# Concepts On Po-Ternary Semirings 

P. Sivaprasad ${ }^{1}$, Dr. D. Madhusudhana Rao ${ }^{2}$ and G. Srinivasa Rao ${ }^{3}$<br>${ }^{1}$ Asst. Prof of Mathematics, Universal College of Engineering \& Technology, Perecherla, Guntur, A. P. India. ${ }^{2}$ Head, Department of Mathematics, V.S.R. \& N.V.R. College, Tenali, A. P. India.<br>${ }^{3}$ Asst. Prof of Mathematics, Tirumala Engineering College, Narasaraopet, A. P. India.


#### Abstract

In this paper, we introduce the concept of zeroid in PO-ternary semirings. We study whether the algebraic structure of ( $T,[]$ ) may determine the order structure of ( $T,+$ ) and vice-versa. Throughout this chapter unless otherwise mentioned $T$ is a po-ternary semiring in which $(T,+)$ is a zeroid. The zeroid of a poternary semiring is denoted by Z. We also study the properties of zeroid PO-ternary semirings and ordered zeroid PO-ternary semirings. We prove that in a PO-ternary semiring every odd power of $x$ is a zeroid if $x$ is a zeroid. We also prove that in a zeroid PO-ternary semiring which is also zero cube Po-ternary semiring, then $T^{3}$ $=\{0\}$.


In section 2, the required preliminaries (concepts, examples and results) are presented.
In section 3, properties of PO-ternary semirings are discussed. We also discuss the examples of totally ordered zeroid $P O$-ternary semirings.
Mathematics Subject Classification : 16Y30, $16 Y 99$.
Key Words : PO-ternary semiring, zeroid, zero cube PO-ternary semiring, mono-ternary semi-ring, zero potent ternary semiring,

## I. Introduction

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results.

The theory of ternary algebraic systems was studied by LEHMER [9] in 1932, but earlier such structures were investigated and studied by PRUFER in 1924, BAER in 1929.

Generalizing the notion of ternary ring introduced by Lister [10], Dutta and Kar [6] introduced the notion of ternary semiring. Ternary semiring arises naturally as follows, consider the ring of integers $Z$ which plays a vital role in the theory of ring. The subset $Z+$ of all positive integers of $Z$ is an additive semigroup which is closed under the ring product,i.e. $Z+$ is a semiring.

### 1.1 PRELIMINARIES

In this section, the required preliminaries (concepts, examples and results) are presented.
Definition 1.1.1: An element $a$ in a PO-ternary semiring ( $\mathrm{T},+,[], \leq$ ) is said to be left positive provided $a x^{2} \geq x$ for every $x$ in T.
Definition 2.2.2: An element $a$ in a PO-ternary semiring (T, $+,[], \leq$ ) is said to be lateral positive provided $x a x \geq x$ for every $x$ in T.
Definition 2.2.3: An element $a$ in a PO-ternary semiring (T, +, [ $], \leq$ ) is said to be right positive provided $x^{2} a \geq x$ for every $x$ in T.
Definition 2.2.4: An element $a$ in a PO-ternary semiring (T, +, [ $], \leq$ ) is said to be two sided positive provided it is both left as well as right positive.
Definition 2.2.5: An element $a$ in a PO-ternary semiring (T, +, [ ], $\leq$ ) is said to be positive provided it is left, lateral as well as right positive.
Definition 2.2.6: An element $a$ in a PO-ternary semiring (T, $+,[], \leq$ ) is said to be left negative provided $a x^{2} \leq x$ for every $x$ in T .
Definition 2.2.7: An element $a$ in a PO-ternary semiring (T, $+,[], \leq$ ) is said to be lateral negative provided $x a x \leq x$ for every $x$ in T.
Definition 2.2.8: An element $a$ in a PO-ternary semiring (T, $+,[], \leq$ ) is said to be right negative provided $x^{2} a$ $\leq x$ for every $x$ in T.
Definition 2.2.9: An element $a$ in a PO-ternary semiring (T, +, [ ], $\leq$ ) is said to be two sided negative provided it is both left as well as right negative.
Definition 2.2.10: An element $a$ in a PO-ternary semiring ( $\mathrm{T},+,[], \leq$ ) is said to be negative provided it is left, lateral as well as right nagative.

NOTE 2.2.11: $\geq$ is the dual of $\leq$.
Definition 2.2.12: Two distinct elements $a, b$ in a PO-ternary semigroup (S, [ ], $\leq$ ) are said to form an anomalous pair if $a^{\mathrm{n}}<b^{\mathrm{n}+1}$ and $b^{\mathrm{n}}<a^{\mathrm{n}+1}$ for all odd $n>0$ where $a, b$ are positive (or) $a^{\mathrm{n}}>b^{\mathrm{n}+1}$ and $b^{\mathrm{n}}>a^{\mathrm{n}+1}$ for all odd $\mathrm{n}>0$ where $a, b$ are negative.
DEFINITION 2.2.13: An element $a$ different from the identity in a non-negatively ordered ternary semigroup
( $\mathrm{T},[], \leq$ ) is said to be $\boldsymbol{O}$-Archimedean if for every $y$ in T there exists a odd natural number $n$ such that $y \leq x^{\mathrm{n}}$.
DEFINITION 2.2.14: A non-negatively ordered ternary semigroup (T, [ ], $\leq$ ) is said to be $\boldsymbol{O}$-Archimedean if every one of its elements different from its identity (if exists) is O-Archimedean.
DEFINITION 2.2.15: Let $\mathrm{a} \in \mathrm{T}$. The least element of the set $\{x \in \mathrm{~N}$ : (there exists $y \in \mathrm{~N}$ ) xax=yay, $x \neq y\}$ is called the index of $a$ and is denoted by $m$, where N is the set of natural numbers.
DEFINITION 2.2.16: The least element of the set $\{x \in \mathrm{~N}: a(m+x) a=a m a\}$ is called the period of $a$ and is denoted by $r$.

### 2.3 PROPERTIES OF PO-TERNARY SEMIRINGS IN WHICH (T, +) IS A ZEROID

In this section, we introduce the structure of zeroid PO-ternary semiring and its properties are studied.
DEFINITION 2.3.1: The set Z of PO-ternary semiring T is said to be zeroid of T provided $\mathrm{Z}=\{a \in \mathrm{~T}: a+b=b$ or $b+a=b$ for some $b \in \mathrm{~T}\}$. The zeroid Z is also denoted by $\mathrm{Z}(\mathrm{T})$.
DEFINITION 2.3.2: A PO-ternary semiring T is said to be zero cube PO-ternary semiring provided $a^{3}=0$ for all $a$ in T.
DEFINITION 2.3.3 : A ternary semiring ( $\boldsymbol{T},+$, ) is said to be mono-ternary semi-ring if either $a+b=a b^{2}$ or $a+b=a^{2} b, \forall a, b \in T$.
EXAMPLE 2.3.4: Let $\mathrm{T}=\{0, x, y\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<0$ and the operations defined as follows:

| + | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | $x$ | 0 | 0 |
| $y$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |


| . | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which $(\mathrm{T},+)$ is a zeroed.
EXAMPLE 2.3.5: Let $\mathrm{T}=\{0, x, y\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<0$ and the operations defined as follows:

| + | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | $y$ | 0 | 0 |
| $y$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |


| . | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which $(\mathrm{T},+$ ) is a zeroed.
EXAMPLE 2.3.6: Let $\mathrm{T}=\{0, x, y\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<0$ and the operations defined as follows:

| + | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | $x$ | $y$ | 0 |
| $y$ | y | y | 0 |
| 0 | 0 | 0 | 0 |


| . | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which $(\mathrm{T},+$ ) is a zeroed.
EXAMPLE 2.3.7: Let $\mathrm{T}=\{0, x, y\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<0$ and the operations defined as follows:

| + | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | $y$ | $y$ | $y$ |
| $y$ | $y$ | $x$ | $x$ |
| 0 | $y$ | $x$ | 0 |


| . | $x$ | $y$ | 0 |
| :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which $(\mathrm{T},+)$ is a zeroed.
EXAMPLE 2.3.8: Let $\mathrm{T}=\{0, x, y, z\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<z<0$ and the operations defined as follows:

| + | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | $x$ | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |


| . | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which $(\mathrm{T},+$ ) is a zeroed.
EXAMPLE 2.3.9: Let $\mathrm{T}=\{0, x, y, z\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<z<0$ and the operations defined as follows:

| + | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | $y$ | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |


| . | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which $(\mathrm{T},+$ ) is a zeroed.
EXAMPLE 2.3.10: Let $\mathrm{T}=\{0, x, y, z\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<z<0$ and the operations defined as follows:

| + | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | $z$ | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |


| . | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which $(\mathrm{T},+)$ is a zeroed.
EXAMPLE 2.3.11: Let $\mathrm{T}=\{0, x, y, z\}$ be a PO-ternary semiring with respect to addition, ternary multiplication and ordering such that $x<y<z<0$ and the operations defined as follows:

| + | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | $x$ | $y$ | $z$ | 0 |
| $y$ | $y$ | $y$ | $z$ | 0 |
| $z$ | $z$ | $z$ | $z$ | 0 |
| 0 | 0 | 0 | 0 | 0 |


| . | $x$ | $y$ | $z$ | 0 |
| :---: | :---: | :---: | :---: | :---: |
| $x$ | 0 | 0 | 0 | 0 |
| $y$ | 0 | 0 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

Then T is a PO-ternary semiring in which ( $\mathrm{T},+$ ) is a zeroed.
EXAMPLE 2.3.12: Let $\mathrm{T}=\{a, b, c\}$ be a PO-ternary semiring with respect to addition, ternary multiplication [ ] and ordering such that $a<b<a+a<a+b<b+a<b+b<c$ and the operations defined as follows:

| ,+[] | $a$ | $b$ | $a+a$ | $a+b$ | $b+a$ | $b+b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a$ | $a+a$ | $a+b$ | $c$ | $c$ | $c$ | $c$ | $c$ |
| $b$ | $b+a$ | $b+b$ | $c$ | $c$ | $c$ | $c$ | $c$ |
| $a+a$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ |
| $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ |

Then $(\mathrm{T},+,[], \leq)$ is a PO-ternary semiring in which $(\mathrm{T},+$ ) is a zeroed. Since $(\mathrm{T},+,[])$ is a mono ternary semiring and $(\mathrm{T},+)$ is not commutative.

EXAMPLE 2.3.13: Let $\mathrm{T}=\{a, b, c, d\}$ be a PO-ternary semiring with respect to addition, ternary multiplication [ ] and ordering such that $a<a+a<c<b<a+b<d$ and the operations defined as follows:

| ,+[] | $a$ | $a+a$ | $c$ | $b$ | $a+b$ | $d$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $a$ | $a+a$ | $c$ | $c$ | $a+b$ | $d$ | $d$ |
| $a+a$ | $c$ | $c$ | $c$ | $d$ | $d$ | $d$ |
| $c$ | $c$ | $c$ | $c$ | $d$ | $d$ | $d$ |
| $b$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ |
| $a+b$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ |

Then $(\mathrm{T},+,[], \leq)$ is a PO-ternary semiring in which $(\mathrm{T},+$ ) is a zeroed. Since $(\mathrm{T},+$ ) is commutative.
EXAMPLE 2.3.14: Let $\mathrm{T}=\{a, b\}$ be a PO-ternary semiring with respect to addition, ternary multiplication [ ] and ordering such that $a+a<a<b$ and the operations defined as follows:

| ,+[] | $a+a$ | $a$ | $b$ |
| :--- | :--- | :--- | :--- |
| $a+a$ | $a+a$ | $a+a$ | $b$ |
| $a$ | $a+a$ | $a+a$ | $b$ |
| $b$ | $b$ | $b$ | $b$ |

Then (T, +, [ ], $\leq$ ) is a PO-ternary semiring in which $(\mathrm{T},+$ ) is a zeroed. Since ( $\mathrm{T},+$ ) is commutative.
Theorem 2.3.15: Let $(\mathrm{T},+,[], \leq)$ be a zero-cube PO-ternary semiring with additive identity 0 . If ( $\mathrm{T},+$ ) is a zeroid then $\mathrm{T}^{3}=\{0\}$.
Proof : Since $(\mathrm{T},+)$ is a zeroid $x+y=y$ or $y+x=y$. Since T is a zero - cube PO-ternary semiring $x^{3}=0, y^{3}=0$, for all $x, y \in \mathrm{~T}$.
Now $x+y=y \Rightarrow(x+y) y^{2}=y^{3} \Rightarrow x y^{2}+y^{3}=y^{3} \Rightarrow x y^{2}+0=0 \Rightarrow x y^{2}=0$,
$x+y=y \Rightarrow y^{2}(x+y)=y^{3} \Rightarrow y^{2} x+y^{3}=y^{3} \Rightarrow y^{2} x+0=0 \Rightarrow y^{2} x=0$,
$x+y=y \Rightarrow y(x+y) y=y^{3} \Rightarrow y x y+y^{3}=y^{3} \Rightarrow y x y+0=0 \Rightarrow y x y=0$,
If $y+x=y$, then $(y+x) y^{2}=y^{3} \Rightarrow y^{3}+x y^{2}=y^{3} \Rightarrow 0+x y^{2}=0 \Rightarrow x y^{2}=0$,
$y+x=y$, then $y y(y+x)=y^{3} \Rightarrow y^{3}+y^{2} x=y^{3} \Rightarrow 0+y^{2} x=0 \Rightarrow y^{2} x=0$,
Also $y+x=y$, then $y(y+x) y=y^{3} \Rightarrow y^{3}+y x y=y^{3} \Rightarrow 0+y x y=0 \Rightarrow y x y=0$.
Therefore $x y^{2}=y x y=y^{2} x=0$ and hence $\mathrm{T}^{3}=\{0\}$.
THEOREM 2.3.16: Let $(T,+,[], \leq)$ be a PO-ternary semiring. If $x \in Z$, where $Z$ is the zeroid of POternary semiring, then every odd power of $x$ is a zeroid.
Proof: Let $x \in \mathrm{Z}$. Then by definition there exists some $y$ in T such that
$x+y=y$ or $y+x=y----(1)$
$\Rightarrow x^{2}(x+y)=x^{2} y \Rightarrow x^{3}+x^{2} y=x^{2} y \Rightarrow x^{3}+s=s$, where $x^{2} y=s \in \mathrm{~T} \Rightarrow x^{3}$ is zeroed.
From (1) $x^{4}(x+y)=x^{4} y \Rightarrow x^{5}+x^{4} y=x^{4} y \Rightarrow x^{5}+s^{1}=s^{1}$, where $x^{4} y=s^{1} \in \mathrm{~T} \Rightarrow x^{5}$ is zeroed.
Continuing in this way, every odd power of $x$ is in Z .
THEOREM 2.3.17: Let $(T,+, \bullet, \leq)$ be a PO-ternary semiring and $(T,+)$ be commutative. Then $(Z,+)$ is a subsemigroup of ( $\mathrm{T},+$ ).
Proof: Let $x, y \in \mathrm{Z} . x \in \mathrm{Z} \Rightarrow$ there exists some $p$ in T such that $x+p=p$ or $p+x=p$
$\mathrm{y} \in \mathrm{Z} \Rightarrow$ there exists some $q$ in T such that $y+q=q$ or $q+y=q$.
Now $(x+p)+(y+q)=p+q \Rightarrow(x+y)+(p+q)=p+q$ (Since ( $\mathrm{T},+$ ) is commutative)
$\Rightarrow(x+y)+s=\mathrm{s}$, where $p+q=s \Rightarrow(x+y) \in \mathrm{Z}$. Therefore Z is a subsemigroup of $(\mathrm{T},+\mathrm{+}$.

## 2.4: ADDITIVELY ZEROPOTENT PO-TERNARY SEMIRING

DEFINITION 2.4.1: A PO-ternary semiring T is said to be a zero potent ternary semiring (or simple called zp-ternary semiring) if $0 \in \mathrm{~T}$ and $2 \mathrm{~T}=\{a+a: a \in \mathrm{~T}\}=0$.

We define order relation $\leq$ on T by $a \leq b$ if and only if $b \in(\mathrm{~T}+a) \cup\{a\}$, then it is easy to see that $\leq$ is a partial order on T which compatible with two operations defined on T . That is $\leq$ is an ordering of the ternary semiring T. Clearly, 0 is a greatest element of T.
THEOREM 2.4.2: Let $T$ be a PO-ternary semiring and $|T| \geq 2$, then an element $a \in T, a \neq 0$, is maximal in $T \backslash\{0\}$ if and only if $T+a=0$.
Proof : Suppose $a \in \mathrm{~T}$ and $a \neq 0$ is maximal in $\mathrm{T} \backslash\{0\}$. To show $\mathrm{T}+a=0$. Let $c \in \mathrm{~T}+a$ then $c=b+a$ for some $b \in \mathrm{~T}$. Suppose $c \neq 0$. Then $c \in(\mathrm{~T}+a) \cup\{a\}$. Which implies $a \leq c$. Since, $a$ is maximal in $\mathrm{T} \backslash\{0\}, a=c$. Therefore $a=c=b+a=b+c=b+(b+c)=2 b+c=0+c=0$, which is a contradiction. Therefore $c=0$. Hence $\mathrm{T}+a=0$.

Conversely, suppose that $\mathrm{T}+a=0$. To show $a$ is maximal in $\mathrm{T} \backslash\{0\}$. Suppose $b \in \mathrm{~T} \backslash\{0\}$ such that $a$ $\leq b$. Then $b \in(\mathrm{~T}+a) \cup\{a\}$. Which implies that $b \in\{0, a\}$. Therefore either $b=0$ or $b=a$. But $b \neq 0$. Therefore $a=b$. Hence $a$ is maximal in $\mathrm{T} \backslash\{0\}$.

NOTE 2.4.3: In the remaining part of the section, we will assume that $\mathrm{T}=\mathrm{T}+\mathrm{T}$.
LEMMA 2.4.4: Let $T$ be a PO-ternary semiring and $|T| \geq 2$. Then $T$ has no minimal elements.
Proof: Suppose $0 \neq a \in \mathrm{~T}$. Then $a=b+c$, for some $b, c \in \mathrm{~T}$. Since T $=\mathrm{T}+\mathrm{T}$. Consequently, $b \leq a$. If $b=a$, then $a=a+c=a+2 c=a+0=0$, a contradiction. Therefore T has no minimal elements.
COROLLARY 2.4.5: If $T$ is an additively zero potent PO-ternary semiring with $T=T+T$ then either $|T|=1$ or $\mathbf{T}$ is infinite.
DEFINITION 2.4.6: An element $a$ of a PO-ternary semiring T is said to be bi-absorbing if it is absorbing for both the operations. i.e., $a+b=b+a=a b c=b a c=b c a=a$, for every $b, c \in \mathrm{~T}$ and $a \leq b, a \leq c$. If such an element exists, it will be denoted by the symbol $0\left(=0_{\mathrm{T}}\right)$.
THEOREM 2.4.7: In a PO-ternary semiring $T$, the only idempotent element is the bi-absorbing element 0 .
Proof: Suppose that $b \leq b^{3}$ for some $b \in \mathrm{~T}$.
Now $b=c+d$ for some $c, d \in \mathrm{~T}$ and $b \leq b^{3}=b b b=b(c+d) b=b c b+b d b$.
Since $c \leq b, d \leq b$ we have $b c d \leq b b d, b c d \leq b c b$.
Now $0 \leq 2 b c d=b c d+b c d \leq b b d+b c b$ and $0=b b d+b c b$.
Finally, $0=b b 0 b b=b b(b b d+b c b) b b=b b d+b c b=b$. Therefore $b=0$.
Thus, 0 is the only idempotent element of T.
THEOREM 2.4.8: Let $T$ be a PO-ternary Semiring. If $a^{k}=a^{l}$ for some $a \in T$ and $1 \leq k \leq l, k, l$ are odd natural numbers, then $a^{k}=0$.
Proof: There are positive integers $m, n$ such that $m(1-k)=2 k+2 n$.
Now if $b=a^{k+n}$, where $k+n$ is a odd positive integer, then $b=a^{k} a^{n}=a^{l} a^{n}=a^{k} a^{l-k} a^{n}=a^{l} a^{l-k} a^{n}=a^{k} a^{l-k} a^{l-k} a^{n}=$ $\ldots \ldots .=a^{k} a^{m(l-k)} a^{n}=a^{k} a^{2 k+2 n} a^{n}=a^{3 k+3 n}=b^{3}$. Then by theorem 2.4.7, $b=0$ and hence $a^{k}=a^{l}=a^{k} a^{l-k}=a^{k} a^{l-k} a^{l-k}=$ $\ldots . .=a^{k} a^{m(l-k)}=a^{k} a^{2 k+2 n}=a^{k} b^{2}=0$. Therefore $a^{k}=0$.
THEOREM 2.4.9: Let $\mathbf{T}$ be a PO-ternary semiring and $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{T}, \boldsymbol{k}, \boldsymbol{l} \geq \mathbf{1}$ be such that $a^{k}=a^{l}+b$ where
$k, l$ are odd natural numbers. Then $a^{2 k}=0$. Moreover, if $2 k \leq l$, then $\boldsymbol{a}^{\boldsymbol{k}}=\mathbf{0}$.
Proof: We have $a^{2 l}+a^{l} b=a^{l}\left(a^{l}+b\right)=a^{l} a^{k}=a^{l+k}=a^{k+l}=a^{k} a^{l}=\left(a^{l}+b\right) a^{l}=a^{2 l}+b a^{l}$. Comsequently, $a^{2 k}=a^{k} a^{k}$ $=\left(a^{l}+b\right)\left(a^{l}+b\right)=a^{2 l}+a^{l} b+b a^{l}+b^{2}=a^{2 l}+b a^{l}+b a^{l}+b^{2}=0$. If $2 k \leq l$, then $a^{2 k}=0$ implies that $a^{l}=0$ and hence $a^{k}=a^{l}+b=0$.
DEFINITION 2.4.10: The elements $x, y$ in a PO-ternary semiring $T$ are said to be incomparable if neither $x \leq y$ nor $y \leq x$ holds.
THEOREM 2.4.11: If $a \in T$ is non-nilpotent element. Then $a, a^{3}, a^{5}, \ldots .$. are pair wise incomparable.
Proof: Suppose that $a \in \mathrm{~T}$ is a non-nilpotent element. Then $a^{n} \neq 0$ for all odd $n \in \mathrm{~N} \cup\{0\}$ (here $a^{0}=a$ ). If $a^{k}=$ $a^{l}$, for some $1 \leq k<l$ then by theorem 2.4.8, $a^{k}=0$, which is a contradiction. If $a^{l} \leq a^{k}$, for $k, l \geq 1$ then $a^{k}=a^{l}$ $+b$, for some $b \in \mathrm{~T}$. By theorem 2.4.9, $a^{2 k}=0$ which is a contradiction. Therefore $a, a^{3}, a^{5}, \ldots$. are pair wise incomparable.

## THEOREM 2.4.12: Let $T$ be a PO-ternary semiring and $a, b \in T$ such that $a b a b a=a$, then $a=0$.

Proof: Suppose that $a b a b a=a$, then $(a b a) b(a b a) b(a b a)=(a b a b a) b(a b a b a)=a b a$. Hence by theorem 2.4.7, $a b a=0$. Consequently, $a=a b a b a=0 b a=0$. Suppose $a=a b a b a+c$ then $a b a=(a b a b a+c) b a=a b a b a b a+$ $c b a$. Hence by theorem 2.4.9, $a b a=0$. Thus $a=0$.

## II. Conclusion

In this paper mainly we studied about concepts in PO-ternary semirings and some special type of PO-ternary semirings.

## References

[1] Arif Kaya and Satyanarayana M. Semirings satisfying properties of distributive type, Proceeding of the American Mathematical Society, Volume 82, Number 3, July 1981.
[2] Chinaram, R., A note on quasi-ideal in i;semirings, Int. Math. Forum, 3 (2008), $1253\{1259$.
[3] Daddi. V. R and Pawar. Y. S.Ideal Theory in Commutative Ternary A-semirings, International Mathematical Forum, Vol. 7, 2012, no. 42, 2085-2091.
[4] Dixit, V.N. and Dewan, S., A note on quasi and bi-ideals in ternarysemigroups, Int. J. Math.Math.Sci. 18, no. 3 (1995), 501-508
[5] Dutta, T.K. and Kar, S., On regular ternary semirings, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and RelatedTopics, World Scienti ${ }^{-}$c, New Jersey, 2003, $343\{355$.
[6] Dutta, T.K. and Kar, S., A note on regular ternary semirings, Kyung-pook Math. J., 46 (2006), 357 \{365.
[7] Jonathan S. Golan. Semirings and Affine Equations over Them: Theory and Applications, Kluwer Academic.
[8] Kar, S., On quasi-ideals and bi-ideals in ternary semirings, Int. J. Math.Math.Sc., 18 (2005), $3015\{3023$.
[9] Lehmer. D. H., A ternary analogue of abelian groups, Amer. J. Math., 59(1932), 329-338.
[10] Lister, W.G., Ternary rings, Trans Amer. Math.Soc., 154 (1971), $37\{55$.
[11] MadhusudhanaRao. D., Primary Ideals in Quasi-Commutative Ternary Semigroups International Research Journal of Pure Algebra - 3(7), 2013, 254-258.

## AUTHORS'S BRIEF BIOGRAPHY:

P. Siva prasad working as Assistant Professor in the department of mathematics, Universal College of Engineering \& Technology, perecharla, Guntur(Dt), Andhra Pradesh, India. He is pursuing Ph.D. under the guidance of Dr. D.Madhusudanarao in AcharyaNagarjuna University. He published more than 3 research papers in popular international Journals to his credit. His area of interests are ternary semirings, ordered ternary semirings, semirings. Presently he is working on Partially Ordered Ternary semirings.


Dr. D. MadhusudhanaRao: He completed his M.Sc. from Osmania University, Hyderabad, Telangana, India. M. Phil. from M. K. University, Madurai, Tamil Nadu, India. Ph. D. from AcharyaNagarjuna University, Andhra Pradesh, India. He joined as Lecturer in Mathematics, in the department of Mathematics, VSR \& NVR College, Tenali, A. P. India in the year 1997, after that he promoted as Head, Department of Mathematics, VSR \& NVR College, Tenali. He helped more than 5 Ph . D's. At present he is guiding 7 Ph . D. Scholars and 3 M . Phil., Scholars in the department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Guntur, A. P.

A major part of his research work has been devoted to the use of semigroups, Gamma semigroups, duo gamma semigroups, partially ordered gamma semigroups and ternary semigroups, Gamma semirings and ternary semirings, Near rings ect. He acting as peer review member to the "British Journal of Mathematics \& Computer Science". He published more than $\mathbf{5 0}$ research papers in different International Journals in the last three academic


He is working as an Assistant Professor in the Department of Applied Sciences \& Humanities, Tirumala Engineering College. He completed his M.Phil. in MadhuraiKamaraj University. He is pursuing Ph.D. under the guidance of Dr.D.Madhusudanarao in AcharyaNagarjuna University. He published more than 10 research papers in popular international Journals to his credit. His area of interests are ternary semirings, ordered ternary semirings, semirings and topology. Presentlyhe is working on Ternary semirings

