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A Serendipity Fixed Point Using Dual F-Contraction

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Abstract: *The concept of Serendipity fixed point was initiated by Powar & Sahu in "Powar P. L. and Sahu G. R. K., (2013), Serendipity fixed point, International Journal of Mathematics and Computer Applications Research (IJMCAR) ISSN: 2249-6955: Vol. 3, Issue 3, 65-78".*

Recently, Wardowski has investigated the existence & uniqueness of fixed point in view of the new concept of contraction viz. F-Contraction in "Wardowski D., (2012), Fixed Points of a new type of contractive mappings in complete Metric Spaces, Fixed Point Theory and Applications, 2012:94". Considering these two new ideas, in the present paper authors have established the existence and uniqueness of Serendipity fixed point with respect to the F-Contraction map. The definition of Serendipity fixed point has been emerged from the concept of weak completeness criterion.

Keywords: *Serendipity Fixed Point, Weak Completeness, F-Contraction, Dual Contraction, Dual F-contraction.*
AMS Subject Classification: *47H10; 54H25; 54E50.*

I. Introduction

The field of fixed point theory has completed its century in 2012. The study of a fixed point and common fixed point satisfying different contractive conditions has been explored by many mathematicians. The fixed point theorems involving the concept of altering distance functions have been initiated by Delbosco in 1976 (cf. [7]) and this was further studied by Khan et al. in 1984 (cf. [10]). The initial concept of altering distance function was generalized by Choudhary [6] and then the concept was extensively used by many researchers (cf. [2], [3], [8], [9], [14]-[18]). The idea of weak complete metric space involving the weak convergence of weak Cauchy sequence has been described in [11]. Pawar and Sahu in 2012(cf. [14]) applied the concept of dual space to assure the existence & uniqueness of Fixed Point. They noticed that the fixed point may not exist for some self maps T defined on a set X but it may exist in the dual space X* of X. To deal with such cases the idea of Serendipity fixed point came in existence and it was studied in [14]. Dariusz Wordwaski in 2012 explored the concept of F-Contraction which is a natural generalization of the Banach contraction criterion. Considering generalized concept, Wordwaski has established the results assuring the existence and uniqueness of the fixed point (cf. [20]). Using the definitions of weak contraction and Banach contraction, the existence and uniqueness theorems for Serendipity fixed points have been proved in [14] with respect to weak completeness instead of completeness. Keeping in view the same concept of weak completeness, in the present paper, authors have proved the existence and uniqueness of Serendipity fixed point with reference to the F-contraction map which is the generalization of the result due to Powar & Sahu (cf. [14]).

II. Preliminaries

In this section, we recall some definitions and concepts which are required for our analysis.

Definition 2.1 [11] Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with the norm defined by

$$||f|| = \mathrm{Sup}_{\substack{x\in X\\ x\neq 0}} \frac{|f(x)|}{||x||} = \operatorname*{Sup}_{\substack{x\in X\\ ||x||=1}} |f(x)|$$

which is called the **dual space** of X and is denoted by X* .

Definition 2.2 [11] Let $\{x_n\}$ be a sequence in a normed linear space X. Sequence $\{x_n\}$ is said to **converge weakly** to x in X if for every linear functional f in X^*(dual space of$X$)$f(x_n) \to f(x)$ as $n \to \infty$ i.e. for every $\varepsilon > 0$ there exists a natural number $n_o \in N$ such that

$$|f(x_n) - f(x)| < \varepsilon, \forall\ n \geq n_o \text{ for all } f \in X^*.$$

Example: 2.1 Consider the Hilbert space $L^2[0, 2\pi]$ which is the space of all square - integrable functions on the interval$[0, 2\pi]$.

The inner product on the space is defined by

$$\prec f, g \succ = \int_0^{2\pi} f(x).g(x)dx$$

The sequence of functions $f_1, f_2, f_3, \ldots$ defined by

$$f_m(x) = sin(mx)$$

converges weakly to the zero function in $L^2[0, 2\pi]$, as the integral

$$\int_0^{2\pi} \sin(mx) . g(x) dx$$

tends to zero for any square integrable function g on $[0, 2\pi]$ when m tends to infinity,

i.e. $\prec f_n, g \succ \longrightarrow \prec 0, g \succ = 0$

However, $sin(mx)$ does not tends to 0 as m $\rightarrow \infty$.

Definition 2.3 [11] A **fixed point** of a mapping $T: X \rightarrow X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is $Tx = x$.

Example 2.2 Let $T: R \rightarrow R$ be defined by $Tx = 1 - x$. Then T has **one fixed point** $\boldsymbol{x = \frac{1}{2}}$.

Remark 2.1 The mapping T may have more than one fixed point (see [14]).

Definition 2.4 [14] A **Serendipity Fixed Point** of a mapping $T: X \rightarrow X$ of a set X into itself is a point $x \in X$ such that there exists real or complex valued function $f \in X^*$ satisfying the condition $f(Tx) = f(x)$.

Example 2.3 Let X be the set of all rational numbers, let $T: X \rightarrow X$ defined by $T(x) = x + 1$ and $f: X \rightarrow R$ defined by $f(x) = x^2$. T has no fixed point but $x = -\frac{1}{2}$ is the only Serendipity fixed point of T.

Definition 2.5 [20] (**F-Contraction**) Let $F: R_+ \rightarrow R$ be a mapping satisfying:

(F1) F is strictly increasing, i.e. for all $\alpha, \beta \in R_+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n\in N}$ of positive numbers $\lim_{n\rightarrow\infty} \alpha_n = 0$ if and only if $\lim_{n\rightarrow\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\rightarrow 0^+} \alpha^k F(\alpha) = 0$.

A mapping $T: X \rightarrow X$ is said to be an **F-Contraction** if there exists $\tau > 0$such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big).$$

Example 2.4 Let $F: R_+ \rightarrow R$ defined by $F(x) = \log(x) + x$ and a mapping $T: R \rightarrow R$ defined by $T(x) = \frac{x}{2}$ then it may be verified easily that T is F-Contraction for $\{\alpha_n\} = \left\{\frac{1}{n}\right\}$(say).

Definition2.6 (**Dual F-Contraction**) Let $F: R_+ \rightarrow R$ be a mapping satisfying (F1), (F2) and (F3). A mapping $T: X \rightarrow X$ and $f \in X^*$(Dual space) then T is said to be a **Dual F- Contraction** if $\exists\ \tau > 0$ such that

$$\forall x, y \in X,\ d(fTx, fTy) > 0 \Rightarrow \tau + F\big(d(fTx\,, fTy)\big) \leq F(d(fx, fy)) \tag{2.1}$$

Example 2.5 Let $F: R_+ \rightarrow R$ defined by $F(\alpha) = \log(\alpha)$

Claim: (F1) holds.

Let $\alpha_1 < \alpha_2$ where $\alpha_1, \alpha_2 \in R_+$

Taking log both sides, we get $\log \alpha_1 < \log \alpha_2 \Rightarrow F(\alpha_1) < F(\alpha_2)$.

Hence, F is strictly increasing.

Claim: (F2) holds.

Let a sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$

$$\lim_{n\rightarrow\infty} F(x_n) = \lim_{n\rightarrow\infty} F(\frac{1}{n}) = \lim_{n\rightarrow\infty} \log(\frac{1}{n}) = \lim_{n\rightarrow\infty} [\log(1) - \log(n)]$$
$$= \lim_{n\rightarrow\infty}[0 - \log(n)] = -\infty$$

Claim: (F3) holds.

There exists $k \in (0, 1)$ such that

$$\lim_{\alpha\rightarrow 0^+} \alpha^k F(\alpha) = \lim_{h\rightarrow 0}(0 + h)^k F(0 + h) = \lim_{h\rightarrow 0} h^k \log h$$
$$= \lim_{h\rightarrow 0} \frac{\log(h)}{h^{-k}} \quad \text{where } k \in (0,1).$$
$$= \lim_{h\rightarrow 0} \frac{\frac{1}{h}}{-kh^{-k-1}} = \lim_{h\rightarrow 0} \frac{h^{k+1}}{-kh} = \lim_{h\rightarrow 0} \frac{h^k}{-k} = 0.$$

A mapping $T: l^\infty \rightarrow l^\infty$ defined by $T(x) = \frac{x}{3}$ and a real functional $f \in X^*$(Dual space) defined by

$f: l^\infty \rightarrow R, f(x) = 2 \mid \xi_j \mid$ where $x = (\xi_i)$

Now $F[d(fx, fy)] = F[|fx - fy|] = \log|fx - fy| = \log\big|2|\xi_j| - 2|\gamma_j|\big|$ for $y = (\gamma_i)$.

$$F[d(fTx, fTy)] = F[|fTx - fTy|] = \log|fTx - fTy| = \log\left|\frac{2}{3}|\xi_j| - \frac{2}{3}|\gamma_j|\right|$$
$$= \log\big|2|\xi_j| - 2|\gamma_j|\big| - \log 3$$
$$= F[d(fx, fy)] - \log 3.$$

We get $F[d(fTx, fTy)] \leq F[d(fx, fy)] - \tau$ with $\tau = \log 3$. Thus T is a dual F-contraction.

III. Main Result

Theorem 3.1 Let X be a normed linear space, T be a selfmap on X and 'd' be the metric defined on R. If $f \in X^*$ (real dual space of X) and f is a bijective mapping satisfying dual F-Contraction for all $x, y \in X$ then T has a unique Serendipity fixed point.

Proof : Let x be an arbitrary point of X and $\{x_n\}$ be asequence of points of X such that

$$Tx_n = x_{n+1}, \forall\, n \in N.$$

Let $\alpha_n = d(y_n, y_{n+1})$, where $y_n = f(x_n) \quad \forall\, n \in N.$ (3.1)

Then using (2.1) we get

$$F(\alpha_n) = F\big[d\big(f(x_n),\ f(x_{n+1})\big)\big] = F\big[d\big(f(Tx_{n-1}),\ f(Tx_n)\big)\big]$$
$$\leq F\big[d\big(f(x_{n-1}),\ f(x_n)\big)\big] - \tau$$
$$\leq F\big[d\big(f(Tx_{n-2}),\ f(Tx_{n-1})\big)\big] - \tau \leq F\big[d\big(f(x_{n-2}),\ f(x_{n-1})\big)\big] - 2\tau$$

In view of (3.1)

$$F(\alpha_n) \leq F(\alpha_{n-1}) - \tau \leq F(\alpha_{n-2}) - 2\tau \leq \cdots \leq F(\alpha_0) - n\tau. \quad (3.2)$$

Now letting $n \to \infty$ in (3.2), we get

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

Using (F2), we get

$$\lim_{n\to\infty} \alpha_n = 0. \quad (3.3)$$

In view of (F3),$\exists\, k \in (0,1)$such that

$$\lim_{\alpha_n \to 0^+} (\alpha_n)^k F(\alpha_n) = 0$$

$$\lim_{n\to\infty} (\alpha_n)^k F(\alpha_n) = 0 \quad (3.4)$$

Consider (3.2) again

$$F(\alpha_n) \leq F(\alpha_0) - n\tau$$

$\Rightarrow$ $$F(\alpha_n) - F(\alpha_0) \leq -n\tau$$

$\Rightarrow$ $$(\alpha_n)^k[F(\alpha_n) - F(\alpha_0)] \leq -(\alpha_n)^k n\tau \leq 0$$

$\Rightarrow$ $$(\alpha_n)^k F(\alpha_n) - (\alpha_n)^k F(\alpha_0) \leq -(\alpha_n)^k n\tau \leq 0\,. \quad (3.5)$$

Letting $n \to \infty$ in (3.5) and using (3.3) and (3.4), we get

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) - \lim_{n\to\infty} \alpha_n^k F(\alpha_0) \leq \lim_{n\to\infty} -\alpha_n^k n\tau \leq 0$$

i.e $$\lim_{n\to\infty} \alpha_n^k n = 0. \quad (3.6)$$

Now, Let us observe from (3.6) that

Given $\epsilon^* > 0$, $\exists\, n_1 \in N$ such that $|\, n\alpha_n^k - 0\,| < \epsilon^* \quad \forall\, n \geq n_1$

$\Rightarrow$ $$n\alpha_n^k < \epsilon^*$$

$\Rightarrow$ $$\alpha_n < \frac{\epsilon}{n^{1\backslash k}} \quad \forall n \geq n_1$$

(3.7)

Claim:$\{y_n\}$ is a Cauchy sequence.

In order to show that $\{y_n\}_{n\in N}$ is a Cauchy sequence, consider $m, n \in N$ such that

$$m > n \geq n_1.$$

Using triangular inequality and relations (3.1) and (3.7), we obtain

$$d(y_m, y_n) \leq d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + \cdots + d(y_{n+1}, y_n).$$
$$\leq \alpha_{m-1} + \alpha_{m-2} + \cdots + \alpha_n$$
$$< \sum_{i=n}^{\infty} \alpha_i \leq \sum_{i=n}^{\infty} \frac{\epsilon}{i^{1\backslash k}}$$

since $k \in (0,1)$ then $\frac{1}{k} > 1$.

By P-series test, it may be noted that the series $\epsilon \sum_{i=n}^{\infty} \frac{1}{i^{1\backslash k}}$ is convergent.

Hence, $\{y_m\}$ is a Cauchy sequence.

Since $\{y_m\}$ is a Cauchy sequence of real numbers and R is complete, therefore it is convergent, i.e. $\exists$ a point y in R such that $y_n \to y$ i.e. $f(x_n) \to y$. Since f is onto and continuous, $\exists\, x \in X$ such that $f(x) = y$ ie.$f(x_n) \to f(x)$.

Hence, $\{x_n\}$ converges weakly to x in X.

We now show that x is Serendipity fixed point of T. Since T is continuous and also every functional f is continuous (see page 224 of [19]), we have

$$\lim_{n\to\infty} f(Tx_n) = f(Tx). \tag{3.8}$$

Using (3.8) and continuity (see page 75 of [19]) of d, we have

$$\begin{aligned} d\big(fTx, f(x)\big) &= \lim_{n\to\infty} d(fTx_n, f(x_n)) \\ &= \lim_{n\to\infty} d(f(x_{n+1}), f(x_n)) \\ &= d\big(f(x), f(x)\big) = 0 \end{aligned}$$

$$\Rightarrow \quad f(Tx) = f(x).$$

Hence, x is a Serendipityfixed point of T.

Claim: x is unique.

Let if possible there exist two Serendipity fixed points viz. $x_1$and x_2 such that $x_1 \neq x_2$.

By the definition of Serendipity fixed point

$$f(Tx_1) = f(x_1) \;\&\; f(Tx_2) = f(x_2). \tag{3.9}$$

Since f is one-one and onto, then $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$ (3.10)

By equation (2.1), (3.9) and (3.10), we get

$$\begin{aligned} \tau &\leq F(d(fx_1, fx_2)) - F(d(fTx_1, fTx_2)) \\ &\leq F(d(fx_1, fx_2)) - F(d(fx_1, fx_2)) = 0 \end{aligned}$$

which is contradiction, because $\tau > 0$. Hence, $x_1 = x_2$.

Thus, T has a unique Serendipity fixed point.

IV. Conclusion

Applying the method of successive approximations, Picard has initiated the method of solving differential equations in which the sequence of solutions converges to a unique solution which is a fixed point. Suppose the sequence of approximate solutions does not converge, in that case, we may apply our technique to obtain a Serendipity fixed point and compute the solution by considering its pre-image in the space X.

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