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Fixed Point Result Satisfying Φ - Maps in G-Metric Spaces

Madhu Shrivastava*, Dr. K.Qureshi*, Dr. A.D.Singh**
*NRI Institute of research science & technology, Bhopal, (M.P.) India **
*Additional Director, Higher Education Department, Govt. of M.P., Bhopal, (M.P.) India**
*Govt. MVM College Bhopal, , (M.P.) India ***

Abstract:- *In this paper , we elaborate some existing result of fixed point theorem, that fulfill the nature of G-metric space and satisfy the Φ-maps. Previously Erdal Karapinar and Ravi Agrawal [24]have modified some existing result of fixed point theory of Samet et al Int.J.Anal(2013:917158,2013) [44]and Jleli-Samet (Fixed point theory application.2012:2010,2012) [45]in a different way.*

I. Introduction

The concept of G-metric spaces was introduced by Mustafa and Sims [25].G-metric spaces is generalization of a metric spaces (X ,d). In this paper they characterized the Banach contraction mapping principal [10] in the context of G-metric spaces .Subsequently many fixed point result on such spaces appeared . Since one is adapted from other.The G-metric spaces is to understand the geometry of three points instead of two , Many result are obtained by contraction condition.

In 2013 , Samet et al [38] and Jleli Samet [39] observed that some fixed point theorems in the context of a G-metric space.in literature can be concluded by some existing results in the setting of (quashi)metric spaces. Also the contraction condition of the fixed point theorem on a G-metric space can be reduced to two variables instead of three. In [20,38 ,39] the authors find d (x,y) = G (x ,y ,y) form a quasi-metric .Erdal Karapinar and Ravi Agrawal modified some existing result to suggest new fixed point theorem , in this way they approach (Samet et al and Jeleli Samet) in a different technique.

2.Definition 2.1 (See [1] Let X be a non-empty set and let$G : X\times X\times X\rightarrow R+$ be a function
Satisfying the following properties:
(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y\neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then the function G is called a generalized metric or, more specifically, a G-metric on X,
and the pair (X,G) is called a G-metric space.
Every G-metric on X defines a metric d_G on X by
$d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all $x, y \in X$.

Example 1 Let (X, d) be a metric space. The function $G: X\times X\times X\rightarrow [0, +\infty)$, defined as
$G(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$
Or
$G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, for all $x, y, z \in X$, is a G-metric on X.

Definition 2.2 Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of
points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if
$\lim_{n,m\rightarrow+\infty} G(x, x_n, x_m) = 0$,
That is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call
x the limit of the sequence and write $x_n\rightarrow x$ or $\lim_{n\rightarrow+\infty} x_n = x$.

Proposition 2.1 Let (X,G) be a G-metric space. The following are equivalent:
(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n, x_n, x)\rightarrow 0$ as $n\rightarrow+\infty$,
(3) $G(x_n, x, x)\rightarrow 0$ as $n\rightarrow+\infty$,
(4) $G(x_n, x_m, x)\rightarrow 0$ as $n,m\rightarrow+\infty$.

Definition 2.3 Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence

if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$,
that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.2 Let (X,G) be a G-metric space. Then the following are equivalent:
(1) the sequence $\{x_n\}$ is G-Cauchy,
(2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$.

Definition 2.4 A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Lemma 2.1 Let (X,G) be a G-metric space. Then
$G(x, x, y) \leq 2G(x, y, y)$ for all $x, y \in X$.

Definition2.5 Let (X,G) be a G-metric space. A mapping $T: X \to X$ is said to beG-continuous if $\{T(x_n)\}$ is G-convergent to $T(x)$ where $\{x_n\}$ is any G-convergent sequence Converging to x.
In [22], Mustafa characterized the well-known Banach contraction mapping principle in the context of G-metric spaces in the following ways.

Theorem 2.1 Let (X,G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:
$G(Tx, Ty, Tz) \leq k\, G(x, y, z)$,
Where $k \in [0,1)$. Then T has a unique fixed point.

Theorem 2.2 Let (X,G) be a complete G-metric space and let $T : X \to X$ be amapping satisfying the following condition for all $x, y \in X$:
$G(Tx, Ty, Ty) \leq k\, G(x, y, y)$,
where $k \in [0,1)$. Then T has a unique fixed point.

Theorem2.3 Let (X,G) be a G-metric space. Let $T : X \to X$ be a mapping suchthat
$G(Tx, Ty, Tz) \leq a\, G(x, y, z) + b\, G(x,Tx,Tx) + c\, G(y,Ty,Ty) + d\, G(z,Tz,Tz)$
for all x, y, z, where a, b, c, d are positive constants such that $k = a+b+c+d < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.4 Let (X,G) be a G-metric space. Let $T : X \to X$ be a mapping such that
$G(Tx, Ty, Tz) \leq k\,[\, G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]$
for all x, y, z, where $k \in [0,\frac{1}{3})$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.5 Let (X,G) be a G-metric space. Let $T : X \to X$ be a mapping suchthat
$G(Tx, Ty, Tz) \leq aG(x, y, z) + b[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]$
for all x, y, z, where a, b are positive constants such that $k = a+b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.6 Let (X,G) be a G-metric space. Let $T : X \to X$ be a mapping suchthat
$G(Tx, Ty, Tz) \leq a\, G(x, y, z) + b \max\{G(x, Tx, Tx),G(y, Ty, Ty),G(z, Tz, Tz)\}$
for all x, y, z, where a, b are positive constants such that $k = a+b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem2.7 Let (X,G) be a G-metric space. Let $T : X \to X$ be a mapping suchthat
$$G(Tx, Ty, Tz) \leq k \max\{G(x, y, z),G(x, Tx, Tx),G(y, Ty, Ty),G(z, Tz, Tz), G(z, Tx, Tx),G(x, Ty, Ty),G(y, Tz, Tz)\}$$
for all x, y, z, where $k \in [0,\frac{1}{2})$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.8 Let (X,G) be a complete G-metric space and let $T : X \to X$ be a given mapping satisfying
$G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z))$
for all $x, y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is continuous with $\phi^{-1}(\{0\}) = 0$. Then there is a unique $x \in X$ such that $Tx = x$

Definition 2.6 A quasi-metric on a nonempty set X is a mapping p : X ×X→[0,∞) such that
(p1) x = y if and only if p(x, y) = 0,
(p2) p(x, y) ≤ p(x, z) + p(z, y),
for all x, y, z ∈ X. A pair (X, p) is said to be a quasi-metric space.

Samet et al. and Jleli-Samet noticed that p(x, y) = pG(x, y) = G(x, y, y) is a quasimetric whenever G : X × X × X→[0,∞) is a G-metric. It is well known that each quasimetric induces a metric. Indeed, if (X, p) is a quasi-metric space, then the function defined by d(x, y) = dG(x, y) = max{p(x, y), p(y, x)} for all x, y ∈ X is a metric on X.

Theorem 2.9 Let (X ,d) be a complete metric space and let T : X →X be a mapping with the property
d(Tx, Ty) ≤ q max {d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)}
for all x ∈ X, where q is a constant such that q ∈ [0,1). Then T has a unique fixed point.
.

Proposition 2.3
(A) If (X,G) is a complete G-metric space, then (X, d) is a complete metric space.
(B) If (X,G) is a sequentially G-compact G-metric space, then (X, d) is a compact metric space.

II. Main Result

Theorem-3.1-Let (X,G) be a complete G-metric space and let f : X → X be a given mapping satisfy for all x,y ϵ X, where ϕ : [0,∞) → [0,∞) is continuous with $\phi^{-1}(\{0\}) = 0$, then there is a unique x ϵ X s.t. fx = x.

$$G(fx, f^2y, f^2z) \le G(x, fy, fz) - \phi(G(x, fy, fz)) \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Proof:- We first show that if the fixed point of the operator f exist , then it is unique , Suppose ,on contrary ,that x and y are two fixed point of f, such that x≠ y , hence G(x ,x ,y) ≠ 0
From equation (1) , we get

$$G(fx, f^2y, f^2y) \le G(x, fy, fy) - \phi(G(x, fy, fy))$$

Which is equivalent to

$$G(x, y, y) \le G(x, y, y) - \emptyset(G(x, y, y))$$

A contradiction hence f has a unique fixed point.
Let $x_0 \epsilon X$, we define a sequence $\{x_n\}$ by $x_n = fx_{n-1}, n\epsilon N$.
If $x_n = x_{n+1}$, for some n, then trivially f has a fixed point.
Taking $x_n = x_{n+1}$, y = z = x_n
Now from equation (1), we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(fx_{n-1}, f^2x_{n-1}, f^2x_{n-1}) \\ &= G(fx_{n-1}, fx_n fx_n) \\ &\le G(x_{n-1}, fx_{n-1}, fx_{n-1}) - \emptyset(G(x_{n-1}, fx_{n-1}, fx_{n-1})) \\ &= G(x_{n-1}, x_n, x_n) - \emptyset(G(x_{n-1}, x_n, x_n)) \\ &< G(x_{n-1}, x_n, x_n) \qquad \ldots\ldots\ldots\ldots\ldots (2) \end{aligned}$$

This shows that $G(x_n, x_{n+1}, x_{n+1})$ is monotone positive decreasing sequence , thus the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ converges to $s \ge 0$.
We shall show that $s = 0$.
Suppose, on contrary that $s > 0$.
Letting $n \to \infty$,in equation (2)
We get $s \le s - \emptyset(s)$

It is a contradiction, Hence conclude that $lim_{n\to\infty} G\{(x_n, x_{n+1}, x_{n+1})\} = 0$
By lemma [2.1], we know that $lim_{n\to\infty} G\{(x_n, x_n, x_{n+1})\} = 0$
Hence

$$lim_{n\to\infty} G\{(x_n, x_{n+1}, x_{n+1})\} \to 0 , n \to \infty \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Now next we show that the $\{x_n\}$ is G-cauchy,on contrary let $\{x_n\}$is not G-cauchy sequence then so there exist $\epsilon > 0$ and subsequence$\{x_{n_k}\}$ $of \{x_n\}$ $with\ n(k) > m(k) > k.$

Such that $G(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon$,$for\ all\ k\epsilon N$ ………………………………….. (4)

More over, corresponding to m_k , we can choose n_k, such that it is the smallest integer with $n_k > m_k$ Satisfying equation (4).

Then that $G(x_{n_k}, x_{m_{k-1}}, x_{m_{k-1}}) < \epsilon$ ……………………………………….(5)

Then we have ,

$$\epsilon \leq G(x_{n_k}, x_{m_k}, x_{m_k})$$
$$\leq G(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}) + G(x_{n_{k-1}}, x_{m_k}, x_{m_k})$$
$$< \epsilon + G(x_{n_{k-1}}, x_{m_k}, x_{m_k})$$

Setting $k \to \infty$ and using equation (3),lim $k \to \infty$ $G(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon$

Now $G(x_{n_k}, x_{m_k}, x_{m_k}) \leq G(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}) + G(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}) + G(x_{m_{k-1}}, x_{m_k}, x_{m_k})$

And

$G(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}) \leq G(x_{n_{k-1}}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_{k-1}}, x_{m_{k-1}})$

Setting $k \to \infty$in above inequality and using (3) and (5)

$lim\ k \to \infty G(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}) = \epsilon$

Now again from equation (1) and (4), we have

$$\epsilon \leq G(x_{n_k}, x_{m_k}, x_{m_k})$$
$$\leq G(fx_{n_{k-1}}, f^2 x_{m_{k-2}}, f^2 x_{m_{k-2}})$$
$$\leq G(x_{n_{k-1}}, fx_{m_{k-2}}, fx_{m_{k-2}}) - \emptyset(G(x_{n_{k-1}}, fx_{m_{k-2}}, fx_{m_{k-2}})$$
$$\leq G(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}) - \emptyset(G(x_{n_{k-1}}, x_{m_{k-1}}, x_{m_{k-1}}))$$

Letting $\to \infty$, we have $\epsilon \leq \epsilon - \emptyset(\epsilon)$, Which is a contradiction, if $\epsilon > 0.$

So , we must have $\epsilon = 0.$ This shows that $\{x_n\}$ is G-cauchy sequence in X. Since X is complete G-metric space .

So there exists zϵ $X,$ such that lim n $\to \infty$ $x_n \to z.$

Now we claim that fz = z.

Consider $G(fz, x_{n+2} x_{n+2}) = G(fz, f^2 x_n, f^2 x_n)$

$$\leq G(z, fx_n, fx_n) - \emptyset(G(z, fx_n, fx_n))$$
$$= G(z, x_{n+1}, x_{n+1}) - \emptyset(G(z, x_{n+1}, x_{n+1}))$$

Let n $\to \infty$, we get $G(fz, z, z) \leq G(z, z, z) - \emptyset(G(z, z, z))$

= 0

Hence G(fz,z,z) = 0, i.e,fz = z.

Hence z is a fixed point.

Theorem 3.2:- Let (X, G) be a G-metric space .Let $f: X \to X$ Be a mapping such that

$$G(fx, fy, fz) \leq kM(x, y, z) \quad for\ all\ x, y, z\ \epsilon X\ and\ k\ \epsilon\ [0,1) and$$

$$M(x, y, z) = max\ \{G(x, y, z), G(f^2 x, fy, fz), G(z, fx, fy), G(y, f^2 x, fy), G(x, fx, fx)$$
$$G(y, fy, fy), G(z, fz, fz), G(fx, f^2 x, fz), G(z, f^2 x, fz), G(fx, f^2 x, fy)\}$$

Then there is a unique $x\epsilon X\ such\ that\ fx = x.$

$\boldsymbol{Proof}$: $Let\ x_0 \epsilon X, We\ define\ \{x_n\}\ in\ the\ following\ fx_n = x_{n+1}, n\epsilon N$

$Taking\ x = x_n, y = z = x_{n+1},\ we\ get$

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq k\ M(x_n, x_{n+1}, x_{n+1})$$

$Where$

$$M(x_n, x_{n+1}, x_{n+1}) = max\{G(x_n, x_{n+1}, x_{n+1}), G(f^2 x_n, fx_{n+1}, fx_{n+1}), G(x_{n+1}, fx_n, fx_{n+1}),$$

$$G(fx_{n+1}, f^2x_n, fx_{n+1})$$
$G(x_n, fx_n, fx_n), G(x_{n+1}, fx_{n+1}, fx_{n+1}), G(x_{n+1}, fx_{n+1}, fx_{n+1}),$
$$, G(fx_n, f^2x_n, fx_{n+1}), G(x_{n+1}, f^2x_n, fx_{n+1}), G(fx_n, f^2x_n, fx_{n+1})\}$$
$$= max\ \{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+2}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+1}, x_{n+2})$$
$$G(x_{n+2}, x_{n+2}, x_{n+2}), G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+2}, x_{n+2}),$$
$$G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+2}, x_{n+2})\}$$
$= max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_{n+1}, x_{n+1}, x_{n+2})\}$

Case (i)-First let $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+1}, x_{n+2})$

By G_5, we get from above

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(fx_n, fx_{n+1}, fx_{n+1}) \\ &\le kM(x_n, x_{n+1}, x_{n+1}) \\ &= kG(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\le k[G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+1}, x_{n+2})\} \end{aligned}$$

Which is a contradiction, since $0 \le k < 1$.

Case-(ii)- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+2}, x_{n+2})$

Then we get $G(x_{n+1}, x_{n+2}, x_{n+2}) = G(fx_n, fx_{n+1}, fx_{n+1}),$

$$\le kM(x_n, x_{n+1}, x_{n+1})$$
$$= kG(x_{n+1}, x_{n+2}, x_{n+2})$$

This is a contradiction, since $0 \le k < 1$.

Case(iii)- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1})$

Then we get, $G(x_{n+2}, x_{n+2}, x_{n+1}) \le kG(x_{n+1}, x_{n+1}, x_n)$

Continuing in this way , we get

$$G(x_{n+2}, x_{n+2}, x_{n+1}) \le k^{n+1}G(x_1, x_1, x_0)$$

Again,

$$\begin{aligned} G(x_m, x_m x_n) &\le G(x_{n+1}, x_{n+1}, x_n) + G(x_{n+2}, x_{n+2}, x_{n+1}) + \cdots \ldots \ldots G(x_{m-1}, x_{m-1}, x_{m-2}) + G(x_m, x_m, x_{m-1}) \\ &\le k^n G(x_1, x_1, x_0) + k^{n+1}G(x_1, x_1, x_0) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + k^{m-1}G(x_1, x_1, x_0) \end{aligned}$$

Let $n, m \to \infty\ we\ get,\ G(x_m, x_m x_n) \to 0.$

Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X,G) is G-complete , then there exist $z\epsilon X$ s.t. $\{x_n\}$ is G-converges to z.Let on contrary that $z \neq fz$.for this let $x_{n+1} = fx_n$

$$\begin{aligned} G(x_{n+1}, fz, fz) &= G(fx_n, fz, fz) \\ &\le kM(x_n, z, z) \end{aligned}$$

Where

$$\begin{aligned} M(x_n, z, z) &= max\ \{(x_n, z, z), G(fz, f^2x_n, fz), G(z, fx_n, fz), G(z, f^2x_n, fz), G(x_n, fx_n, fx_n), \\ &\quad (z, fz, fz), G(x_n, fz, fz), G(fx_n, f^2x_n, fz), (z, f^2x_n, fz), (fx_n, f^2x_n, fz)\} \\ &= max\ \{(x_n, z, z), G(fz, x_{n+2}, fz), G(z, x_{n+1}, fz), G(z, x_{n+2}, fz), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad (z, fz, fz), G(x_n, fz, fz), G(fx_n, x_{n+2}, fz), (z, x_{n+2}, fz), (x_{n+1}, x_{n+2}, fz)\} \end{aligned}$$

Letting $n\to\infty$, since G is continuous, we get

$G(z, fz, fz) \le kG(z, fz, fz)$

Or

$$\begin{aligned} G(z, fz, fz) &\le kG(z, z, fz) \\ &\le k[G(z, fz, fz) + G(fz, z, fz)] \\ &= k[2G(z, fz, fz) \end{aligned}$$

$so\ G(z, fz, fz) \le 2kG(z, fz, fz)$

This is a contradiction.

Since $0 \le k < 1$.So $fz = z$.

Uniqueness:-Next we show that uniqueness of z of f.Suppose on contrary, there exist another common fixed point $u\epsilon X\ with\ z \neq u$.

We get $G(z, z, u) = G(fz, fz, fu)$

$$\le kM(z, z, u)$$

We get a contradiction, since $0 \leq k < 1$. *Thus* $z = u$ *is a fixed point.*

Example:-Let $X = [0, \infty)$, $G: XxXxX \to R$ *be defined by*

$$G(x, y, z) = \begin{cases} 0, & if\ x = y = z \\ max\{x, y, z\}, & othrwise \end{cases}$$

Then (X,G) is a complete G-metric space

Let $f: X \to X$ be defined by

$$\begin{cases} \frac{1}{3}x, & if\ 0 \leq x < \frac{1}{2} \\ \frac{1}{9}x^3, & if\ \frac{1}{2} < x \leq 1 \end{cases}$$

And $\emptyset(t) = \frac{2}{3}\ t$, *for all* $t \epsilon [0, \infty)$

Solution:- First we examine the following cases:

Let $0 \leq x, y < \frac{1}{2}$, then

$$G(fx, f^2y, f^2y) = max\left\{\frac{1}{3}x, \frac{1}{9}y, \frac{1}{9}y\right\}$$
$$\leq \frac{1}{3} max\left\{x, \frac{1}{3}y, \frac{1}{3}y\right\}$$

Let $\frac{1}{2} \leq x, y < 1$, then

$$G(fx, f^2y, f^2y) = max\left\{\frac{1}{9}x^3, \frac{1}{81}y^9 \frac{1}{81}y^9\right\}$$
$$\leq \frac{1}{3} max\{x, \frac{1}{9}y^3, \frac{1}{9}y^3\}$$

Let $0 \leq x < \frac{1}{2} \leq y < 1$, then

$$G(fx, f^2y, f^2y) = \max\left\{\frac{1}{3}x, \frac{1}{81}y^9 \frac{1}{81}y^9\right\}$$
$$\leq \frac{1}{3} \max\{x, \frac{1}{9}y^3, \frac{1}{9}y^3\}$$

Let $0 \leq y < \frac{1}{2} \leq x < 1$, then

$$G(fx, f^2y, f^2y) = \max\left\{\frac{1}{9}x^3, \frac{1}{9}y, \frac{1}{9}y\right\}$$
$$\leq \frac{1}{3} \max\{x, \frac{1}{3}y, \frac{1}{3}y\}$$

Hence f has a unique fixed point.

Here (0,0,0) is a fixed point.

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