-1}\left((i,j) - scl_{\delta_I}(B)\right) \subseteq f^{-1}\left(f((i,j) - scl_{\delta_I}(f^{-1}(B)))\right) = (i,j) - scl_{\delta_I}(f^{-1}(B))$. Thus $f^{-1}\left((i,j) - scl_{\delta_I}(B)\right) \subseteq (i,j) - scl_{\delta_I}(f^{-1}(B))$. Hence $(i,j) - scl_{\delta_I}(f^{-1}(B)) = f^{-1}\left((i,j) - scl_{\delta_I}(B)\right)$.

Corollary 4.6. If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, I)$ is an p- δ_I^*-semi-homeomorphism, then $(i,j) - scl_{\delta_I}(f(B)) = f\left((i,j) - scl_{\delta_I}(B)\right)$ for every $B \subseteq X$.

Proof. Obvious from the Lemma 2.3 and Theorem 4.5

Corollary 4.7. If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, I)$ is a p - δ_I^*-semi-homeomorphism then $f((i,j) - scl_{\delta_I}(B)) = \left((i,j) - scl_{\delta_I}(f(B))\right)$ for every $B \subseteq X$.

Proof: For any set $B \subseteq X$,

$(i,j) - sint_{\delta_I}(B) = ((i,j) - scl_{\delta_I} f(B^c))^c$.

$f\left((i,j) - sint_{\delta_I}(B)\right) = f(((i,j) - scl_{\delta_I} f(B^c))^c)$.

$= (f((i,j) - scl_{\delta_I}(B^c)))^c$

Then by corollary 4.6, we see that,

$f\left((i,j) - sint_{\delta_I}(B)\right) = \left((i,j) - scl_{\delta_I}(f(B^c))\right)^c$

$= \left((i,j) - sint_{\delta_I}(f(B))\right)$.

Theorem 4.8 If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, I)$ and $g : (Y, \sigma_1, \sigma_2, I) \to (Z, \gamma_1, \gamma_2, I)$ are p- δ_I^*-semi-homeomorphism then the composition gB f : $(X, \tau_1, \tau_2, I) \to (Z, \gamma_1, \gamma_2, I)$ is also p- δ_I^*-semi-homeomorphism.

Proof: Let U be (i, j)-δ_I- semi-open in Z. Since f and g are p- δ_I^*-semi-homeomorphisms, $f^{-1}(g^{-1}(U)) = (gB\, f)^{-1}(U)$ is (i, j)-δ_I- semi-open in X. This implies that gB f is (i, j)-δ_I- semi-irresolute. Again, let G be an

(i,j)-δ_I- semi-open in X. Since f is p- δ_I^*-semi-homeomorphism, $(f^{-1})^{-1}(G) = f\,(G)$ is (i,j)-δ_I- semi-open in Y. Since g is p- δ_I^*-semi-homeomorphism, $(g^{-1})^{-1}(f(G)) = g(f(G))$ = (gB f) (G) = $((gB\,f)^{-1})^{-1}(G)$ is (i,j)-δ_I- semi-open in Z. This implies that $(gB\,f)^{-1}$ is (i,j) - δ_I - semi-irresolute. Since f and g are p- δ_I^*-semi-homeomorphisms, f and g are bijective and so gB f is bijective. Hence gB f is p- δ_I^*-semi-homeomorphism.

Theorem 4.9. The set p- $S^*_{\delta_I}H(X)$ is a group under composition of functions.
Proof: Let f , g ∈ p- $S^*_{\delta_I}H(X)$. Then fB g ∈ p- $S^*_{\delta_I}H(X)$ by Theorem 4.8. Since f is bijective, f^{-1} ∈ p- $S^*_{\delta_I}H(X)$. This completes the proof.

Theorem 4.10. If f : (X, τ_1, τ_2, I) → (Y, σ_1, σ_2, I) is an p- δ_I^*-semi-homeomorphism. Then f induces an isomorphism from the group p- $S^*_{\delta_I}H(X)$ onto the group p- $S^*_{\delta_I}H(Y)$.
Proof: Let f ∈ p- $S^*_{\delta_I}H(X)$. Then define a map χ_f : p- $S^*_{\delta_I}H(X)$ → p- $S^*_{\delta_I}H(Y)$
by $\chi_f(h) = fBhBf^{-1}$ for every h ∈ p- $S^*_{\delta_I}H(X)$. Let h_1, h_2 ∈ p- $S^*_{\delta_I}H(X)$. Then $\chi_f(h_1\,B\,h_2)$ = f B(h_1 B h_2)Bf^{-1} = f B(h_1 B f^{-1}BfB h_2) B f^{-1}

$$= (f\,B\,h_1\,B\,f^{-1})B(fB\,h_2B\,f^{-1})$$
$$= \chi_f(h_1)B\,\chi_f(h_2).$$

Since $\chi_f(f^{-1}B\,h\,Bf) = h$, $\chi_f$is onto. Now, $\chi_f(h)$ = I implies fBhBf^{-1}= I. That implies h =I. This proves that χ_f is one-one. This shows χ_f is an isomorphism.

Theorem 4.11. p- δ_I^*-semi-homeomorphism is an equivalence relation in the collection of all ideal bitopological spaces,
Proof: Reflexive and Symmetric properties are obvious and Transitive property follows from Theorem 4.8.

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