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Fixed Point in 2-Metric Space through Rational Expression

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Abstract: *In this paper we have established fixed point theorem in 2-metric space which generalizes many previous results.*
Key Words: *Fixed point theorem, 2-metric space, Euclidean space, Rational expression.*

I. Introduction

After the introduction of concept of 2-metric space by Gahler [3] many authors [1] [2] [3] etc. establishes an analogue of Banach. Contraction principle in 2-metric space. First Kannan [6] and later on Jaggi and Das [5] established the fixed point theorem for non-continuous maps in metric space.

In this paper we have extended this idea to 2-metric space in more general form by increasing the number of terms in R.H.S.

II. Preliminaries

Now we give some basic definitions and well known results that are needed in the sequel.

Definition (2.1): [3] let X be an non empty set and $d:XxXxX\rightarrow R_+$.

If for all x, y, z and u in X, we have

(d_1) d(x,y,z)=0 if at least two of x, y, z are equal.

(d_2) for all x≠y there exist a point z in X st. d(x,y,z)≠0.

(d_3) d(x,y,z)=d(x,z,y) = d(y,z,x) = ………………… and so on.

(d_4) d(x,y,z)≤d(x,y,u)+d(x,u,z)+d(u,y,z)

Then d is called a 2-metric on X and the pair (X,d) is called a 2-metric space.

Definition (2.2): A sequence $\{x_n\}_{n\in N}$ in a 2-metric space (X,d) is said to be a Cauchy sequence if $\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m, a) = 0$ for all $a \in X$.

Definition (2.3): A sequence $\{x_n\}_{n\in N}$ in a 2-metric space (X,d) is said to be a convergent if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$. The point x is called the limit of the sequence.

Definition (2.4): A 2-metric space (X,d) is said to be complete if every Cauchy sequence is X in convergent.

III. Main Result

Theorem (3.1): Let *(X,d)* be a complete 2-metric space and *T* be a self mapping of *X* satisfying.

$$d(Tx,Ty,a) \le \alpha\, min\left\{ \frac{d(x,Tx,a)[1+d(y,Ty,a)]}{1+d(Tx,Ty,a)}, \frac{d(x,Tx,a)[1+d(x,Tx,a)]}{1+d(x,y,a)}, \frac{d(y,Ty,a)[1+d(x,Tx,a)]}{1+d(x,y,a)}, d(x,y,a) \right\} \quad ....(3.3.1)$$

for all *x, y, a* in *X*, where $d(Tx,Ty,a)\neq 0$ and $d(x,y,a)\neq 0$ and $\alpha,\beta,\gamma,\delta\geq 0$ where $\alpha+\beta+\gamma+\delta\geq 0$ and $\alpha+\beta+\gamma+\delta<1$.

Proof: Let x_0 be an arbitrary point of *X*. We define a sequence $\{x_n\}$ as follows:

$x_0, x_1 = Tx_0, x_2 = Tx_1,$ ………….. $x_n = Tx_{n-1}, x_{n+1} = Tx_n,$ ……….

Now first we shall show that $d(x_n,x_{n+1},x_{n+2})=0$

$$d(x_n, x_{n+1}, x_{n+2}) = d(x_n, Tx_n, Tx_{n+1})$$

$$= d\left(Tx_n, T\,x_{n+1}, x_n\right)$$

$$\leq \alpha\ min\left\{\frac{d\left(x_n, T\,x_n, x_n\right)\left[1+d\left(x_{n+1}, T\,x_{n+1}, x_n\right)\right]}{1+d\left(Tx_n, T\,x_{n+1}, x_n\right)},\right.$$

$$\frac{d\left(x_n, T\,x_n, x_n\right)\left[1+d\left(x_n, T\,x_n, x_n\right)\right]}{1+d\left(x_n, x_{n+1}, x_n\right)}$$

$$\frac{d\left(x_{n+1}, T\,x_{n+1}, x_n\right)\left[1+d\left(x_n, T\,x_n, x_n\right)\right]}{1+d\left(x_n, x_{n+1}, x_n\right)}$$

$$d\left(x_n, x_{n+1}, x_n\right)$$

or, $d\left(x_{n+1}, x_{n+2}, x_n\right) \leq \alpha\ min\left\{\frac{d\left(x_n, x_{n+1}, x_n\right)\left[1+d\left(x_{n+1}, x_{n+2}, x_n\right)\right]}{1+d\left(x_{n+1}, x_{n+2}, x_n\right)}\right.$

$$d\left(x_n, x_{n+1}, x_n\right)\frac{\left[1+d\left(x_n, x_{n+1}, x_n\right)\right]}{1+d\left(x_n, x_{n+1}, x_n\right)}$$

$$d\left(x_{n+1}, x_{n+2}, x_n\right)\frac{\left[1+d\left(x_n, x_{n+1}, x_n\right)\right]}{1+d\left(x_n, x_{n+1}, x_n\right)}$$

$$\left. d\left(x_n, x_{n+1}, x_n\right)\right\}$$

or, $d\left(x_{n+1}, x_{n+2}, x_n\right) \leq 0$

which implies that $d(x_n,x_{n+1},x_{n+2})=0$

Now, $d(x_1,x_2,a) = d(Tx_0,Tx_1,a)$

$$\leq \alpha\ min\left\{\frac{d\left(x_0, T\,x_0, a\right)\left[1+d\left(x_1, T\,x_1, a\right)\right]}{1+d\left(Tx_0, T\,x_1, a\right)}\right.$$

$$\frac{d\left(x_0, T\,x_0, a\right)\left[1+d\left(x_0, T\,x_0, a\right)\right]}{1+d\left(x_0, x_1, a\right)}$$

$$\left.\frac{d\left(x_1, T\,x_1, a\right)\left[1+d\left(x_0, T\,x_0, a\right)\right]}{1+d\left(x_0, x_1, a\right)}, d\left(x_0, x_1, a\right)\right\}$$

$$\leq \alpha\ min\left\{\frac{d\left(x_0, x_1, a\right)\left[1+d\left(x_1, x_2, a\right)\right]}{1+d\left(x_1, x_2, a\right)}\right.$$

$$\frac{d\left(x_0, x_1, a\right)\left[1+d\left(x_0, x_1, a\right)\right]}{1+d\left(x_0, x_1, a\right)}$$

$$\frac{d\left(x_1, x_2, a\right)\left[1+d\left(x_0, x_1, a\right)\right]}{1+d\left(x_0, x_1, a\right)}, d\left(x_0, x_1, a\right)$$

$$= \alpha\ min\left\{d\left(x_0, x_1, a\right), d\left(x_1, x_2, a\right)\right\}$$

Now, if $d(x_1,x_2,a)$ is minimum then,
$d(x_1,x_2,a) \leq \alpha d(x_1,x_2,a)$.which is a contradiction.
So, we have, $d(x_1,x_2,a) \leq \alpha\ d(x_0,x_1,a)$ ………………………….. (3.3.2)
By similar argument we get from 3.3.2
$d(x_2,x_3,a) \leq \alpha\ d(x_1,x_2,a) \leq \alpha^2 d(x_0,x_1,a)$
Therefore proceedings in the similar way we get,
$d(x_n,x_{n+1},a) \leq \alpha^n d(x_0,x_1,a)$.
Now we shall show that $\{x_n\}$ is a cauchy sequence.

Since,

$$d(x_n,x_{n+m},a) \le d(x_n,x_{n+1},x_{n+m}) + d(x_n,x_{n+1},a) + d(x_{n+1},x_{n+2},x_{n+m}) + d(x_{n+1},x_{n+2},a) + \ldots + \ldots + \ldots + \ldots + d(x_{n+m+2},x_{n+m-1},x_{n+m}) + d(x_{n+m-1},x_{n+m},a)$$

$$= \sum_{k=1}^{n+m-2} d(x_k,x_{k+1},x_{k+2}) + \sum_{k=1}^{n+m-1} d(x_k,x_{k+1},a)$$

Now, we have

$$d(x_n,x_{n+1},x_{n+m}) \le hd(x_{n+1},x_n,x_{n+m}) \ \cdots \ h^n d(x_0,x_1,x_{n+m})$$

and also, $d(x_n,x_{n+1},a) \le hd(x_{n+1},x_n,a) \ \cdots \ h^n d(x_0,x_1,a)$

Now $\sum_{k=1}^{n+m-2} d(x_k,x_{k+1},x_{n+m}) \le \left[h^{n+m-2} + h^{n+m-1} + \ldots\ldots h^n\right] d(x_0,x_1,x_{n+m})$

$$\le \frac{h^{n+m-2}}{1-h} d(x_0,x_1,a)$$

$$\to 0 \text{ as } n \to \infty$$

Thus $\{x_n\}_{n\in N}$ is a Cauchy sequence. Since X is complete, there exists a point u in X such that $\lim_{n\to\infty} x_n = u$.

Now shall prove that u is the fixed point of T.

$$d(u,Tu,a) \le d(u,Tu,x_n) + d(u,x_n,a) + d(x_n,Tu,a)$$

$$\le d(u,Tu,x_n) + d(u,x_n,a) + d(Tx_{n-1},Tu,a)$$

$$\le d(u,Tu,x_n) + d(u,x_n,a) + \alpha \min\left\{ \frac{d(x_{n-1},Tx_{n-1},a)[1+d(u,Tu,a)]}{1+d(Tx_{n-1},Tu,a)}, \frac{d(x_{n-1},Tx_{n-1},a)[1+d(x_{n-1},Tx_{n-1},a)]}{1+d(x_{n-1},u,a)}, \frac{d(u,Tu,a)[1+d(x_{n-1},Tx_{n-1},a)]}{1+d(x_{n-1},u,a)}, d(x_{n-1},u,a) \right\}$$

$$= d(u,Tu,x_n) + d(u,x_n,a) + \alpha \min\left\{ \frac{d(x_{n-1},x_n,a)[1+d(u,Tu,a)]}{1+d(x_n,Tu,a)}, \frac{d(x_{n-1},x_n,a)[1+d(x_{n-1},x_n,a)]}{1+d(x_{n-1},u,a)}, \frac{d(u,Tu,a)[1+d(x_{n-1},x_n,a)]}{1+d(x_{n-1},u,a)}, d(x_{n-1},u,a) \right\}$$

When $n \to \infty$, $x_n \to u$, $x_{n-1} \to u$, we get

$$d(u,Tu,a) \le \alpha d(u,Tu,a)$$

i.e. $(1-\alpha)d(u,Tu,a) \le 0$ which implies that $Tu=u$.

Now we show that u is the unique fixed point of T.

If not so, Let $v \neq u$ and $Tv=v$.
Then $d(u,v,a) = d(Tu,Tv,a)$

$$\leq \alpha \min \left\{ \frac{d(u,Tu,a)[1+d(v,Tv,a)]}{1+d(Tu,Tv,a)}, \frac{d(u,Tu,a)[1+d(u,Tu,a)]}{1+d(u,v,a)}, \frac{d(u,Tv,a)[1+d(u,Tu,a)]}{1+d(u,v,a)}, d(u,v,a) \right\}$$

or, $d(u,v,a) \leq \alpha d(u,v,a)$
or, $(1-\alpha)d(u,v,a) \leq 0$ which implies $u = v$.
Thus $u=v$ i.e. u is the unique fixed point of T.
Corollary 1.2.4: Let (X,d) be a complete 2-metric space.
Let T^p be a sequence of self mapping of X satisfying

$$d(T^p x, T^p y, a) \leq \alpha \min \left\{ \frac{d(x,T^p x,a)[1+d(y,T^p y,a)]}{1+d(T^p x,T^p y,a)}, \frac{d(x,T^p x,a)[1+d(x,T^p y,a)}{1+d(x,y,a)}, \frac{d(y,T^p y,a)[1+d(x,T^p x,a)]}{1+d(x,y,a)}, d(x,y,a) \right\}$$

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