The fascinating primes

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Abstract: *In this article some of the interesting results on prime numbers are surveyed which made prime numbers very fascinating.*
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I.   Introduction

What makes prime numbers very fascinating ?
Prime numbers are first introduced at school to find H.C.F. and L.C.M.
As soon as the student looks at the first result "Fundamental theorem of Arithmetic and unique factorisation theorem it gives first pleasure for student to play with prime numbers.
What makes the game difficult?
There are infinitely many primes and there are large gaps among the prime numbers. That is, for any given number k there are k consecutive composite numbers:
$(k + 1)! + 2, (k + 2)! + 3, . . . , (k + 1)! + (k + 1)$

This irregular distribution makes the study of prime numbers more fascinating, interesting, as well as difficult.

II.   Some Important Theorems

The use of analytical methods in proving the results in number theory began with Euler. The following result is due to Euler:

2.1 Theorem: The sum

$$\sum_{p,\text{prime}} \frac{1}{p}$$

diverges.

This theorem itself gives another proof to say there are infinitely many primes. Apart from the irregularity that was mentioned we still get several surprises. Let us use this theorem to compare their density. For this consider the following three series.

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges} \tag{1}$$

$$\sum_{p,\text{prime}} \frac{1}{p} \quad \text{diverges} \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges} \tag{3}$$

The first result Ramanujan himself investigated in his note books. By p-test the third series converges. So, in some sense, the density of primes is more than the density of sequence of squares $1^2, 2^2, 3^2, ...,$ So there can be a good guess that for any given natural number n>1 there is a prime p such that $n<p<n^2$. But, there is a more powerful and surprising result due to Bertrand. The following result was verified by Bertrand for large numbers and conjectured, but was proved later by Chebychev.

2.2 Theorem:(Bertrand's theorem) For any given natural number n>1 there is a prime p such that n<p<2n
The proof of this theorem too needs heavy analysis.

Consider any non-trivial arithmetic sequence as given below:
a, a+d, a+2d, …   for some d>0 and (a,d)=1. Then once again you can find infinitely many primes in this sequence. This theorem is known as Dirichlet's theorem.

III. Prime Number Theorem

By observing the above results, the main question arises that if there is any simple formula to analyse the distribution or the density of primes. One of the most profound theorems in mathematics which shows an order in the disorder distribution of primes is known as the Prime number theorem. Before stating the theorem let us ask the following question first which is more motivating.

3.1 Question: What is the probability that the number you choose from first x positive integers is a prime?

The function that of main interest to answer this question is Prime number function which is defined for positive integers x by

$$\pi(x) = \text{number of primes} \leq x$$

3.2 Prime Number Theorem: If $\pi(x)$ is the prime number function, then

$$\lim_{n \to \infty} \frac{\pi(x)}{x/\log x} = 1$$

In the above equation clearly $\pi(x) \to \infty$ as $x \to \infty$. The prime number theorem asserts that asymptotically, $\pi(x)$ is given by $\frac{x}{\log x}$.

The answer to the question we raised above is a different way of expressing the prime number theorem. Therefore, the probability of randomly choosing a prime less than or equal to a positive integer x, is asymptotically given by $\frac{1}{\log x}$. The proof of prime number theorem using the theory of complex functions opened a new branch in number theory introduced by Riemann which is known as analytic number theory.

The aim of this article is to motivate an interested reader in number theory, to search for these results and do further research. The proofs and other details are available in many advanced books in number theory. Some reference books are given below to help the reader to search for these results.

References

Books:
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