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# Fuzzy Subspace, Balanced Fuzzy Set and Absorbing Fuzzy Set in Fuzzy Vector Space

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***Abstract:*** *In this paper, we have studied fuzzy subspace, balanced fuzzy set, and absorbing fuzzy set over a fuzzy vector space. We examine the properties of the balance fuzzy set, and absorbing fuzzy set, and established some independent results, under the linear mapping from one fuzzy vector space to another fuzzy vector space.*
***Keywords:****Fuzzy vector space, Fuzzy balanced set, Fuzzy absorbing set.*

## I. Introduction:

The concept of fuzzy set was introduced by Zadeh [6],and the notion of fuzzy vector space was defined and established by KATSARAS,A.K. and LIU,D.B [2].Using the definition of fuzzy vector space, balanced fuzzy set and absorbing fuzzy set, over the fuzzy vector space. We established the result that under the linear mapping, these balanced fuzzy set, and absorbing fuzzy set, remains the same.

### 1. Fuzzy Vector Space

**Definition 1.1** : Let $X$ be a vector space over $K$ ,where $K$ is the space of real or complex numbers, then the vector space equipped with addition (+) and scalar multiplication defined over the fuzzy set (on $X$ ) as below is called a fuzzy vector space.

Addition (+) : Let $A_1,.........,A_n$ be the fuzzy sets on vector space $X$ , let $f : X^n \to X$ , such that $f(x_1,.......,\mathrm{x}_n) = x_1 + ........ + x_n$ ,we define

$A_1 + ......... + A_n = f\left(A_1,.......\mathrm{A}_n\right)$, by the extension principle

$$\mu_{f(A_1,......,A_n)}(y) = \sup_{\substack{x_1,...x_n \\ y=f(x_1,...x_n)}} \left\{\mu_{A_1}(x_1),.......,\mu_{A_n}(x_n)\right\}$$

Obviously, when sets $A_1,.......,A_n$ are ordinary sets ,the gradation function used in the sum are taken as characteristic function of the set.

Scalar multiplication (.) : If $\alpha$ is a scalars and $B$ be a fuzzy set on $X$ and $g : X \to X$ ,such that $g(x) = \alpha x$ ,then using extension principle we define $\alpha B$ as $\alpha B = g(B)$ ,where

$\mu_{g(B)}(y) = \sup\limits_{\substack{y=g(x) \\ y=\alpha x}} \{\mu_B(x)\}$, if $y = \alpha x$ holds

$\mu_{g(B)}(y) = 0$ , if $y \neq \alpha x$ ,for any $x \in X$

i.e $\mu_{\alpha B}(y) = \sup\limits_{\substack{x \\ y=\alpha x}} \mu_B(x)$, if $y \in X$

$\mu_{\alpha B}(y) = 0$, if $y \neq \alpha x$ ,for any $x$

**THEOREM 1.1** :*If E and F are vector spaces over K,f is a linear mapping from E to F and A,B are fuzzy sets on E,then*

$f(A+B) = f(A) + f(B)$

$f(\alpha A) = \alpha f(A)$, for all scalars α

i.e $f(\alpha A + \beta B) = \alpha f(A) + \beta f(B)$, where α, β, are scalars

Proof: Proof is straight forward.

**Definition 1.2**: *If A is a fuzzy set in a vector space E and x* $\in$ *X, we define x +A as x + A = {x} +A.*

**THEOREM 1**.2 :*If $f_x$: E → E (vector space) such that $f_x(y) = x + y$, then if B is a fuzzy set in E and A is an ordinary subset of E, the following holds*

$$x+B=f_x(B)$$

$$\mu_{x+B}(z)=\mu_B(z-x)$$

$$A+B=\bigcup_{x\in A}(x+B)$$

Proof : Proof is straight forward.

**THEOREM 1.3**: *If $A_1,\dots,A_n$, are fuzzy sets in vector space E and $\alpha_1,\dots,\alpha_n$ are scalars* $\alpha_1A_1+\dots\dots\alpha_nA_n\subset A$ *iff for $x_1,\dots,x_n$ in E, we have*

$$\mu_A(\alpha_1x_1+\dots\dots+\alpha_nx_n)\ge\min\{\mu_{A_1}(x_1),\dots\dots,\mu_{A_n}(x_n)\}$$

Proof : Proof is obvious.

## 2. Fuzzy Subspace

*A fuzzy set F in a vector space E is called fuzzy subspace of E if (i) F + F ⊂ F (ii)α F ⊂ F, for every scalars α.*

**THEOREM 2.1:***If F is a fuzzy set in a vector space E, then the followings are equivalent*

***(i)*** *F is a subspace of E*

***(ii)*** *For all scalars k,m, kF + mF ⊂ F*

***(iii)*** *For all scalars k,m , and all x, y ∈ E*

$$\mu_F(kx+my)\ge\min\{\mu_F(x),\mu_F(y)\}$$

**Proof** : It is obvious

**THEOREM 2.2 :***If E and F are vector spaces over the same field and f is a linear mapping from E to F and A is subspace of E. Then f(A) is a subspace of F and if B is a subspace of F. Then $f^{-1}(B)$ is a subspace of E.*

**Proof :** Let k, m, be scalars and f is a linear mapping from E to F, then for any fuzzy set A in E

$$kf(A)+mf(A)=f(kA)+f(mA)=f(kA+mA)\subset f(A)$$

As kA + mA⊂ A, since A is a subspace.

∴ f(A) is a subspace of F

Also , $\mu_{f^{-1}(B)}(kx+my)=\mu_B(f(kx+my))$

$\mu_{f^{-1}(B)}(kx+my)=\mu_B(kf(x)+mf(y))$ , since f is a linear mapping

$\mu_{f^{-1}(B)}(kx+my)\ge\min\{\mu_B(f(x)),\mu_B(f(y))\}$ , as B is a subspace.

$$\mu_{f^{-1}(B)}(kx+my)\ge\min\{\mu_{f^{-1}(B)}(x),\mu_{f^{-1}(B)}(y)\}$$

i.e $f^{-1}(B)$ , is a subspace of F

**THEOREM 2.3***: If A, B, are fuzzy subspace of E and K is a scalars. Then A + B and K A are fuzzy subspaces.*

**Proof:** Proof is obvious.

## 3. Balanced Fuzzy Set

*A fuzzy set A in a vector space E is said to be balanced if α A ⊂ A, for all scalars α with I x I ≤ 1.*

**THEOREM 3.1:** Let *A be a fuzzy set in a vector space E. Then the following assertions are equivalent.*

**(i)** A is balanced

**(ii)** $\mu_A(\alpha x)\ge\mu_A(x)$, for all scalars α with I α I ≤ 1

**(iii)** For each α∈ [0,1], the ordinary set $A_\alpha$ given by

$A_\alpha=\{x\in E:\mu_A(x)\ge\alpha\}$, is balanced

**Proof :**(i) ⟹ (ii)

Suppose A is balanced i.e α A ⊂ A, for all scalars α with I α I ≤ 1.

i.e $\mu_A(x) \geq \mu_{\alpha A}(x)$, for all scalars α with I α I ≤ 1, taking α x for x

$\therefore \mu_A(\alpha x) \geq \mu_{\alpha A}(\alpha x) = \mu_A(x)$, where α≠0 …………….. (i)

$i.e \mu_A(\alpha x) \geq \mu_A(x)$, for all scalars α, with I α I≤ 1 and x∈ E

If α = 0,from (i)

$\mu_A(\alpha x) \geq \mu_{\alpha A}(\alpha x) = \mu_{0A}(0x) = \sup_{y \in E} \mu_A(y)$

$\therefore \mu_A(\alpha x) \geq \mu_A(x)$, where α = 0

Suppose, (ii) ⟹ (iii)

$i.e \mu_A(\alpha x) \geq \mu_A(x)$, for all α with I α I ≤ 1 and x ∈ E

Let $A_\alpha = \{x \in E : \mu_A(x) \geq \alpha\}, \alpha \in [0,1]$

Now, $tA_\alpha = \{tx : x \in A_\alpha\}$, with I t I ≤ 1, let $x \in A_\alpha$

Since $\mu_A(\alpha x) \geq \mu_A(x) \geq \alpha$, with I α I ≤ 1

$tx \in A_\alpha$*, when I t I ≤ 1*

$\therefore tA_\alpha \subset A_\alpha$, with I t I ≤ 1

$\Rightarrow A_\alpha$, is balanced

(iii) ⟹ (i) Let x∈E, and let $\mu_A\left(x/k\right) = \alpha$ ,where I k I≤ 1

$\therefore \frac{x}{k} \in A_\alpha$, where $A_\alpha = \{y : \mu_A(y) \geq \alpha\}$

Now $kA_\alpha = \{kx : x \in A_\alpha\}$ ,Since $\frac{x}{k} \in A_\alpha$ , $k.\frac{x}{k} \in A_\alpha$ i.e $x \in A_\alpha \because kA_\alpha \subset A_\alpha$, as $A_\alpha$ is balanced i.e

$\mu_{kA}(x) \geq \mu_A\left(x/k\right) = \alpha$

$\therefore \mu_{kA}(x) \leq \mu_A(x)$, for all scalars k with I k I≤ 1, and x ∈ E

$\therefore kA \subset A$, A is balanced.

**THEOREM 3.2:** *Let E, F be vector spaces over k and let f: E → F be a linear mapping. If A is balanced fuzzy set in E. Then f(A) is balanced fuzzy set in F. Similarly $f^{-1}(B)$ is balanced fuzzy set in Whenever B is balanced fuzzy set in F.*

Proof : Let E, F be vector spaces over k and f : E → F be a linear mapping. Suppose A is balanced fuzzy set in E.

Now α.f(A) = f (αA) ⊂ f(A), for all scalars α with I α I ≤ 1

i.e α. f(A) ⊂ f(A),hence f(A) is balanced [∵αA ⊂ A ]

Again suppose B is a balanced fuzzy set in F

∴α B ⊂ B, for all scalars α with I α I ≤ 1

Now, let M = α $f^{-1}$(B), therefore, f(M) = f (α $f^{-1}$(B)) = α f ($f^{-1}$(B)) ⊂α B ⊂ B

∴ M ⊂ $f^{-1}$(B), hence α $f^{-1}$(B) ⊂ $f^{-1}$(B), therefore $f^{-1}$(B), is balanced fuzzy set in E.

**THEOREM 3.3:** *If A, B are balanced fuzzy sets in a vector space E over K. Then A + B is balanced fuzzy set in E.*

**Proof** : Let A,B are balanced fuzzy sets in E. Therefore α A ⊂ A, and α B ⊂ B, for all scalars α with I α I≤ 1, Now α (A + B) = α A + α B ⊂ A + B, hence A + B is balanced fuzzy set in E.

**THEOREM 3.4** :*If $\{A_i\}_{i\in I}$ ,is a family of balanced fuzzy sets in vector spaces E. Then A = ∩ $A_i$ , is balanced fuzzy set in E*

Proof : Since $\{A_i\}_{i\in I}$, is a family of balanced fuzzy sets in E

$\alpha A_i \subset A_i$ , for all scalars α with I α I≤ 1

that is, $\mu_{A_i}(\alpha x) \geq \mu_{A_i}(x)$, for all scalars α with I α I≤ 1
Now let, $A = \cap A_i$

$\mu_A(y) = \inf\limits_{i \in I} \mu_{A_i}(y)$, for all y ∈ E

$\therefore \mu_A(\alpha x) = \inf\limits_{i \in I} \mu_{A_i}(\alpha x)$, take y = α x

$\mu_A(\alpha x) \geq \inf\limits_{i \in I} \mu_{A_i}(x) = \mu_A(x)$, for all scalars α with I α I≤ 1, and x∈ E

$\therefore A = \bigcap\limits_{i \in I} A_i$, is balanced fuzzy set in E

## 4. Absorbing Fuzzy Set

**Definition** :A fuzzy set A in a vector space E is said to be absorbing if $E = \bigcup\limits_{k>0} kA$.
**THEOREM 4.1 :** For a fuzzy subset of A of a vector space E, the following are equivalent.

**(i)** A is absorbing

**(ii)** For each x ∈ E, $\sup\limits_{k>0} \mu_A(kx) = 1$

**(iii)** For each α with 0≤ α ≤ 1,the ordinary set $A_\alpha = \{x \in E : \mu_A(x) \geq \alpha\}$, is absorbing

Proof : (i) ⟹(ii)

Let $E = \bigcup\limits_{k^{'}>0} k^{'}A$

$\therefore 1 = \mu_E(x) = \mu_{\bigcup\limits_{K'>0} K'A}(x)$, for all x ∈ E

$\therefore 1 = \mu_E(x) = \sup\limits_{k'>0} \mu_A\left(x/k'\right)$, Now put $\frac{1}{k'} = k$

$= \sup\limits_{k>0} \mu_A(kx) = 1$

(ii)⟹ (iii),
Suppose for each x ϵ E

$\sup\limits_{t>0} \mu_A(tx) = 1$,let $A_\alpha = \{x \in E : \mu_A(x) \geq \alpha\}$, 0 ≤ α ≤ 1

$\therefore tA_\alpha = \{ty : \mu_A(y) \geq \alpha\}$

Now , $\sup\limits_{t>0} \mu_A(tx) = 1$. Hence for every x there is $\frac{1}{t} > 0$,such that

$\mu_A\left(\frac{1}{t}x\right) \geq \alpha$, where 0 ≤ α ≤ 1 , Let x = t y, then $\mu_A(y) \geq \alpha$

Hence $x \in tA_\alpha$, for some t > 0,
∴$A_\alpha$ is absorbing
(iii) ⟹ (i)

Suppose for 0 ≤α ≤ 1, $A_\alpha = \{x \in E : \mu_A(x) \geq \alpha\}$, is absorbing
Then for each x ∈ E, x ∈ t $A_\alpha$ , for some , t > 0
Then x = t y, where , $\mu_A(y) \geq \alpha$

$\mu_{\bigcup\limits_{t>0} tA}(x) = \sup\limits_{t>0} \mu_{tA}(x)$

$\mu_{\bigcup\limits_{t>0} tA}(x) = \sup\limits_{t>0} \mu_A\left(x/t\right)$

$\mu_{\bigcup\limits_{t>0} tA}(x) = \sup \mu_A(y) \geq \alpha$ [∵0 ≤ α ≤ 1]

$\mu_{\bigcup_{t>0} tA}(x) = 1$, $\therefore \bigcup_{t>0} tA = E$ ,Therefore A is absorbing.

**THEOREM 4.2:** *Let E, F be a vector space over K and f : E → F, be a linear mapping. If A is an absorbing fuzzy set in F. Then $f^{-1}(A)$ is an absorbing fuzzy set in E.*

Proof : Suppose A is an absorbing fuzzy set in F. Let x ∈ E

$\sup_{k>0} \mu_{f^{-1}(A)}(kx) = \sup_{k>0} \mu_A(f(kx))$ [∵ $\mu_{f^{-1}(A)}(x) = \mu_A(f(x))$, ∀ x∈ E ]

$\sup_{k>0} \mu_{f^{-1}(A)}(kx) = \sup_{k>0} \mu_A(kf(x))$ by extension principle, [∵ f is linear ]

$\sup_{k>0} \mu_{f^{-1}(A)}(kx) = 1$, as f(x) ∈ F, and A is absorbing fuzzy set in F

∴ $f^{-1}(A)$ is an absorbing fuzzy set in E.

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