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On A Certain Class of Multivalent Functions with Negative Coefficients

C. Selvaraj[1] And G.Thirupathi[2]
[1]*Department of Mathematics, Presidency College, Chennai – 600 005, Tamilnadu, India.*
[2]*Department of Mathematics, RMK Engineering College, R.S.M. Nagar, Kavaraipettai – 601 206. Tamilnadu, India.*

Abstract: *In the present paper, we introduce the class* $S_P^*(\alpha,\beta,\gamma,A,B,\delta)$ *of p - valent functions in the unit disc* $U=\{z:|z|<1\}$*. We obtain coefficient estimate, distortion and closure theorems, radii of close-to-convexity and* ε *- neighborhood for this class.*

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I. Introduction And Definition

Let A_p be the class of functions analytic in the open unit disc $U=\{z:|z|<1\}$ of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p} \tag{1.1}$$

and let $A_1=A$.

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $f(z)$ is subordinate to $g(z)$ in U, if there exists an analytic function $w(z)$ in U such that $|w(z)|<|z|$ and $f(z)=g(w(z))$, denoted by $f(z)\prec g(z)$. If $g(z)$ is univalent in U, then the subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

For the functions $f(z)$ of the form (1.1) and $g(z)=z^p+\sum_{n=1}^{\infty}b_{n+p}z^{n+p}$, the hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}b_{n+p}\,z^{n+p}.$$

A function f belonging to A_p is said to be p-valently starlike of order β if it satisfies

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>\beta \qquad (z\in U),$$

for some $\beta(0\le\beta<p)$. We denote by $S_P^*(\beta)$ the subclass of A_p consisting of functions which are p-valently starlike of order β in U.

Recently, M.K. Aouf et. al. [1] introduced the operator $\Re_{\beta,p}^{\alpha,\gamma}:A_p\to A_p$ as follows:

$$\begin{aligned}\Re_{\beta,p}^{\alpha,\gamma}f(z)&=\frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)}\frac{1}{z^p}\int_0^z\left(1-\frac{t}{z}\right)^{\alpha-\gamma}t^{\beta-1}f(t)\,dt\\&=z^p+\frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)}\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+\beta+n)}{\Gamma(p+\alpha+\beta+n-\gamma+1)}\right]a_{n+p}z^{n+p}\end{aligned} \tag{1.2}$$

$$\left(\beta > -p; \alpha > \gamma - 1; \gamma \in \square; p \in \square; z \in U\right).$$

From (1.2), it is easy to verify that

$$z\left(\Re_{\beta,p}^{\alpha+1,\gamma} f(\mathrm{z})\right)' = \left(\alpha+\beta+p-\gamma+1\right)\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z}) - \left(\alpha+\beta-\gamma+1\right)\Re_{\beta,p}^{\alpha+1,\gamma} f(\mathrm{z}). \tag{1.3}$$

Remark:1.1. If we let $\gamma = 1$, then this operator $\Re_{\beta,p}^{\alpha,\gamma}$ reduces to the operator introduced and studied by Liu and Owa [2] and $Q_{\beta,1}^{\alpha} = Q_{\beta}^{\alpha}$ introduced and studied by Jung et.al.[3]. For other choices of α and β then the operator $\Re_{\beta,p}^{\alpha,\gamma}$ reduces to the familiar other well- known integral operators introduced and discussed by various authors [4, 5, 6, 7].

Let $T_p(n)$ be the subclass of A_p, consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \geq 1). \tag{1.4}$$

Motivated by the earlier investigations of Aouf [8], Darwish and Aouf [9], Magesh et. al. [10], Guney, H.O and Sumer Eker.S [11] and Mahzoon [12], we investigate, in the present paper, the various properties and characteristics of analytic p-valent functions belonging to the subclass $S_P^*(\alpha,\beta,\gamma,A,B,\delta)$.

Definition: 1.1. A function $f \in T_p(n)$ is said to in the class $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ if it satisfies the following differential condition:

$$\frac{zF'(z)}{F(z)} \prec \frac{p+\left[pB+(A-B)(p-\delta)\right]}{1+Bz}, \tag{1.5}$$

where

$$F(\mathrm{z}) = (1-\lambda)\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right) + \lambda z\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)'.$$

The condition (1.5) is equivalent to

$$\left|\frac{\frac{zF'(z)}{F(z)} - p}{\left[pB+(A-B)(p-\delta)\right] - B\frac{zF'(z)}{F(z)}}\right| < 1, \tag{1.6}$$

where the parameters $\alpha, p, \delta, \lambda, \gamma$ are constrained as follows:
$\alpha > \gamma - 3,\ \beta > -p,\ \gamma \in \square,\ 0 \leq \delta < p,\ -1 \leq B < A \leq 1,\ -1 \leq B < 0,\ 0 \leq \lambda \leq 1$ and $p \in \square$.

II. Coefficient Estimates

Theorem: 2.1. A function $f(z)$ defined by (1.4) is in $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ if it satisfies the following inequality:

$$\sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right]a_{n+p} \leq (A-B)(p-\delta)(1+\lambda(\mathrm{p}-1)), \tag{2.1}$$

where $$\phi(n,\alpha,\beta,\gamma) = \frac{\Gamma(p+\alpha+\beta-\gamma+1)\Gamma(p+\beta+n)}{\Gamma(p+\beta)\Gamma(p+\alpha+\beta+n-\gamma+1)}\left[1+\lambda(n+p-1)\right] \tag{2.2}$$

$0 \leq \delta < p,\ -1 \leq B < A \leq 1,\ -1 \leq B < 0$ and $0 \leq \lambda \leq 1.$

Equality holds for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)\left[1+\lambda(p-1)\right]}{(p+\beta)\left[(1-B)+(A-B)(p-\delta)\right]\left[1+\lambda p\right]} z^{p+1}.$$

Proof: Assume that the inequality (2.1) holds true and let $|z| = 1$. Then we obtain

$$\left| \frac{\frac{zF'(z)}{F(z)} - p}{\left[pB+(A-B)(p-\delta)\right] - B\frac{zF'(z)}{F(z)}} \right|$$

$$= \frac{\left|\sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\, n\, a_{n+p} z^n\right|}{\left|(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1)) + \sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\left[nB-(A-B)(p-\delta)\right] a_{n+p} z^n\right|}$$

$$\le (A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))$$

by hypothesis. Hence, by the maximum modulus theorem, we have $f \in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$. Conversely, assume that $f(z) \in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$, then in the view of (1.2) and (1.5), we get

$$\left| \frac{\frac{zF'(z)}{F(z)} - p}{\left[pB+(A-B)(p-\delta)\right] - B\frac{zF'(z)}{F(z)}} \right|$$

$$= \frac{\left|\sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\, n\, a_{n+p} z^n\right|}{\left|(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1)) + \sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\left[nB-(A-B)(p-\delta)\right] a_{n+p} z^n\right|} < 1$$

Since $\mathrm{Re}(z) \le |z|$ for all z , we have

$$\mathrm{Re}\left\{ \frac{\sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\, n\, a_{n+p} z^n}{(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1)) + \sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\left[nB-(A-B)(p-\delta)\right] a_{n+p} z^n} \right\} < 1.$$

Choosing values of z on the real axis and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right] a_{n+p} \le (A-B)(p-\delta)(1+\lambda(\mathrm{p}-1)).$$

The proof is completed.

Corollary: 2.1. Let the function $f(z)$ defined by (1.4) be in $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$. Then

$$a_{n+p} \le \frac{(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))}{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)}$$

for $n \ge 1$. Equality holds for the function $f(z)$ of the form

$$f\left(z\right)=z^{p}-\frac{\left(A-B\right)\left(p-\delta\right)\left[1+\lambda\left(p-1\right)\right]}{\left[n\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\phi\left(n,\alpha,\beta,\gamma\right)}z^{n+p}. \tag{2.3}$$

III. Distortion Bounds

Theorem: 3.1. A function $f\left(z\right)$ defined by (1.4) is in $S_{P}^{*}\left(\alpha,\beta,\gamma,A,B,\lambda,\delta\right)$. Then for $\left|z\right|=r$, we have

$$r^{p}-\frac{\left(p+\alpha+\beta-\gamma+1\right)\left(A-B\right)\left(p-\delta\right)\left[1+\lambda\left(p-1\right)\right]}{\left(p+\beta\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]}r^{p+1}\leq\left|f\left(z\right)\right|\leq \tag{3.1}$$

$$r^{p}+\frac{\left(p+\alpha+\beta-\gamma+1\right)\left(A-B\right)\left(p-\delta\right)\left[1+\lambda\left(p-1\right)\right]}{\left(p+\beta\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]}r^{p+1}$$

for $z\in U$. The result is sharp.

Proof: Since $f\left(z\right)$ belongs to the class $S_{P}^{*}\left(\alpha,\beta,\gamma,A,B,\lambda,\delta\right)$, in view of Theorem 2.1, we obtain

$$\frac{\left(\beta+p\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]}{\left(p+\alpha+\beta-\gamma+1\right)}\sum_{n=1}^{\infty}a_{n+p}\leq$$

$$\sum_{n=1}^{\infty}\phi\left(n,\alpha,\beta,\gamma\right)\left[n\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]a_{n+p}\leq\left(A-B\right)\left(p-\delta\right)\left[1+\lambda(\mathrm{p}-1)\right]$$

which is equivalent to

$$\sum_{n=1}^{\infty}a_{n+p}\leq\frac{\left(p+\alpha+\beta-\gamma+1\right)\left(A-B\right)\left(p-\delta\right)\left[1+\lambda(\mathrm{p}-1)\right]}{\left(\beta+p\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]} \tag{3.2}$$

Using (1.4) and (3.2), we obtain

$$\begin{aligned}\left|f\left(z\right)\right| &\leq \left|z\right|^{p}+\left|z\right|^{p+1}\sum_{n=1}^{\infty}a_{n+p}\\ &\leq r^{p}+r^{p+1}\sum_{n=1}^{\infty}a_{n+p}\\ &\leq r^{p}+\frac{\left(p+\alpha+\beta-\gamma+1\right)\left(A-B\right)\left(p-\delta\right)\left[1+\lambda(\mathrm{p}-1)\right]}{\left(\beta+p\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]}r^{p+1}.\end{aligned}$$

Similarly,

$$\left|f\left(z\right)\right|\geq r^{p}-\frac{\left(p+\alpha+\beta-\gamma+1\right)\left(A-B\right)\left(p-\delta\right)\left[1+\lambda(\mathrm{p}-1)\right]}{\left(\beta+p\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]}r^{p+1}.$$

This completes the proof of Theorem 3.1.

Theorem: 3.2. A function $f\left(z\right)$ defined by (1.4) is in $S_{P}^{*}\left(\alpha,\beta,\gamma,A,B,\lambda,\delta\right)$. Then for $\left|z\right|=r$, we have

$$pr^{p-1}-\frac{\left(p+\alpha+\beta-\gamma+1\right)\left(A-B\right)\left(p-\delta\right)\left[1+\lambda\left(p-1\right)\right]\left(p+1\right)}{\left(p+\beta\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]}r^{p}\leq\left|f'\left(z\right)\right|\leq \tag{3.3}$$

$$pr^{p-1}+\frac{\left(p+\alpha+\beta-\gamma+1\right)\left(A-B\right)\left(p-\delta\right)\left[1+\lambda\left(p-1\right)\right]\left(p+1\right)}{\left(p+\beta\right)\left[\left(1-B\right)+\left(A-B\right)\left(p-\delta\right)\right]\left[1+\lambda p\right]}r^{p}$$

for $z\in U$. The result is sharp.

Proof: Since $f\left(z\right)$ belongs to the class $S_{P}^{*}\left(\alpha,\beta,\gamma,A,B,\lambda,\delta\right)$, in view of Theorem 2.1, we obtain

$$\frac{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}{(p+\alpha+\beta-\gamma+1)(p+1)}\sum_{n=1}^{\infty}(n+p)a_{n+p} \le$$

$$\sum_{n=1}^{\infty}\phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right]a_{n+p} \le (A-B)(p-\delta)[1+\lambda(\text{p}-1)]$$

which is equivalent to

$$\sum_{n=1}^{\infty}(n+p)a_{n+p} \le \frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\text{p}-1)](p+1)}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]} \tag{3.4}$$

Using (1.4) and (3.4), we obtain

$$\begin{aligned}|f'(z)| &\le p|z|^{p-1}+|z|^{p}\sum_{n=1}^{\infty}(n+p)a_{n+p}\\ &\le p r^{p-1}+r^{p}\sum_{n=1}^{\infty}(n+p)a_{n+p}\\ &\le p r^{p-1}+\frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\text{p}-1)](p+1)}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}\, r^{p}.\end{aligned}$$

Similarly,

$$|f'(z)| \ge p r^{p-1}-\frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\text{p}-1)](p+1)}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}\, r^{p}.$$

This completes the proof.

IV. Closure Theorems

Theorem: 4.1. Let the functions

$$f_j(z)=z^{p}-\sum_{n=1}^{\infty}a_{n+p,j}\,z^{n+p} \qquad (a_{n+p,j}\ge 0) \tag{4.1}$$

be in the class $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ for every $j=1,2,3,\cdots,m$. Then the function $h(z)$ defined by

$$h(z)=\sum_{j=1}^{\infty}c_j f_j(z) \qquad (c_j\ge 0) \tag{4.2}$$

is also in the same class $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$, where

$$\sum_{j=1}^{m}c_j=1. \tag{4.3}$$

Proof: By means of the definition of $h(z)$, we can write

$$h(z)=z^{p}-\sum_{n=1}^{\infty}\left(\sum_{j=1}^{m}c_j a_{n+p,j}\right)z^{n+p}. \tag{4.4}$$

Now, since $f_j(z)\in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ for every $j=1,2,3,\cdots,m,$ we obtain

$$\sum_{n=1}^{\infty}\phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right]a_{n+p,j} \le (A-B)(p-\delta)(1+\lambda(\text{p}-1)), \tag{4.5}$$

for every $j=1,2,3,\cdots,m,$, by virtue of Theorem 2.1. Consequently, with the aid of (4.5) we can see that

$$\sum_{n=1}^{\infty}\phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right]\left(\sum_{j=1}^{m}c_j a_{n+p,j}\right)$$

$$=\sum_{j=1}^{m} c_j \left\{ \sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right] a_{n+p,j} \right\}$$

$$\leq \left(\sum_{j=1}^{m} c_j \right)(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1)) = (A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))$$

This proves that the function $h(z)$ belongs to the class $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$.

Theorem: 4.2. Let

$$f_p(z)=z^p \tag{4.6}$$

and

$$f_{n+p}(z)=z^p-\frac{(A-B)(p-\delta)\left[1+\lambda(p-1)\right]}{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)} z^{n+p} \tag{4.7}$$

for $-1 \leq B < A \leq 1,\ -1 \leq B < 0,\ 0 \leq \delta < p$ and $\phi(n,\alpha,\beta,\gamma)$ is defined by (2.2). Then $f(z)$ is in the class $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty} \zeta_{n+p} f_{n+p}(z) \qquad (\zeta_{n+p} \geq 0) \tag{4.8}$$

and

$$\sum_{n=0}^{\infty} \zeta_{n+p} = 1. \tag{4.9}$$

Proof: Assume that

$$f(z)=\sum_{n=0}^{\infty} \zeta_{n+p} f_{n+p}(z) = z^p-\sum_{n=1}^{\infty} \frac{(A-B)(p-\delta)\left[1+\lambda(p-1)\right]}{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)} \zeta_{n+p} z^{n+p} \tag{4.10}$$

Then we get

$$\sum_{n=1}^{\infty} \phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right]$$
$$\times \frac{(A-B)(p-\delta)\left[1+\lambda(p-1)\right]}{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)} \zeta_{n+p}$$
$$\leq (A-B)(p-\delta)\left[1+\lambda(p-1)\right].$$

By virtue of Theorem 2.1 this shows that $f(z)$ is in the class $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$.

Conversely, assume that $f(z)$ belongs to the class $S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$. Again, by virtue of Theorem 2.1, we have

$$a_{n+p} \leq \frac{(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))}{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)}.$$

Next, setting

$$\zeta_{n+p} \leq \frac{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))} a_{n+p}$$

and

$$\zeta_p = 1-\sum_{n=1}^{\infty} \zeta_{n+p},$$

we have the representation (4.8). This completes the proof of the theorem.

V. Inclusion And Neighborhood Results

In this section, we prove certain relationship for functions belonging to the class

$S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ and also, we determine the neighborhood properties of functions belonging to the subclass $S_P^*(\rho,\alpha,\beta,\gamma,A,B,\lambda,\delta)$.

Following the works of Goodman [13], Ruschweyh [14] and Altintas et. al. [15, 16], we define the $(n,\varepsilon)-$ neighborhood of a function $f\in T_p(n)$ by

$$N_{n,\varepsilon}(f)=\left\{g\in T_p(n):g(z)=z^p-\sum_{n=1}^{\infty}b_{n+p}z^{n+p}\ \ and\ \sum_{n=1}^{\infty}(n+p)\left|a_{n+p}-b_{n+p}\right|\le\varepsilon\right\}. \tag{5.1}$$

In particular, for the function $e(z)=z^p\ (p\in\square)$

$$N_{n,\varepsilon}(e)=\left\{g\in T_p(n):g(z)=z^p-\sum_{n=1}^{\infty}b_{n+p}z^{n+p}\ \ and\ \sum_{n=1}^{\infty}(n+p)\left|b_{n+p}\right|\le\varepsilon\right\}. \tag{5.2}$$

A function $f\in T_p(n)$ defined by (1.4) is said to be in the class $S_P^*(\rho,\alpha,\beta,\gamma,A,B,\lambda,\delta)$ if there exists a function $h\in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right|<p-\rho \qquad (z\in U,\ 0\le\rho<p). \tag{5.3}$$

Theorem: 5.1. Let

$$\varepsilon=\frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\text{p}-1)](p+1)}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}. \tag{5.4}$$

Then $\qquad S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)\subseteq N_{n,\varepsilon}(e).$ (5.5)

Proof: Let $f\in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$. Then in view of assertion (2.1) of Theorem 2.1, we have

$$\frac{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}{(p+\alpha+\beta-\gamma+1)}\sum_{n=1}^{\infty}a_{n+p}\le$$

$$\sum_{n=1}^{\infty}\phi(n,\alpha,\beta,\gamma)\left[n(1-B)+(A-B)(p-\delta)\right]a_{n+p}\le(A-B)(p-\delta)[1+\lambda(\text{p}-1)].$$

$$\Rightarrow\ \sum_{n=1}^{\infty}a_{n+p}\le\frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\text{p}-1)]}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]} \tag{5.6}$$

Applying assertion (2.1) of Theorem 2.1 in conjunction with (5.6), we obtain

$$\frac{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}{(p+\alpha+\beta-\gamma+1)}\sum_{n=1}^{\infty}a_{n+p}\le(A-B)(p-\delta)[1+\lambda(\text{p}-1)],$$

$$\frac{(p+1)(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}{(p+\alpha+\beta-\gamma+1)}\sum_{n=1}^{\infty}a_{n+p}\le(p+1)(A-B)(p-\delta)[1+\lambda(\text{p}-1)].$$

Hence,

$$\sum_{n=1}^{\infty}(n+p)a_{n+p}\le\frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\text{p}-1)](p+1)}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}=\varepsilon,$$

which by virtue of (5.2) establishes the inclusion relation (5.5).

Theorem: 5.2. Let

$$\rho = p - \frac{\varepsilon}{p+1} \times \tag{5.7}$$

$$\left[\frac{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]-(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\mathrm{p}-1)]}\right]$$

Then $\quad N_{n,\varepsilon}(h) \subseteq S_P^*(\rho,\alpha,\beta,\gamma,A,B,\lambda,\delta).$ (5.8)

Proof: Suppose that $f \in N_{n,\varepsilon}(h)$, we can find from (5.1) that

$$\sum_{n=1}^{\infty}(n+p)\left|a_{n+p}-b_{n+p}\right| \le \varepsilon$$

which readily implies the following coefficient inequality,

$$\sum_{n=1}^{\infty}\left|a_{n+p}-b_{n+p}\right| \le \frac{\varepsilon}{p+1}. \qquad (n \in \mathbb{N}) \tag{5.9}$$

Next, since $f \in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$ in the view of (5.6), we have

$$\sum_{n=1}^{\infty} b_{n+p} \le \frac{(p+\alpha+\beta-\gamma+1)(A-B)(p-\delta)[1+\lambda(\mathrm{p}-1)]}{(\beta+p)\left[(1-B)+(A-B)(p-\delta)\right][1+\lambda p]}. \tag{5.10}$$

Using (5.9), (5.10) together with (5.3), we get the required assertion.

VI. Radii Of Close-To-Convexity, Starlikeness And Convexity

Theorem: 6.1. Let $f \in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$. Then f is p-valently close-to-convex of order $\eta\ (0 \le \eta < p)$ in $|z| < R_1$, where

$$R_1 = \inf_n \left\{\left[\frac{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))}\left(\frac{p-\eta}{n+p}\right)\right]^{\frac{1}{n}}\right\} \tag{6.1}$$

and $\phi(n,\alpha,\beta,\gamma)$ is defined by (2.2).

Theorem: 6.2. Let $f \in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$. Then f is p-valently starlike of order $\eta\ (0 \le \eta < p)$ in $|z| < R_2$, where

$$R_2 = \inf_n \left\{\left[\frac{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))}\left(\frac{p-\eta}{n+p-\eta}\right)\right]^{\frac{1}{n}}\right\} \tag{6.2}$$

and $\phi(n,\alpha,\beta,\gamma)$ is defined by (2.2).

Theorem: 6.3. Let $f \in S_P^*(\alpha,\beta,\gamma,A,B,\lambda,\delta)$. Then f is p-valently convex of order $\eta\ (0 \le \eta < p)$ in $|z| < R_3$, where

$$R_3 = \inf_n \left\{\left[\frac{\left[n(1-B)+(A-B)(p-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{(A-B)(p-\delta)(1+\lambda(\mathrm{p}-1))}\left(\frac{p(p-\eta)}{(n+p)(n+p-\eta)}\right)\right]^{\frac{1}{n}}\right\}. \tag{6.3}$$

In order to establish the required results in Theorems 6.1, 6.2 and 6.3, it is sufficient to show that

$$\left|\frac{f'(\mathrm{z})}{z^{p-1}} - p\right| \le p-\eta \qquad \text{for} \qquad |z| < R_1,$$

$$\left|\frac{zf'(\text{z})}{f(\text{z})} - p\right| \le p - \eta \qquad \text{for} \qquad |z| < R_2 \text{ and}$$

$$\left|\left[1 + \frac{zf''(\text{z})}{f'(\text{z})}\right] - p\right| \le p - \eta \qquad \text{for} \qquad |z| < R_3,$$

respectively.

Remark 6.1: The results in Theorems 6.1, 6.2 and 6.3 are sharp with the extremal function f given by (2.3). Furthermore, taking $\eta = 0$ in Theorems 6.1, 6.2 and 6.3, we obtain radius of close-to-convexity, starlikeness and convexity respectively.

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