IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 2 Ver. IV (Mar - Apr. 2015), PP 01-07
www.iosrjournals.org

Some Categorical Aspects of Rings

Dhanjit Barman
Department of Mathematics, Gauhati University.

Abstract: *An arbitrary ring with unity can be thought of as a category with one object. In this paper we have shown how an arbitrary ring with unity can be thought as a category with one object. Also we have defined quotient category of a ring. The categorical approach to the fundamental theorem of homomorphism of ring theory has been provided. Moreover the isomorphism theorems of ring have been proved categorically.*
Keywords*: Category , Cokernel, Congruence relation, Functor, Kernel, Morphism , Quotient category,* ***Ring*** *(category of rings).*

.

I. Introduction

Here we discussed some categorical aspects of Rings in details. We have proved the fundamental theorem of homomorphism of rings categorically. We have also provided categorical proof of isomorphism theorems of ring .

Preliminaries

For notions of category theory we shall in general follow the notation and terminology of Popescu [6]. However, we do deviate somewhat.

For **C** a category and A, B objects of **C**, Mor(A, B) denotes the set of morphisms from A to B.
It will be shown that an arbitrary ring with unity can be thought of as a category with one object.

Next we shall use the definition of quotient category from Mitchel [3] and **quotient category of a ring** will be defined.

If R and S are ringss, regarded as categories, then we can consider arbitrary functors between them f: R→S. It is obvious that a functor between rings is exactly the same thing as a ring homomorphism.

We will also think the fundamental theorem of homomorphism of ring theory in categorical way.

Lastly the 2[nd] and 3[rd] isomorphism theorems of ring will be proved categorically by using the fact that "**every morphism in the category of rings(Ring) has a cokernel."**

Main Results:-

1. In the category of rings (Ring) every morphism has a kernel.

Proof: Let us consider the morphism f: R →S.
Let kerf = K be the kernel of f.
Let us consider the diagram

$$\mathbf{K} \xrightarrow{i} \mathbf{R} \xrightarrow{f} \mathbf{S}$$ **,where i is inclusion map.**

Clearly foi = **0** , **0**: K→S being zero morphism.
Suppose that g: M → R be another morphism such that
fog = **0**.....................................(i)
Let us define j: M → K by j(m) = g(m) for all m∈M.
This is well defined as………..
f(g(m)) = (fog)(m)
= **0**(m) [from (i)]
= > g(m)∈ K
Now (ioj)(m) = i(j(m))
= j(m)
= g(m) for all m∈M .
= > ioj = g.
If j': M →K be another morphism such that ioj'= g.
Then ioj = ioj'
= > i(j(m)) = i(j'(m)) for all m ∈M.
= > j(m) = j'(m) [i is inclusion]
= > j= j'
Thus j is unique.

Hence i: K → R is a kernel of f : R → S.

1. In the category of rings(Ring) every morphism has a cokernel.

Proof: Let f : R→S be morphism in **Ring.** Let us consider the diagram

$$R \xrightarrow{f} S \xrightarrow{p} S/J$$ **, where J is the ideal generated by f(R).**

Let us consider a morphism g: S→T such that gof= **0.**

Let us define j: S/J → T by j(s+J) = g(s).

It is well defined as ………..

x+J = y+J for x ,y ∈S

= > x-y∈ J

= > x-y is a finite sum of elements of the form sf(r) ,where r ∈R and s ∈S.

Since g(s f(r)) = g(s)g(f(r))

= $g(s)(gof)(r)g(s^{-1})$

= g(d)**0**(r)

= $g(r)e_T$

= e_T

Thus $g(x-y) = e_T$

= > $g(x)-g(y) = e_T$

= >g(x) = g(y)

= > j(x+J)= j(y+J).

Now (jop)(s) = j(p(s))

= j(s+J)

= g(s) for all s ∈S.

= > jop= g.

Also 'j' is unique as p is epimorphism.

Hence p: S →S/J is cokernel of f: R→S.

2. A Ring With Unity Can Be Thought Of As Category With One Object :

Let us consider an arbitrary ring with unity (R,+,.).

Let us consider the collection R' as follows------

i) **Ob**R'={R}

ii) The only set **Mor(R,R**) and the morphisms are the elements of R i.e. r∈R <=> r:R→R.

iii) The composition in **Mor**(R,R) is defined as , if r:R→R ,s: R→R then sor: R→R is defined as sor=s.r

Then we have the following-------

a) For r,s,t∈ Mor(R,R),

to(sor) = to(s.r)

= t.(s.r)

= (t.s).r

= (tos) or

"o" is associative.

b) let "u" be the unity in R i.e. u:R→R and for r:R→R, s: R→R we have rou=r.u=r and uos=u.s=s

Therefore u:R→R is the identity morphism in **Mor**(R,R).

(we shall frequently write 1_R for u :R→R)

Hence **R'** is a category.

Here onwards we call the category corresponding to the ring R as R'.

3. Quotient Category Of A Ring :

Let K be an ideal of R.

Let us define a relation" ≈ " in **Mor** (R,R) as follows----

For any r,s ∈ **Mor** (R,R),

r≈s < => r-s ∈ K.

Then we have the followings…………..

(i) r≈r as r-r = 0∈K, so ≈ is reflexive.

(ii) let r ≈ s then r-s∈ K

=>s-r= -(r-s)∈ K

=> s≈r , so ≈ is symmetric.
(iii) Let r≈s and s≈t then we have
r-s=∈ K and s-t∈ K
=> (r-s) +(s-t) ∈K
=> r-t ∈K
=> r≈t, so ≈ is transitive.
Thus ′ ≈ ′ is an equivalence relation.
Next assume that r≈s and r' ≈s' then
r- s ∈ K => rs' – ss' ∈ K and
r' –s' ∈ K => rr' - rs' ∈ K
from which it follows that
(rr' – rs') + (rs' – ss') = rr' – ss' ∈ K
=> rr'≈ss'
Also r''≈s'' and r ≈ s
= >r''-s'' ∈ K => r''r – s''r ∈ K and r – s ∈ K => s''r - s''s ∈ K
from which it follows that
(r''r – s''r) + (s''r – s''s) = r''r – s''s ∈ K
=> r''r ≈ s''s

Hence ′ ≈ ′ is a congruence relation on **Mor** (R,R).
Next we define quotient category R'/≈ (= $Q_{R'}$) of R' as follows--------
i) **Ob**($Q_{R'}$) =**Ob**(R'),
ii) **Mor** $Q_{R'}$ ={the equivalence classes E(r) : r∈ **Mor** (R,R)}
where E(r) = {s ∈ **Mor**(R,R)| s≈r }.
Let us define composition in Mor($Q_{R'}$) as E(x)oE(y)=E(xoy), which is well defined as……..
If E(x)= E(a) and E(y)=E(b)
then x≈a and y≈b.
=>x-a ∈ K and y-b ∈K.
=> xy - ay∈K and ay – ab ∈ K
Now, (xy - ay) + (ay – ab) ∈K
=> xy – ab ∈ K
=>xy ≈ ab
=> E(xoy)=E(aob)

4. Categorical Approach To The Fundamental Theorem Of Homomorphism Of Ring Theory :
Let f: R→ S be a homomorphism of the ring R on to the ring S.
Then K=kerf is an ideal of R.
Clearly f : R' →S' will be a full functor which is surjective on object. (where R' and S' are the corresponding categories of the rings R and S respectively.)
Let us consider the quotient category $Q_{R'}$ of R'.
Let us define F: $Q_{R'}$ → S' by
F(R) =S and
F(E(r)) =f(r), where r: R→R and f(r): S→ S, which is well defined as………
E(r) =E(s) => r ≈s
=> r – s ∈ K
=> f(r - s) = 0
=> f(r) = f(s).
Now,
i) F(E(s)oE(r)) = F(E(sor))
= f(s.r)
= f(s).f(r)
= F(E(s)) o F(E(r)).
ii) F(E(u)) = f(u)
$=1_S$
$=1_{F(R)}$
Therefore 'F' is a covariant functor.
Conversely, let us define G: S' → $Q_{R'}$ by

$G(S) = R$
$G(f(r))=E(r)$, which is well defined as K=kerf is an ideal of R.

Now,
i) $G(f(s)of(r)) = G(f(sor))$
$= E(sor)$
$= E(s)oE(r)$
$= G(f(s)) \text{ o } G(f(r))$.

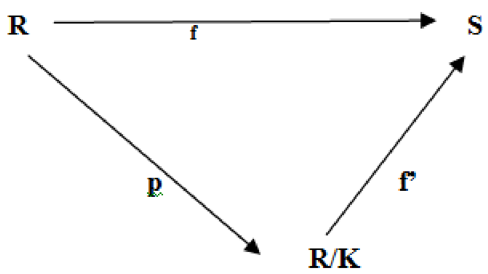
ii) $G(1_S) = G(f(u))$
$= E(u)$
$= 1_R$
$= 1_{G(S)}$.

Therefore 'G' is a covariant functor.

Thus $FoG(f(r)) = F(G(f(r)))$
$= F(E(r))$
$= f(r)$
$= Id_{S'}(f(r))$.

i.e. $FoG = Id_{S'}$.
Similarly it can be proved that $GoF = IdQ_{R'}$.
Hence $Q_{R'} \cong S'$.

5. Lemma : Let f: R→S be a ring homomorphism such that f kills K (i.e. $f(K)= 0_S$)where K is an ideal of R. Then there exists a unique homomorphism f' : R/K → S with f' o p = f , i.e. the diagram



commutes.

Where p : R → R/K is natural homomorphism.

Proof : Let K be an ideal of R. Then we have
K → R → R/K [where the elements of R/K are the equivalence classes of the form E(r) for all r∈R
and p:R→R/K is natural homomorphism and i: K→R is inclusion]

such that p o i = u , where u: K→R/K is a zero homomorphism.
Because, $(poi)(k) = p(i(k))$
$= p(k)$
$= K$
$=$ zero element in R/K.
$= u(k)$.

=> $poi = u$.

Let f: R→S be a ring homomorphism such that foi = u i.e.
$(foi)(k) = u(k)$
=> $f(i(k)) = 0_S$
=> $f(k) = 0_S$
=> $f(K) =0_S$.
=> f kills K.

Next let us define f': R/K → S by
$f'(E(r)) = f(r)$.

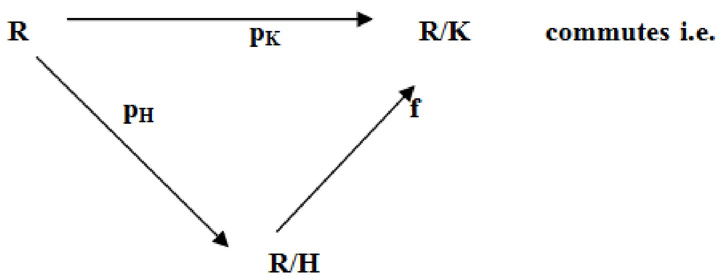
which is well defined as……..
if $E(r) = E(s)$
then $r \approx s$
=> $r - s \in K$

=> $f(r - s) = 0_S$ (since f kills N)
=> $f(r) - f(s) = 0_S$
=> $f(r) = f(s)$.
Also $(f' o p)(r) = f'(p(r))$
$= f'(E(r))$
$= f(r)$.
=> $f' o p = f$.
Suppose ,if possible ,f'': R/K → S be another homomorphism such that
f'' o p = f.
Then f''op = f'op
=> $f'' = f'$ (since p is surjective).
Hence f' is unique.

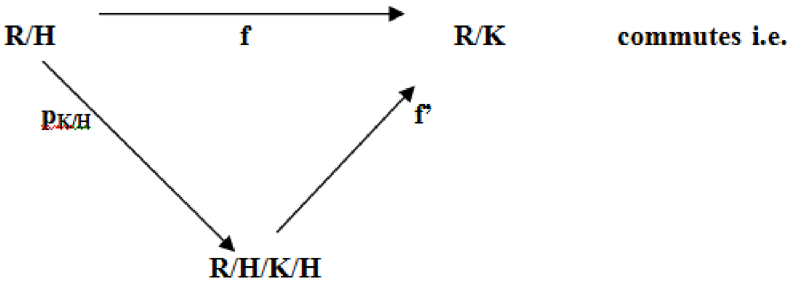
6. Categorical Proof Of Isomorphism Theorems Of Ring.

Theorem 1: **Let H ,K are ideals of R such that H ⊆ K.**
Then R/H/K/H ≅ R/K.

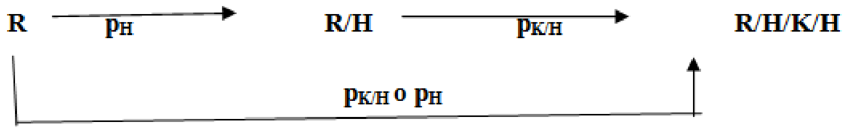
Proof : As K is an ideal of R so p_K : R→ R/K is a ring homomorphism and it kills H (since H ⊆ K). Therefore by above lemma we have a unique ring homomorphism f: R/H→R/K such that the following diagram



$f \circ p_H = p_K$ ……………………………………………………………………………...(i)
Now f: R/H → R/K is a ring homomorphism which kills K/H.
Therefore by above lemma there exists a unique ring homomorphism f' : R/H/K/H → R/K such that the following diagram



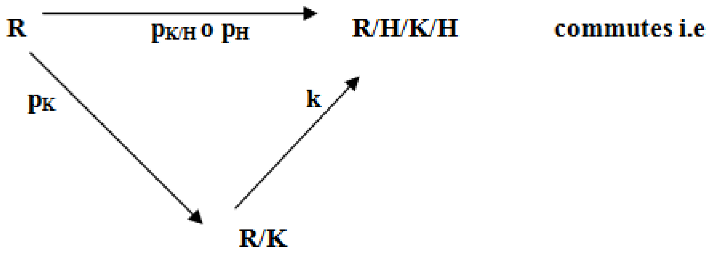
$f' \circ p_{K/H} = f$ ……………………………………………………………………..(ii)
But we have also



which kills K.

So we have a unique ring homomorphism k : R/K → R/H/K/H such that

The following diagram…….



$kop_K = p_{K/H}$ o p_H ……………………………………………………………..(iii)

From (ii) we have k o f' o $p_{K/H}$ o p_H = k o f o p_H
= k o p_K [from (i)]
= $p_{K/H}$ o p_H [from (iii)]
Therefore kof' = $Id_{R/H/K/H}$ …………………………………………….(iv)
Similarly from (iii) we have f' o k o p_K = f' o $p_{K/H}$ o p_H
= f o p_H [from (ii)]
= p_K [from (i)]

Thus f' o k = $Id_{R/K.}$ ……………………………………...(v)
From (iv) and (v) we have

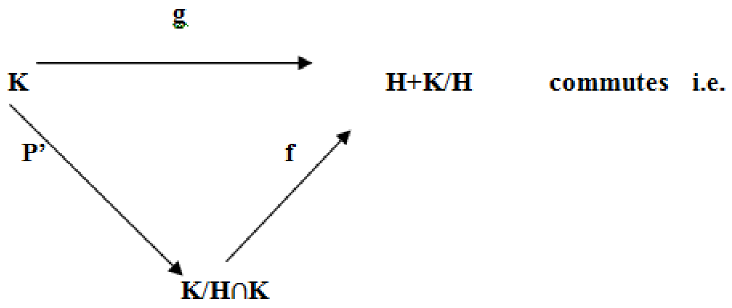
R/K ≅ R/H/K/H.

Theorem 2: **Let H , K are ideals of R. Then**
H+K/H ≅ K/H∩K.

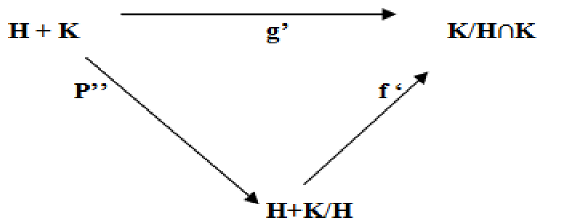
Proof : As H is an ideal of R so it is ideal of H+K.
So we may compose the inclusion i : K → H+K with the natural homomorphism p'' : H+K → H+K / H to get a
homomorphism

g : K → H+K / H [i.e. p''oi = g].

It kills H∩K. Therefore by above lemma we have a unique ring homomorphism f : K/H∩K → H+K/H such that the
following diagram



fop' = g ………………………………………………………………..(i)
Also g' : H+K → K/H∩K is a ring homomorphism which kills H.
So by above lemma we have a unique homomorphism f' : H+K/H → K/H∩K such that the following diagram



commutes i.e.

$f' \circ p'' = g'$……………………….(ii)

So from (i) we have

$$f' \circ f \circ p' = f'og$$
$$= p'$$

Therefore $f' \circ f = Id_{K/H\cap K}$.. (iii)

Also from (ii) we have

$$fof'op'' = fog'$$
$$= p''$$

Thus $fof' = Id_{H+K/H}$. ………………………………………………(iv)

From (iii) and (iv) we have

$$H+K / H \cong K / H\cap K.$$

II. Conclusion

In this paper we have used some categorical notions to prove some theorems of Ring theory.Basically kernel of a morphism and cokernel of a morphism play an important role in this case . This can be extended to the product of two rings i.e the product of two rings can be proved as a category of one object and the elements of the product as its morphisms.

Acknowledged by

Dr. Khanindra Chandra Chowdhury
Department of Mathematics Gauhati University

References

[1]. Anderson,Frank W.& Fuller,Kent R., Rings and Categories of Modules,Springer-Verlag New York berlin Heidelberg London paris Tokyo Hong Kong Barcelona Budapast.
[2]. Mac Lane, S.,1971: Categories for the Working Mathematician, Springer-Verlag New York Berlin.
[3]. Mitchel, Barry.1965: Theory of Categories, Academic Press New Yorkand London.
[4]. Krishnan,V.S.,198TH1: An introduction to Category Theory,North Holland NewYork Oxford.
[5]. Schubert, Horst,1972: Categories, Springer-Verlag Berlin Heidelberg New York.
[6]. Popescu,N.,1973: Abelian Categories with Applications to Rings and Modules, Academic Press, London & New York.
[7]. Awodey, Steve.,2006: CategoryTheory,Second Edition, Clarendon Press,Oxford.
[8]. Borceux, Francis.1994: Hand Book of Categorical Algebra,Cambridge University Press
[9]. Simmons, Harold.,2011: An Introduction to Category Theory, Cambridge University Press.
[10]. Freyd, P.,1965: Abelian Categories, An Introduction to theTheory of Functors, A Harper International Edition, 0 jointly published by Harper & Row,NewYork,Evaston & London and JOHN WEATHERHILL INC. TOKYO.
[11]. Pareigis, Bodo.,1970:Categories and Functors,Academic Press New York, London.
[12]. m.Fokkinga,Maarlen.,1994: A Gentle Introduction to Category Theory.,University of Twente,dept INF.
[13]. Verlag,Heldermann.,Category Theory at work, Research and Exposition in Mathematics., Volume 18.
[14]. VanOostem,Jaap., 1995: Basic Category Theory
[15]. Atitah,M.F.'Macdonald,I.G.,1969: Introduction to Commutative Algebra, University of Oxford,Addision-Wesley Publishing Company. Hovey, Mark. 1998: Monoidal model categories, preprint.
[16]. Kelly,G.M., Basic Concept of Enriched Category Theory.,University of Sydney,Cambridge University Press, LondonMathematical Society
[17]. Lecture Notes Series 64,1992.
[18]. Lawvere,F.M., The category of Categories as a foundation for mathematics,proc. Conf. Categorical Algebra(La Jolla,Calif,1965) Springer,New York,1966.
[19]. Buchsbaum,D.A.,Exact Categories and Duality,Trans. Amer. Math. Soc 80 (1955)
[20]. Freyd, p., Abelian Categories. An Introduction to the Theory of Functors.Harper's Series in Modern Math.,Harper and Row, New York,1964
[21]. Watkins,John J.,Topics in Commutative Ring Theory,Princeton University Press,Princeton and Oxford.
[22]. Lambek,J., Lectures on Rings and Modules,McGill University.
[23]. Stenstrom, Bo., Algebra, Lectures on Rings and Modules, Stockholms Universitet Matematisca institutionen,November 2001.Khanna Vijay K.,Acourse in Abstract Algebra, Third Edition,Vikas Publishing House Pvt Ltd.
[24]. Singh,s., Zameeruddin,Q.,Modern Algebra, Vikas Publishing House Pvt Ltd.
[25]. Goodearl,K.R., Ring Theory,NONSINGULAR RINGS AND MODULES,Marcel Dekker INC., New York and Basel.
[26]. Musili,c., Introduction to Rings and Modules, Second Revised Edition,Narosa Publishing House.