

2^{nd} -Highly Irregular Fuzzy Graphs

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Abstract: In this paper, 2^{nd} -Highly irregular fuzzy graphs and 2^{nd} -Highly totally irregular fuzzy graphs are defined. Comparative study between 2^{nd} -Highly irregular fuzzy graph and 2^{nd} -Highly totally irregular fuzzy graph is done. Property of 2^{nd} -Highly irregular fuzzy graph and 2^{nd} -Highly totally irregular fuzzy graph are discussed. 2^{nd} -Highly irregularity on fuzzy graphs whose underlying graphs are cycle C_n , a Barbell graph $B_{n,m}$, Sub ($B_{n,m}$) are studied.

keywords: degree of a vertex in fuzzy graph, irregular fuzzy graph, highly irregular fuzzy graph, highly totally irregular fuzzy graph.

I. Introduction

In 1736, Euler first introduced the concept of graph theory. Graph theory is a very useful tool for solving combinatorial problems in different areas such as operations research, optimization, topology, geometry, number theory and computer science. Fuzzy set theory was first introduced by Zadeh in 1965. Fuzzy set theory has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest. The first definition of fuzzy graph was introduced by Haufmann in 1973 based on Zadeh's fuzzy relations in 1971. In 1975, Rosenfeld introduced the concept of fuzzy graphs. Now, fuzzy graphs have been witnessing a tremendous growth and finds application in many branches of engineering and technology. A. Nagoorgani and S.R. Latha introduced irregular fuzzy graphs, total degree and totally irregular fuzzy graphs. They also introduced the concept of neighbourly irregular fuzzy graphs and highly irregular fuzzy graphs. These motivates us to introduce 2^{nd} – highly irregular fuzzy graphs and 2^{nd} – highly totally irregular fuzzy graphs and discussed some of its properties.

II. Preliminaries

We present some known definitions related to fuzzy graphs for ready reference to go through the work presented in this paper.

Definition 2.1 A fuzzy graph $G:(\sigma, \mu)$ is a pair of functions (σ, μ) , where $\sigma: V \rightarrow [0,1]$ is a fuzzy subset of a non empty set V and $\mu: VXV \rightarrow [0,1]$ is a symmetric fuzzy relation on σ such that for all u, v in V , the relation $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ is satisfied. A fuzzy graph G is called complete fuzzy graph if the relation $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ is satisfied.

Definition 2.2 Let $G:(\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. The degree of a vertex u in G is denoted by $d(u)$ and is defined as $d(u) = \sum \mu(uv)$, for all $uv \in E$.

Definition 2.3 Let $G:(\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. The total degree of a vertex is denoted by $td(u)$ and is defined as $td(u) = \sum \mu(uv) + \sigma(u)$, for all $u \in V$. It can also be defined as $td(u) = d(u) + \sigma(u)$.

Definition 2.4 Let $G:(\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be an irregular fuzzy graph if at least two vertices of G has distinct degrees.

Definition 2.5 Let $G:(\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be highly irregular fuzzy graph if each vertex in G is adjacent to the vertices with distinct degrees.

Definition 2.6 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be highly totally irregular fuzzy graph if each vertex in G is adjacent to the vertices with distinct total degrees.

Definition 2.7 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be strongly irregular fuzzy graph if every pair of vertices have distinct degrees.

Definition 2.8 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be strongly totally irregular fuzzy graph if every pair of vertices have distinct total degrees.

Definition 2.9 The 2^{nd} – neighbourhood of a vertex v in G is defined as $N_2(v) = \{u : u \text{ is at a distance } 2 \text{ away from } v\}$.

III. 2^{nd} -Highly Irregular Fuzzy Graphs

Definition 3.1 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be a 2^{nd} - Highly Irregular Fuzzy Graph if the vertices in the 2^{nd} - neighbourhood of each vertex have distinct degrees (or) each vertex u in $N_2(v)$ has distinct degrees.

Example 3.2 Consider a fuzzy graph on $G^*(V,E)$, a cycle of length 5.

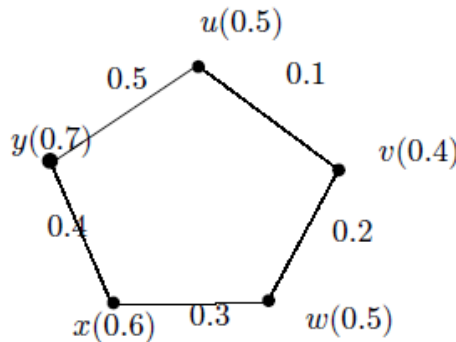


Figure.1

Here, $d(u) = 0.6, d(v) = 0.3, d(w) = 0.5, d(x) = 0.7, d(y) = 0.9$

$N_2(u) = \{x, w\}$ and also $d(x) \neq d(w)$.

$N_2(v) = \{y, x\}$ and also $d(y) \neq d(x)$.

$N_2(w) = \{u, y\}$ and also $d(u) \neq d(y)$.

$N_2(x) = \{u, v\}$ and also $d(u) \neq d(v)$.

$N_2(y) = \{v, w\}$ and also $d(v) \neq d(w)$.

So, the vertices in the 2^{nd} – neighbourhood of each vertex have distinct degrees. Hence G is 2^{nd} -Highly irregular fuzzy graph.

Definition 3.3 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be a 2^{nd} -Highly Totally Irregular Fuzzy Graph if the vertices in the 2^{nd} - neighbourhood of each vertex have distinct total degrees (or) each vertex u in $N_2(v)$ has distinct total degrees.

Example 3.4 Consider a fuzzy graph on $G^*(V,E)$.

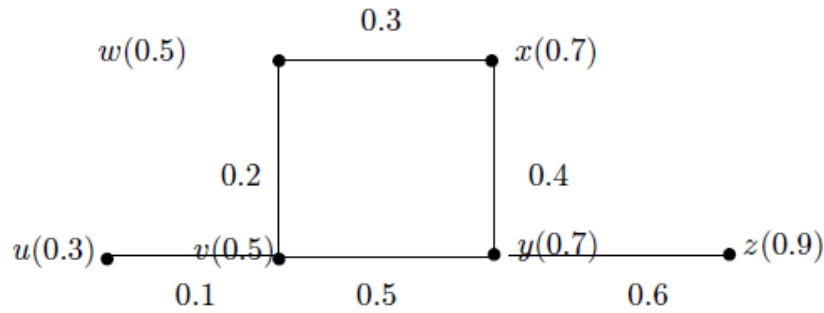


Figure.2

Here, $d(u) = 0.1, d(v) = 0.8, d(w) = 0.5, d(x) = 0.7, d(y) = 1.5$ and $d(z) = 0.6$.

Also, $td(u) = 0.4, td(v) = 1.3, td(w) = 1, td(x) = 1.4, td(y) = 2.2$ and $td(z) = 1.5$.

$N_2(u) = \{w, y\}$ and also $td(w) \neq td(y)$.

$N_2(v) = \{x, z\}$ and also $td(x) \neq td(z)$.

$N_2(w) = \{u, y\}$ and also $td(u) \neq td(y)$.

$N_2(x) = \{v, z\}$ and also $td(v) \neq td(z)$.

$N_2(y) = \{u, w\}$ and also $td(u) \neq td(w)$.

$N_2(z) = \{x, v\}$ and also $td(x) \neq td(v)$.

So, the vertices in the 2^{nd} -neighbourhood of each vertex have distinct total degrees. Hence G is 2^{nd} -Highly totally irregular fuzzy graph.

Remark 3.5 A 2^{nd} -highly irregular fuzzy graph need not be 2^{nd} -highly totally irregular fuzzy graph.

Example 3.6 Consider a fuzzy graph on $G^* : (V, E)$, a cycle of length 5.

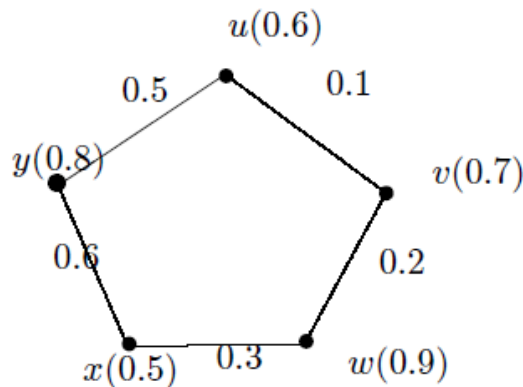


Figure.3

Here, $d(u) = 0.6, d(v) = 0.3, d(w) = 0.5, d(x) = 0.9, d(y) = 1.1$.

Also, $td(u) = 1.2, td(v) = 1, td(w) = 1.4, td(x) = 1.4, td(y) = 1.9$.

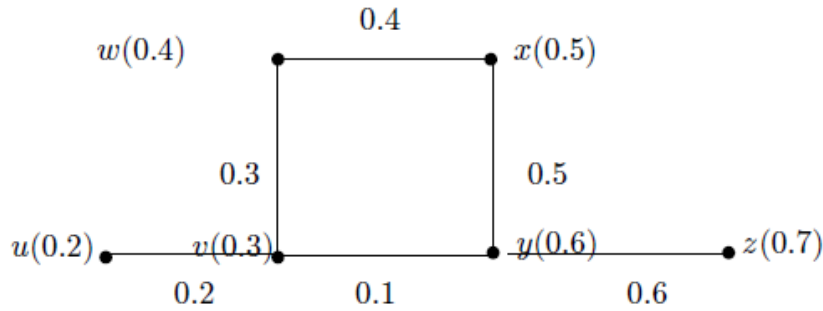
The vertices in the 2^{nd} -neighbourhood of each vertex has distinct degrees. Hence G is 2^{nd} -Highly

irregular fuzzy graph.

Here, $N_2(u) = \{w, x\}$ and $td(w) = td(x)$. Hence G is not 2^{nd} -Highly totally irregular fuzzy graph.

Remark 3.7 A 2^{nd} -Highly totally irregular fuzzy graph need not be 2^{nd} -Highly irregular fuzzy graph.

Example 3.8 Consider a fuzzy graph on $G^*(V, E)$.



Here, $d(u) = 0.2, d(v) = 0.6, d(w) = 0.7, d(x) = 0.9, d(y) = 1.2$ and $d(z) = 0.6$.

Also, $td(u) = 0.4, td(v) = 0.9, td(w) = 1.1, td(x) = 1.4, td(y) = 1.8$ and $td(z) = 1.3$.

The vertices in the 2^{nd} -neighbourhood of each vertex have distinct total degrees. Hence G is 2^{nd} -highly totally irregular fuzzy graph.

Here, $N_2(x) = \{v, z\}$ and $d(v) = d(z)$. Hence G is not 2^{nd} -Highly irregular fuzzy graph.

Theorem 3.9 Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. If G is 2^{nd} -Highly irregular fuzzy graph and σ is a constant function, then G is 2^{nd} -Highly totally irregular fuzzy graph.

Proof. Let $G : (\sigma, \mu)$ be a 2^{nd} -Highly irregular fuzzy graph. Then the vertices in the 2^{nd} -neighbourhood of each vertex have distinct degrees. Let v and w be the vertices in the 2^{nd} -neighbourhood of the vertex u such that $d(v) = k_1$ and $d(w) = k_2$, where $k_1 \neq k_2$. Also, assume that $\sigma(u) = c$, for all $u \in V$. Suppose, $td(v) = td(w) \Rightarrow d(v) + \sigma(v) = d(w) + \sigma(w) \Rightarrow k_1 + c = k_2 + c \Rightarrow k_1 = k_2$. Which is a contradiction. So, $td(v) \neq td(w)$. Hence the vertices in the 2^{nd} -neighbourhood of the vertex u have distinct total degrees provided σ is a constant function. This is true for all the vertices in G . Hence G is 2^{nd} -Highly totally irregular fuzzy graph.

Theorem 3.10 Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. If G is 2^{nd} -Highly totally irregular fuzzy graph and σ is a constant function, then G is 2^{nd} -Highly irregular fuzzy graph.

Proof. Let $G : (\sigma, \mu)$ be a 2^{nd} -Highly totally irregular fuzzy graph. Then the vertices in the 2^{nd} -neighbourhood of each vertex have distinct total degrees. Let v and w be the vertices in the 2^{nd} -neighbourhood of the vertex u such that $td(v) = k_1$ and $td(w) = k_2$, where $k_1 \neq k_2$. Also, assume that $\sigma(u) = c$, for all $u \in V$. Suppose, $td(v) \neq td(w) \Rightarrow d(v) + \sigma(v) \neq d(w) + \sigma(w) \Rightarrow d(v) \neq d(w)$. Hence the vertices in the 2^{nd} -neighbourhood of the vertex u have distinct degrees provided σ is a constant function. This is true for all the vertices in G . Hence G is 2^{nd} -Highly irregular fuzzy graph.

Remark 3.11 The theorems 3.9 and 3.10 jointly yield the following result. Let G be a fuzzy graph on $G^*(V, E)$. If σ is a constant function, then G is 2^{nd} -Highly irregular fuzzy graph if and only if G is 2^{nd} -Highly totally irregular fuzzy graph.

Remark 3.12 Let $G: (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. If G is both 2^{nd} -Highly irregular fuzzy graph and 2^{nd} -Highly totally irregular fuzzy graph, then σ need not be a constant function.

Example 3.13 Consider a fuzzy graph $G: (\sigma, \mu)$ on $G^*(V, E)$.

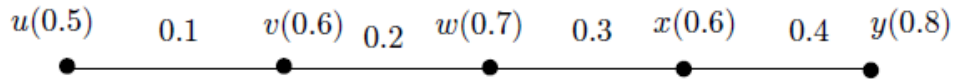


Figure.5

Here, G is both 2^{nd} –highly irregular fuzzy graph and 2^{nd} –highly totally irregular fuzzy graph. But σ is not constant function.

Theorem 3.14 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. If G is strongly irregular fuzzy graph then G is 2^{nd} -highly irregular fuzzy graph.

Proof: Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Assume that G is strongly irregular fuzzy graph. Then ever pair of vertices in G have distinct degrees \Rightarrow all the vertices in G have distinct degrees. So,the vertices in the 2^{nd} -neighbourhood of each vertex has distinct degrees in G . Hence G is 2^{nd} -highly irregular fuzzy graph.

Theorem 3.15 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. If G is strongly totally irregular fuzzy graph then G is 2^{nd} -highly totally irregular fuzzy graph.

Proof: Proof is similar to the above Theorem 3.14.

Remark 3.16 The converse of above Theorems 3.14 and 3.15 need not be true.

Example 3.17 Consider a fuzzy graph on $G^*(V,E)$.

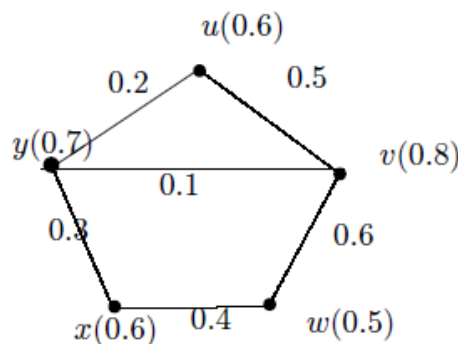


Figure. 6

Here, $d(u)=0.7, d(v)=1.2, d(w)=1, d(x)=0.7, d(y)=0.6$ and

$td(u)=1.3, td(v)=2, td(w)=1.5, td(x)=1.3, td(y)=1.3$.

So, the vertices in the 2^{nd} - neighbourhood of each vertex have distinct degrees. Hence G is 2^{nd} -highly irregular fuzzy graph. But the vertices u and x have the same degrees. Hence G is not strongly irregular fuzzy graph.

Also, the vertices in the 2^{nd} - neighbourhood of each vertex have distinct total degrees. Hence G is 2^{nd} -highly totally irregular fuzzy graph. But the vertices u and x have the same total degrees. Hence G is not strongly totally irregular fuzzy graph.

irregular fuzzy graph.

IV 2^{nd} -Highly Irregularity on cycle with some specific membership functions

Theorem 4.1: Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$, an even cycle of length n . If σ takes distinct membership values and μ is constant function or alternate edges takes same membership values, then G is 2^{nd} -highly totally irregular fuzzy graph.

Proof: Case1: Suppose μ is a constant function, $d(u)=k$, for all $u \in V$. We have $td(u)=d(u) + \sigma(u)$. Since σ takes distinct membership values, the vertices in the 2^{nd} -neighbourhood of each vertex have distinct total degrees. Hence G is 2^{nd} -highly totally irregular fuzzy graph.

Case2: Alternate edges takes same membership values

$$\text{Let } \mu(e_i) = \begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even} \end{cases}$$

Let the vertices in the 2^{nd} -neighbourhood of the vertex u be w and y . Since σ takes distinct membership values, let $\sigma(w)=c_1$ and $\sigma(y)=c_2$. Now, $td(w)=c_1+k_1+k_2$ and $td(y)=c_2+k_1+k_2 \Rightarrow td(w) \neq td(y)$. So, the vertices in the 2^{nd} -neighbourhood of the vertex u have distinct total degrees. This is true for each vertex in G . Hence G is 2^{nd} -highly totally irregular fuzzy graph.

Theorem 4.2: Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$, a cycle of length n . If the membership values of the edges e_1, e_2, \dots, e_n are respectively c_1, c_2, \dots, c_n such that $c_1 < c_2 < \dots < c_n$, then G is 2^{nd} -highly irregular fuzzy graph.

Proof: Let e_1, e_2, \dots, e_n be the edges of the cycle C_n taking membership values c_1, c_2, \dots, c_n
 $d(v_1)=c_1+c_n$. For $i = 2, 3, \dots, n$, $d(v_i) = c_{i-1} + c_i$.

Here, all the vertices have distinct degrees. So, the vertices in the 2^{nd} -neighbourhood of each vertex have distinct degrees. Hence G is 2^{nd} -highly irregular fuzzy graph.

Remark 4.3: Even if the membership values of the edges e_1, e_2, \dots, e_n are respectively c_1, c_2, \dots, c_n such that $c_1 < c_2 < \dots < c_n$ then G is not 2^{nd} -highly totally irregular fuzzy graph.

Example 4.4: Consider a fuzzy graph on $G^*(V,E)$, a cycle of length 6.

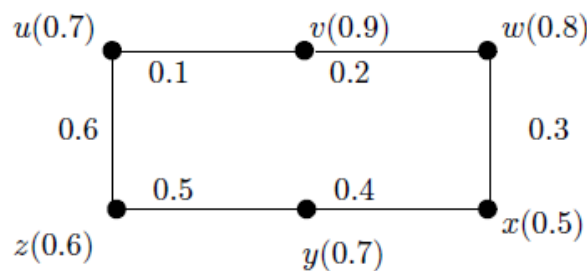


Figure.7

Here, $N_2(z)=\{v,x\}$ such that $td(v)=td(x)$. Hence G is not 2^{nd} -highly totally irregular fuzzy graph.

Theorem 4.5: Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$, a cycle of length n . If the membership values of the edges e_1, e_2, \dots, e_n are respectively c_1, c_2, \dots, c_n such that $c_1 > c_2 > \dots > c_n$, then G is 2^{nd} -highly irregular fuzzy graph.

Proof: Proof is similar to above Theorem 4.2

Remark 4.6: Even if the membership values of the edges e_1, e_2, \dots, e_n are respectively c_1, c_2, \dots, c_n such that $c_1 > c_2 > \dots > c_n$, then G is not 2^{nd} -highly totally irregular fuzzy graph.

Example 4.7: Consider a fuzzy graph on $G^*(V,E)$, a cycle of length 6.

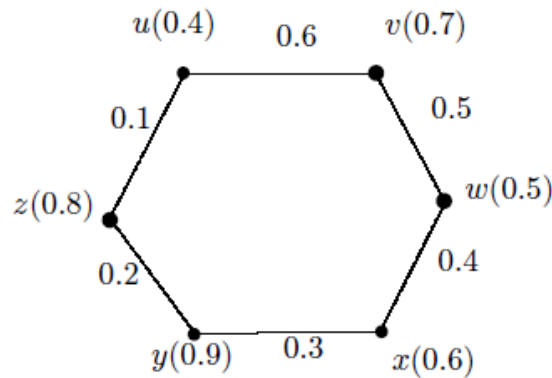


Figure.8

Here, $N_2(u)=\{w,y\}$ such that $td(w)=td(y)$.
Hence G is not 2nd-highly totally irregular fuzzy graph.

Remark 4.8: Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$, a cycle of length n . If the membership values of the edges e_1, e_2, \dots, e_n are respectively c_1, c_2, \dots, c_n are all distinct, then G need not be 2nd-highly irregular fuzzy graph.

Example 4.9: Consider a fuzzy graph on $G^*(V,E)$, a cycle of length 6.

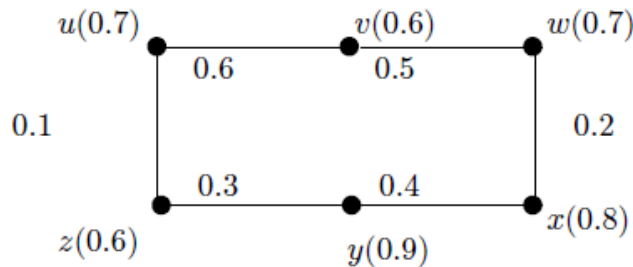


Figure.9

Here, $N_2(u)=\{w, y\}$ and also $d(w)=d(y)$. Hence G is not 2nd-highly irregular fuzzy graph.

V. 2nd-Highly irregularity on Barbell graph with specific membership functions

Theorem 5.1: Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$ which is a barbell graph $B_{m,n}$. If μ is constant function and σ takes distinct membership values, then G is 2nd-highly totally irregular fuzzy graph.

Proof: Let v_1, v_2, \dots, v_n be the vertices adjacent to vertex x and u_1, u_2, \dots, u_n be the vertices adjacent to vertex y and xy is the middle edge. Since μ is constant function, $\mu(uv) = c$, for all $uv \in E$. Then all the pendant edges have same degrees. Since the vertices takes distinct membership values, the total degrees of all the vertices are distinct. So, the vertices in the 2nd-neighbourhood of each vertex have distinct total degrees. Hence G is 2nd-highly totally irregular fuzzy graph.

Remark 5.2: Even if μ is constant function and σ takes distinct membership values, then G need not be 2nd-highly irregular fuzzy graph.

Example 5.3: Consider a fuzzy graph on $G^*(V,E)$, a barbell graph $B_{3,2}$

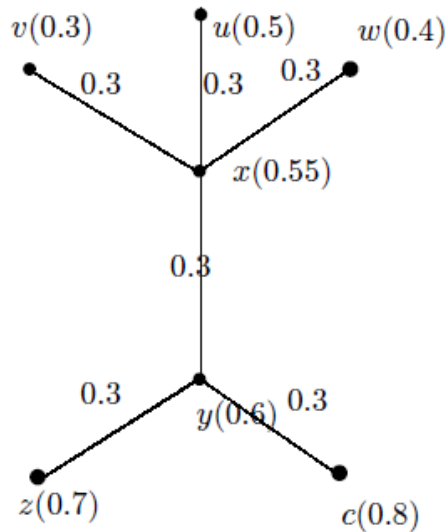


Figure.10

Here, $N_2(u)=\{v,w,y\}$ such that $d(v)=d(w)$. Hence G is not 2nd-highly irregular fuzzy graph.

Remark 5.4: The converse of Theorem 5.1 need not be true. For Example, in above Figure.10, if we replace membership value of edge xy by 0.4 then $\mu(xy)=0.4$.

Example 5.5: Consider a fuzzy graph on $G^*(V,E)$, a barbell graph $B_{3,2}$

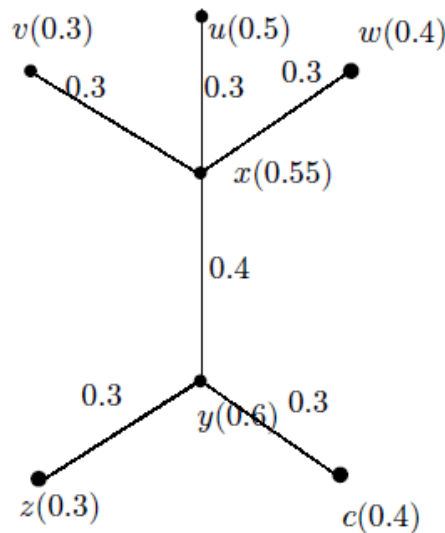


Figure.11

The graph is 2nd-highly totally irregular fuzzy graph. But neither μ is a constant function nor σ takes distinct membership values.

Theorem 5.6: Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$ which is a barbell graph $B_{m,n}$. If the pendant edges have same membership values less than or equal to the membership values of middle edge and σ takes distinct membership values, then G is 2nd-highly totally irregular fuzzy graph.

Proof: If the pendant edges have same membership values then,

$$\text{Let } \mu(e_i) = \begin{cases} c_1 & \text{if } e_i \text{ is pendant edge} \\ c_2 & \text{if } e_i \text{ is middle edge} \end{cases}$$

If $c_1=c_2$, then μ is constant function and hence by theorem 5.1, G is 2^{nd} -highly totally irregular fuzzy graph. If $c_1 < c_2$, $d(v_i) = c_1$ if e_i is pendant edge. But since σ takes distinct membership values, total degrees of all the vertices are distinct. So, the vertices in the 2^{nd} - neighbourhood of each vertex has distinct total degrees. Hence G is 2^{nd} -highly totally irregular fuzzy graph.

Remark 5.7: Even if the pendant edges have same membership value less than membership value of middle edge, G need not be 2^{nd} -highly irregular fuzzy graph, since the degrees of pendant edges are same.

Theorem 5.8: Let $G: (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$ which is a sub($B_{n,m}$). If μ is constant function and σ takes distinct membership values, then G is 2^{nd} -highly totally irregular fuzzy graph.

Proof: Let v_1, v_2, \dots, v_n be the vertices adjacent to vertex x and u_1, u_2, \dots, u_n be the vertices adjacent to vertex y and xy is the middle edge. Subdivide each edge of $B_{n,m}$. Then the additional edges are $xw_i, w_i v_i$ ($1 \leq i \leq n$) and $yt_i, t_i u_i$ ($1 \leq i \leq n$) and two more edges xs, sy . If μ is constant function say $\mu(uv)=c$, for all $uv \in E$. The vertices w_i, t_i are all have same degrees. But since σ takes distinct membership values, the vertices in the 2^{nd} - neighbourhood of each vertex has distinct total degrees. Hence G is 2^{nd} -highly totally irregular fuzzy graph.

Remark 5.9: Even if μ is constant function and σ takes distinct membership values, then G need not be 2^{nd} -highly irregular fuzzy graph.

Example 5.10: Consider a fuzzy graph on $G^*(V, E)$

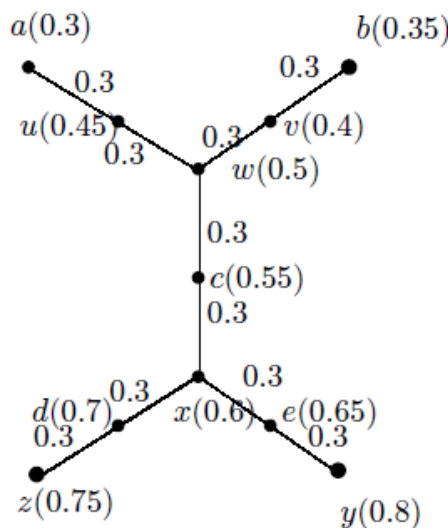


Figure.12

Here, μ is constant function and σ takes distinct membership values. But $N_2(u)=\{v,c\}$ such that $d(v)=d(c)$. Hence G is not 2^{nd} -highly irregular fuzzy graph.

Remark 5.11: The converse of Theorem 5.8 need not be true.

Example 5.12: Consider a fuzzy graph on $G^*(V, E)$.

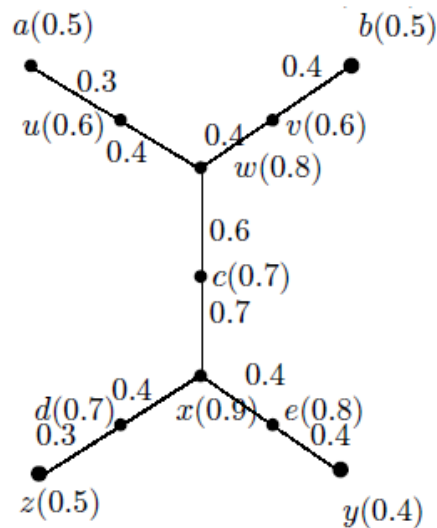


Figure.13

The graph is 2^{nd} -highly totally irregular fuzzy graph. But neither μ is a constant function nor σ takes distinct membership values.

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