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Modified Variational Iteration Method for Solution Fractional Dispersion Equation

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Abstract: *This paper deals with the boundary and initial value problems for the fractional dispersion equation model by using the modified variational iteration method. The fractional derivative is described in Caputo's sense. Tested for some examples and the obtained results demonstrate efficiency of the proposed method. The results were presented in tables and figure using the MathCAD 12 and Matlab software package.*
Key words: *Modified variational iteration method, Fractional dispersion equation, Lagrange multiplier.*

I. Introduction

The fractional calculus is used in many fields of science and engineering [1, 2, 3]. He's variational iteration method for solving kinds of partial differential equations, integer or fractional, order. This method is based on the use of Lagrange multipliers of a parameter in a functional.
In most of these equations analytical solutions are either quite difficult or impossible to achieve, so approximations and numerical techniques must be used. variational iteration method [4–11] are relatively new approach to provide an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because it provides immediate and visible symbolic terms of analytic solutions, and numerical approximate solutions to both linear and nonlinear differential equations.
In the present work, we apply the modified variational method for solving two-dimensional fractional dispersion equation and compare the results with exact solution.

II. Idea of Modified Variational Iteration Method

In this section, we will explain modified variational iteration method for solving partial differential equation. Consider the general nonlinear differential equation

$$Lu(x,t)+Nu(x,t)=g(x,t),$$

where L is a linear differential operator, N is a nonlinear operator, and g an inhomogeneous term. According to modified variational iteration method, we can construct a correct functional as follows:

$$u_0(x,t)=\sum_{i=0}^{2}k_i(x)t^i$$

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(Lu_n(\tau)+Nu_n(\tau)-g(\tau))d\tau$$

and λ is a Lagrange multiplier which can be identified optimally via the variational theory. The subscript n indicates the n th approximation and $\tilde{u}_n$ is considered as a restricted variation $\delta\tilde{u}_n = 0$.

III. Modified Variational Iteration Method For Solving Two-Dimensional Fractional Dispersion Equation

We consider the two-dimensional fractional dispersion equation of the form:

$$\frac{\partial u(x,y,t)}{\partial t}=a(x,y)\frac{\partial^{\alpha}u(x,y,t)}{\partial x^{\alpha}}+b(x,y)\frac{\partial^{\beta}u(x,y,t)}{\partial y^{\beta}}+q(x,y,t) \quad (1)$$

on a finite rectangular domain $x_L < x < x_H$ and $y_L < y < y_H$, with fractional orders $1<\alpha<2$ and $1<\beta<2$, where the diffusion coefficients $a(x,y)>0$ and $b(x,y)>0$. The 'forcing' function $q(x, y, t)$ can be used to represent sources and sinks. We will assume that this fractional dispersion equation has a unique and sufficiently smooth solution under the following initial and boundary conditions. Assume the initial

condition $u(x,y,0) = f(x,y)$ for $x_L < x < x_H$ and $y_L < y < y_H$, and Dirichlet boundary condition $u(x,y,t) = S(x,y,t)$ on the boundary (perimeter) of the rectangular region $x_L < x < x_H$ and $y_L < y < y_H$, with the additional restriction that $S(x_L, y, t) = S(x, y_L, t) = 0$. Eq.(1) also uses Caputo fractional derivatives of order α and β, such that:
The Caputo fractional derivative operator D^α of order α is defined in the following form [12,13]:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

where $m - 1 < \alpha < m, m \in \mathrm{N}, x > 0.$

Applying modified variational iteration method in (1):

We will create new initial condition is: $u_0(x,y,t) = \sum_{i=0}^{2} k_i(x,y) t^i$

Next

$$u_{n+1}(x,y,t) = u_n(x,y,\tau) + \int_0^t \lambda(\frac{\partial u_n(x,y,\tau)}{\partial \tau} - a(x,y)\frac{\partial^\alpha u(x,y,\tau)}{\partial x^\alpha} + b(x,y)\frac{\partial^\beta u(x,y,\tau)}{\partial y^\beta} + q(x,y,\tau))d\tau$$

Where $\lambda = \frac{(-1)^m (\tau - t)^{m-1}}{(m-1)!}$

IV. Numerical Application

For finding the analytical solution of fractional dispersion equation, we will apply modified variational iteration method:

$$\frac{\partial u(x,y,t)}{\partial t} = a(x,y)\frac{\partial^{1.8} u(x,y,t)}{\partial x^{1.8}} + b(x,y)\frac{\partial^{1.6} u(x,y,t)}{\partial y^{1.6}} + q(x,y,t)$$

with the coefficient function: $a(x,y) = \Gamma(2.2) x^{2.8} y/6$, and $b(x,y) = 2x^{2.6} y/\Gamma(4.6)$,

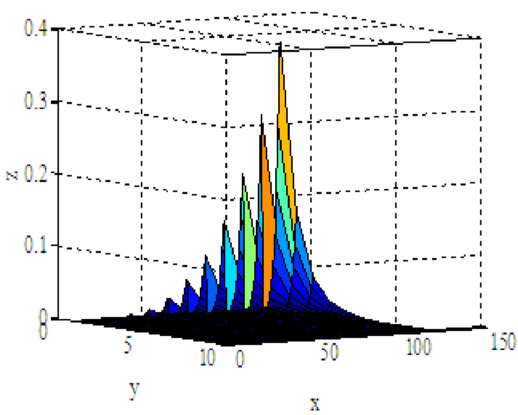
and the source function: $q(x,y,t) = -(1+2xy)e^{-t} x^3 y^{3.6}$,

subject to the initial condition $u(x,y,0) = x^3 y^{3.6}$, $0 < x < 1$,
and Dirichlet boundary conditions : $u(x,0,t) = u(0,y,t) = 0$, $u(x,1,t) = e^{-t} x^3$, and $u(1,y,t) = e^{-t} y^{3.6}$, $t \geq 0$. Note that the exact solution to this problem is: $u(x,y,t) = e^{-t} x^3 y^{3.6}$.

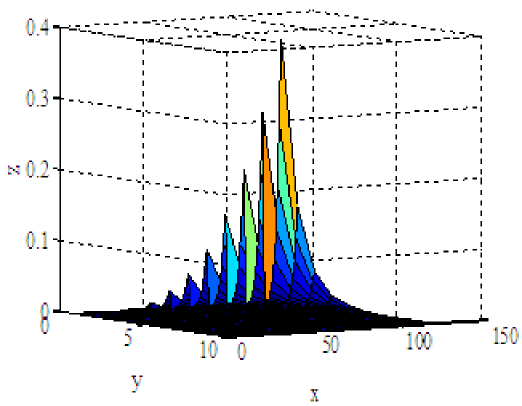
Table 1 displays the analytical solutions for fractional dispersion equation obtained for different values and comparison between exact solution and analytical solution. Figure 1 show the plot of the numerical and the exact solution surface for fractional dispersion equation respectively

Table1. Some of comparison between exact solution and analytical solution when $\alpha = 1.8, \beta = 1.6$ for fractional dispersion equation

x = y	t	Exact Solution	Modified Variational Iteration Method	$\|u_{ex}-u_{MVIM}\|$
0	4	0	0	0.00000000000
0.1	4	0.000000004601	0.000000004601	0.00000000000
0.2	4	0.000000446300	0.000000446300	0.00000000000
0.3	4	0.000064840000	0.000064840000	0.00000000000
0.4	4	0.000043290000	0.000043290000	0.00000000000
0.5	4	0.000188800000	0.000188800000	0.00000000000
0.6	4	0.000629000000	0.000629000000	0.00000000000
0.7	4	0.001740000000	0.001740000000	0.00000000000
0.8	4	0.004200000000	0.004200000000	0.00000000000
0.9	4	0.009137000000	0.009137000000	0.00000000000
1	4	0.018000000000	0.018000000000	0.00000000000
0	5	0.000000000000	0.000000000000	0.00000000000
0.1	5	0.000000001692	0.000000001692	0.00000000000
0.2	5	0.000000164200	0.000000164200	0.00000000000
0.3	5	0.000002385000	0.000002385000	0.00000000000
0.4	5	0.000015930000	0.000015930000	0.00000000000
0.5	5	0.000006946000	0.000006946000	0.00000000000
0.6	5	0.000231400000	0.000231400000	0.00000000000
0.7	5	0.000640000000	0.000640000000	0.00000000000
0.8	5	0.001545000000	0.001545000000	0.00000000000
0.9	5	0.003361000000	0.003361000000	0.00000000000
1	5	0.006738000000	0.006738000000	0.00000000000
0	6	0.000000000000	0.000000000000	0.00000000000
0.1	6	0.000000000623	0.000000000623	0.00000000000
0.2	6	0.000000006040	0.000000006040	0.00000000000
0.3	6	0.000000877500	0.000000877500	0.00000000000
0.4	6	0.000005859000	0.000005859000	0.00000000000
0.5	6	0.000025550000	0.000025550000	0.00000000000
0.6	6	0.000085120000	0.000085120000	0.00000000000
0.7	6	0.000235400000	0.000235400000	0.00000000000
0.8	6	0.000568400000	0.000568400000	0.00000000000
0.9	6	0.001237000000	0.001237000000	0.00000000000
1	6	0.002479000000	0.002479000000	0.00000000000



(a)



(b)

Fig. 1: (a) Numerical solution (b) Exact solution

V. Conclusion

Analytical solutions for fractional dispersion equation obtained for different values of a using modified variational iteration method has been described in addition to shown. It is clear that the modified variational iteration method is in high agreement with the exact solutions.

References

[1]. Podlubny I.," Fractional Differential Equations", San Diego: Academic Press, 1999.

[2]. Miller KS, Ross B. "An Introduction to the Fractional Calculus and Fractional Differential Equations", NewYork: Wiley, 1993.

[3]. Shimizu N, Zhang W. "Fractional calculus approach to dynamic problems of viscoelastic materials", JSMESeries C–Mechanical Systems, Machine Elements and Manufacturing,42:825-837, 1999.

[4]. Iman. I. Gorial**,"** Modified Variational Iteration Method of Solution the Fractional Partial Differential Equation Model", IOSR Journal of Mathematics (IOSR-JM), Volume 11, Issue 2 Ver. IV (Mar - Apr. 2015), PP 84-87.

[5]. J. H. He, "Variational iteration method for delay diferential equations," Communications in Nonlinear Science and Numerical Simulation, vol. 2, no. 4, pp. 235–236, 1997.

[6]. J. H. He, "Approximate solution of non linear differential equations with convolution product nonlinearities," Computer Methods in Applied Mechanics and Engineering, vol. 167, pp. 69–73, 1998.

[7]. J. H. He, "Approximate analytical solution for seepage low with fractional derivatives in porous media," Computer Methods in Applied Mechanics and Engineering, vol. 167, no. 1-2, pp. 57–68,1998.

[8]. J. H. He, "Variational iteration method a kind of non-linear analytical technique: some examples," International Journal of Non-Linear Mechanics, vol. 34, no. 4, pp. 699–708, 1999.

[9]. Mustafa Inc, "The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method", J. Math. Anal. Appl. 345, 476–484, 2008.

[10]. A.M. Wazwaz, "The variational iteration method for analytic treatment for linear and Nonlinear ODEs", Applied Mathematics and Computation 212, 120–134, 2009.

[11]. A. Sevimlican, "An Approximation to Solution of Space and Time Fractional Telegraph Equations by He's Variational Iteration Method", Mathematical Problems in Engineering, 2010.

[12]. Podlubny I., "An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications," Fractional Differential Equations, Mathematics in Science and Engineering, Vol. 198, Academic Press, San Diego, 1999.

[13]. Oldham K. B. and Spanier J., "Fractional Calculus: Theory and Applications, Differentiation and Integration to Arbitrary Order", Academic Press, Inc., New York-London, 234 Pages, 1974.