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Some New Properties of Fuzzy General Set Functions

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Abstract: *In this paper we study the fuzzy measure, fuzzy integral and prove some new properties of them. Also we discuss the relation between the types of fuzzy measures and fuzzy integration. Finally we prove the Radon-Nikodym theorem on fuzzy measure space*
Keywords: *Fuzzy Measure, Fuzzy Integral **,** fuzzy signed measure**.***

I. Preliminaries

In this section, we the concepts of the family of subsets of a set will be given and some important properties of them, which are used in this paper.

Definition(1.1)[1]

A family F of subsets of a set Ω is called a field (or algebra) on Ω if,

(1) $\Omega \in F$ (2) If $A \in F$, then $A^c \in F$ (3) If $A_1, A_2, \cdots, A_n \in F$,then $\bigcup_{i=1}^{n} A_i \in F$,

If (3) is replaced by the closure under countable union, that is,

(4) If $A_n \in F$, $n = 1, 2, \cdots$,then $\bigcup_{n=1}^{\infty} A_n \in F$

F is called a σ -field (σ - algebra) on a set Ω .

A measurable Space is a pair (Ω, F) , where Ω is a set and F is a σ -field on Ω . A subset A of Ω is called measurable(measurable with respect to the σ -field F) if $A \in F$, i.e. any member of F is called a measurable set.

Remark

It is clear to show that . If $\{F_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of σ -field on a set Ω with $\Lambda \neq \phi$, then $F = \bigcap_{\lambda \in \Lambda} F_\lambda$ is a σ -field on Ω .

Definition(1.2)[1]

Let G be a family of subsets of a set Ω .The smallest σ -field containing G called the σ -field generated by G and it is denoted by $\sigma(G)$

Definition(1.3)[1]

Let (Ω, τ) be a topological space. The σ -field generated by τ is called the Borel σ -field and it is denoted by $\beta(\Omega)$, i.e. $\beta(\Omega) = \sigma(\tau)$ .The member of $\beta(\Omega)$ are called Borel sets of Ω .

Definition(1.4)[1]

Let G be a family of subsets of a set Ω , and let $A \subset \Omega$. The restriction (or trace) of G on A is the collection of all sets of the form $A \cap B$, were $B \in G$,and it is denoted by G_A (or $A \cap G$)

$G_A = A \cap G = \{A \cap B : B \in G\}$

G_A is a family of subsets of A . The σ -field $\sigma(G_A)$ generated by G_A some time denoted by $\sigma_A(A \cap G)$, i.e. $\sigma(G_A) = \sigma_A(A \cap G)$

Definition(1.5)[1]

Let $\{A_n\}$ be a sequence of subsets of a set Ω . The set of all points which belong to infinitely many sets of the sequence $\{A_n\}$ is called the upper limit (or limit superior) of $\{A_n\}$ and is denoted by A^* and defined by

$$A^* = \limsup_{n \to \infty} A_n = \{x \in A_n : \text{for infinitely many } n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \lim_{n \to \infty} \bigcup_{k=n}^{\infty} A_k$$

Thus $x \in A^*$ iff for all n, then $x \in A_k$ for some $k \geq n$

The lower limit (or limit inferior) of $\{A_n\}$, denoted by A_* is the set of all points which belong to almost all sets of the sequence $\{A_n\}$, and defined by

$$A_* = \lim_{n\to\infty} \inf A_n = \{x \in A_n : \text{ for all but finitely many } n\} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \lim_{n\to\infty} \bigcap_{k=n}^{\infty} A_k$$

Thus $x \in A_*$ iff for some n, then $x \in A_k$ for all $k \geq n$

Definition(1.6)[1]

A sequence $\{A_n\}$ of subsets of a set Ω is said to converge if $\lim_{n\to\infty} \sup_n A_n = \lim_{n\to\infty} \inf_n A_n = A$ (say), and A is said to be the limit of $\{A_n\}$, we write $A = \lim_{n\to\infty} A_n$ or $A_n \to A$

Definition(1.7)[1]

A sequence $\{A_n\}$ of subsets of a set Ω is said to be increasing if $A_n \subset A_{n+1}$ for $n = 1,2,\cdots$. It is said to be decreasing if $A_{n+1} \subset A_n$ for $n = 1,2,\cdots$. A monotone sequence of sets is one which either increasing or decreasing.

Definition(1.8)[1]

Let $\{A_n\}$ be sequence of subsets of a set Ω. We say that

(1) A_n increase to A, " write $A_n \uparrow A$ " if $\{A_n\}$ is an increasing sequence and $\bigcup_{n=1}^{\infty} A_n = A$.

(2) A_n decrease to A, " write $A_n \downarrow A$ " if $\{A_n\}$ is a decreasing sequence and $\bigcap_{n=1}^{\infty} A_n = A$.

II. Fuzzy Measures

In this section we study the fuzzy measure, fuzzy integral and prove some new properties of them.

Definition (2.1)[1]

Let (Ω, F) is measurable space. A set function $\mu : F \to [0,\infty]$ is said to be

- finite if, $\mu(A) < \infty$ for each $A \in F$
- Semi-finite, if for each $A \in F$ with $\mu(A) = \infty$, there exists $B \in F$ with $B \subseteq A$ and $0 < \mu(B) < \infty$
- Bounded if, $Sup\{|\mu(A)| : A \in F\} < \infty$
- σ - finite if, for each $A \in F$, there is a sequence $\{A_n\}$ of sets in F such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for all n.
- additive if, $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in F$ and $A \cap B = \phi$.
- finitely additive if, $\mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k)$, whenever $A_1, A_2, \cdots, A_n$ are disjoint sets in F.
- σ -additive (sometimes called completely additive, or Countably additive) f,

$$\mu(\bigcup_{n=1}^{\infty} A_k) = \sum_{n=1}^{\infty} \mu(A_n),$$

whenever $\{A_n\}$ is a sequence of disjoint sets in F.

- null additive if, $\mu(A \cup B) = \mu(A)$ whenever $A, B \in F$ such that $A \cap B = \phi$ and $\mu(B) = 0$.
- Measure, if μ is σ - additive and $\mu(A) \geq 0$ for all $A \in F$.
- Probability, if μ is a measure and $\mu(\Omega) = 1$
- Continuous from below at $A \in F$, if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$,where $A_n \in F$ for all $n \in \mathbb{N}$, and $A_n \uparrow A$
- Continuous from above at $A \in F$, if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$,where $A_n \in F$ for all $n \in \mathbb{N}$, and $A_n \downarrow A$

- Continuous at $A \in \mathcal{F}$, if it is continuous both from below and from above at A

Definition (2.2)[5]

Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mu : \mathcal{F} \to [0, \infty]$ is said to be a fuzzy measure on $(\Omega, \mathcal{F})$ if it satisfies the following properties:

(1) $\mu(\phi)=0$ (2) If $A, B \in \mathcal{F}$ and $A \subseteq B$, then $\mu(A) \le \mu(B)$

- A fuzzy measure space is a triple $(\Omega, \mathcal{F}, \mu)$,where $(\Omega, \mathcal{F})$ is measurable space and μ is a fuzzy measure on $(\Omega, \mathcal{F})$.
- A fuzzy measure μ on $(\Omega, \mathcal{F})$ is called regular if $\mu(\Omega) = 1$.

Remark

Every measure on a measurable space $(\Omega, \mathcal{F})$ is a fuzzy measure but the converse need not be true as follows :

Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\phi, \{1, 2\}, \{3, 4\}, \Omega\}$, $(\Omega, \mathcal{F})$ is a measurable space. Define $\mu : \mathcal{F} \to [0, \infty]$ by $\mu(\phi) = \mu(\{1,2\}) = \mu(\{3,4\}) = 0$, $\mu(\Omega) = 1$

Clearly μ is a fuzzy but not measure on $(\Omega, \mathcal{F})$.

Now we introduce the concept of the fuzzy integral with respect to fuzzy measure and prove some new properties of this integral. Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space , $F(\Omega)$ denote the set of all $\mathcal{F}$ -measurable functions, $F^{+}(\Omega) = \{f \in F(\Omega) : f \ge 0\}$. for any $f \in F(\Omega)$, we write $F_t = \{x \in \Omega : f(x) \ge t\}$, i.e. $F_t = f^{-1}[t, \infty)$ and $F_{\bar{t}} = \{x \in \Omega : f(x) > t\}$, where $t \in [0, \infty]$

Definition(2.3)[3]

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space and let $f \in F(\Omega)$, $A \in \mathcal{F}$ .The fuzzy integral of f over A with respect to μ is defined as

$$\int_A f \, d\mu = \int_0^\infty \mu(A \cap F_t) dt \ ,$$

The function f is said to be μ - integrable (or simply integrable if μ is understood) on A if $\int_\Omega f d\mu$ is finite

We now prove to some properties of the fuzzy integral

Theorem(2.4)

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space and let $f, g \in F(\Omega)$, $A, B \in \mathcal{F}$, $a \in \square$.

(1) If $\mu(A) = 0$, then $\int_A f \, d\mu = 0$

(2) If $f \le g$, then $\int_A f \, d\mu \le \int_A g \, d\mu$

(3) $\int_A a \, d\mu = \min\{a, \mu(A)\}$

(4) If $A \subseteq B$, then $\int_A f \, d\mu \le \int_B f \, d\mu$

(5) $\int_A \max\{f, g\} d\mu \ge \max\{\int_A f \, d\mu, \int_A g \, d\mu\}$

(6) $\int_A \min\{f, g\} d\mu \le \min\{\int_A f \, d\mu, \int_A g \, d\mu\}$

(7) $\int_{A \cup B} f \, d\mu \ge \max\{\int_A f \, d\mu, \int_B f \, d\mu\}$

(8) $\int_{A \cap B} f \, d\mu \le \min\{\int_A f \, d\mu, \int_B f \, d\mu\}$

Proof :

(1) Let $A \in \mathcal{F}$ and $\mu(A) = 0$

Since $A \cap F_t \subseteq A \Rightarrow \mu(A) \subseteq \mu(A \cap F_t)$

Since $\mu(A) = 0 \Rightarrow \mu(A \cap F_\alpha) = 0$

Since $\int_A f\, d\mu = \int_0^\infty \mu(A \cap F_t)dt \quad \Rightarrow \quad \int_A f\, d\mu = 0$

(2) Let $F_t = \{x \in \Omega : f(x) \ge t\}$ and $G_t = \{x \in \Omega : g(x) \ge t\}$

Since $f \le g \quad \Rightarrow \quad F_t \subseteq G_t \quad \Rightarrow \quad A \cap F_t \subseteq A \cap G_t \quad \Rightarrow \quad \mu(A \cap F_\alpha) \subseteq \mu(A \cap G_\alpha)$

Since $\int_A f\, d\mu = \int_0^\infty \mu(A \cap F_t)dt$ and $\int_A g\, d\mu = \int_0^\infty \mu(A \cap G_t)dt \Rightarrow \int_A f\, d\mu \le \int_A g\, d\mu$

(3) since $F_t = \{x \in \Omega : a \ge t\} = \begin{cases} \phi, & a < t \\ \Omega, & a \ge t \end{cases}$

Since $\int_A f\, d\mu = \int_0^\infty \mu(A \cap F_t)dt$

$\Rightarrow \quad \int_A f\, d\mu = \int_0^\infty \mu(A \cap \phi)dt \vee \int_0^\infty \mu(A \cap \Omega)dt = \int_0^\infty \mu(\phi)dt \vee \int_0^\infty \mu(A)dt = \min\{a, \mu(A)\}$

(4) since $A \subseteq B \quad \Rightarrow \quad A \cap F_t \subseteq B \cap F_t \quad \Rightarrow \quad \mu(A \cap F_t) \le \mu(B \cap F_t)$

$\int_A f\, d\mu = \int_0^\infty \mu(A \cap F_t)dt \le \int_0^\infty \mu(B \cap F_t)dt = \int_B f\, d\mu$

(5) since $f \le \max\{f, g\}$ and $g \le \max\{f, g\}$

$\int_A \max\{f, g\}d\mu \ge \int_A f\, d\mu$ and $\int_A \max\{f, g\}d\mu \ge \int_A g\, d\mu$

$\Rightarrow \quad \int_A \max\{f, g\}d\mu \ge \max\{\int_A f\, d\mu, \int_A g\, d\mu\}$

(6) since $f \ge \min\{f, g\}$ and $g \ge \min\{f, g\}$

$\int_A \min\{f, g\}d\mu \le \int_A f\, d\mu$ and $\int_A \min\{f, g\}d\mu \le \int_A g\, d\mu$

$\Rightarrow \quad \int_A \min\{f, g\}d\mu \le \min\{\int_A f\, d\mu, \int_A g\, d\mu\}$

(7) since $A \subseteq A \cup B$ and $B \subseteq A \cup B$

$\int_{A\cup B} f\, d\mu \ge \int_A f\, d\mu$ and $\int_{A\cup B} f\, d\mu \ge \int_B f\, d\mu \quad \Rightarrow \quad \int_{A\cup B} f\, d\mu \ge \max\{\int_A f\, d\mu, \int_B f\, d\mu\}$

(8) since $A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\int_{A\cap B} f\, d\mu \le \int_A f\, d\mu$ and $\int_{A\cap B} f\, d\mu \le \int_B f\, d\mu \quad \Rightarrow \quad \int_{A\cap B} f\, d\mu \le \min\{\int_A f\, d\mu, \int_B f\, d\mu\}$

Theorem(2.5)

Let $F_\alpha^n = \{x \in \Omega : f_n(x) \ge \alpha\}$, $F_{\bar{\alpha}}^n = \{x \in \Omega : f_n(x) > \alpha\}$ and $A \in \mathrm{F}$

(1) If $f_n \downarrow f$ on A , then $A \cap F_\alpha^n \downarrow A \cap F_\alpha$

(2) If $f_n \uparrow f$ on A , then $A \cap F_\alpha^n \uparrow A \cap F_{\bar{\alpha}}$

Proof :

(1) since $f_n \downarrow f \quad \Rightarrow \quad F_\alpha^n \downarrow F_\alpha \quad \Rightarrow \quad A \cap F_\alpha^n \downarrow A \cap F_\alpha$

(2) since $f_n \uparrow f \quad \Rightarrow \quad F_\alpha^n \uparrow F_{\bar{\alpha}} \quad \Rightarrow \quad A \cap F_\alpha^n \uparrow A \cap F_{\bar{\alpha}}$

Theorem(2.6)

Let μ be a continuous fuzzy measure on (Ω, F), and let $f \in F(\Omega)$, $A \in \mathrm{F}$. If $\int_A f\, d\mu = 0$, then $\mu(A \cap \{x \in \Omega : f(x) > 0\}) = 0$

Proof:

Assume $\mu(A \cap \{x \in \Omega : f(x) > 0\}) = b > 0$

Since $A \cap \{x \in \Omega : f(x) \geq \frac{1}{n}\} \uparrow A \cap \{x \in \Omega : f(x) > 0\}$. By using the continuity from below of μ, we have $\lim_{n \to \infty} \mu\{A \cap \{x \in \Omega : f(x) \geq \frac{1}{n}\}\} = b$.

So, there exists n_0 such that $\mu\{A \cap F_{\frac{1}{n_0}}\} = \mu\{A \cap \{x \in \Omega : f(x) \geq \frac{1}{n_0}\}\} \geq \frac{b}{2}$

Consequently, we have $\int_A f \, d\mu = \int_0^\infty \mu(A \cap F_t)dt \geq \min\{\frac{1}{n_0}, \frac{b}{2}\} > 0$.

This contradicts $\int_A f \, d\mu = 0$

Theorem(2.7)

$\int_A f \, d\mu = \int_0^\infty \mu(A \cap F_{\bar{t}})dt$,

Proof :

Since $F_{\bar{t}} \subseteq F_t \Rightarrow A \cap F_{\bar{t}} \subseteq A \cap F_t \Rightarrow \mu(A \cap F_{\bar{t}}) \leq \mu(A \cap F_t)$ for all $t \in [0, \infty]$

$\Rightarrow \int_A f \, d\mu = \int_0^\infty \mu(A \cap F_t)dt \geq \int_0^\infty \mu(A \cap F_{\bar{t}})dt$

Let $\int_A f \, d\mu > \int_0^\infty \mu(A \cap F_{\bar{t}})dt = b$

$\Rightarrow$ there exists $\varepsilon > 0$ such that $\int_0^\infty \mu(A \cap F_t)dt > b + \varepsilon$

$\Rightarrow$ there exists β such that $\mu(A \cap F_\beta) > b + \varepsilon \Rightarrow \mu(A \cap F_{\overline{b+\varepsilon}}) \geq \mu(A \cap F_\beta) = b + \varepsilon$

Therefore, $\mu(A \cap F_{\bar{t}}) \geq \mu(A \cap F_{\overline{b+\varepsilon}}) = b + \varepsilon > b$

This contradiction, so $\int_A f \, d\mu = \int_0^\infty \mu(A \cap F_{\bar{t}})dt$

Theorem(2.8)

The function f is integrable on A iff there exists $\beta \in [0, \infty)$ such that $\mu(A \cap F_\beta) < \infty$.

Proof :

Suppose that there exists $\beta \in [0, \infty)$ such that $\mu(A \cap F_\beta) < \infty$.

Take $\mu(A \cap F_\beta) = t \Rightarrow t < \infty$

$\Rightarrow \mu(A \cap F_t) \leq \mu(A \cap F_\beta)$ for any $t > \beta$

Since $\int_A f \, d\mu = \int_0^\infty \mu(A \cap F_t)dt \Rightarrow \int_A f \, d\mu < \infty$

Conversely, if for any $\alpha \in [0, \infty)$ such that $\mu(A \cap F_t) = \infty \Rightarrow \int_A f \, d\mu = \infty$

This contradiction , so exists $\beta \in [0, \infty)$ such that $\mu(A \cap F_\beta) < \infty$.

Definition(2.9)[7]

Let $(\Omega, \mathrm{F}, \mu)$ is the fuzzy measure space , $f, f_n \in F(\Omega)$, $n \in \mathbb{N}$ and $A \in \mathrm{F}$, we say that $\{f_n\}$

- Converges in fuzzy measure to f on A , denoted by $f_n \xrightarrow{\mu} f$ on A , if $\lim_{n \to \infty} \mu\{\{x \in \Omega : |f_n(x) - f(x)| < \varepsilon\} \cap A\} = \mu(A)$
  For any given $\varepsilon > 0$.
- Converges in fuzzy mean to f on A , denoted by $f_n \xrightarrow{m} f$ on A , if $\lim_{n \to \infty} \int_A |f_n - f| d\mu = 0$.

Theorem(2.10)

Let $(\Omega, \mathrm{F}, \mu)$ be a fuzzy measure space, $f, f_n \in F(\Omega)$, $n \in \mathbb{N}$ and $A \in \mathrm{F}$. If $f_n \xrightarrow{m} f$, then $f_n \xrightarrow{\mu} f$

III. Fuzzy signed measure

In this section, we introduce the concept of the fuzzy signed measure, fuzzy positive (rsp. negative) set,. Finally we prove the Hahn-Decomposition theorem on fuzzy measure space.

Definition(3.1)

Let (Ω, F) be a measurable space. A set function $\lambda : \mathrm{F} \to \mathrm{R}^*$ is called a fuzzy signed measure on (Ω, F) if it satisfies the following properties:

(1) $\lambda(\phi) = 0$ (2) If $A, B \in \mathrm{F}$ and $A \subseteq B$, then $\lambda(A) \le \lambda(B)$

Remark

Every fuzzy measure on a measurable space (Ω, F) is a fuzzy signed measure but the converse need not be true

Theorem(3.2)

Let $(\Omega, \mathrm{F}, \mu)$ be a fuzzy measure space, and let $f : \Omega \to \mathrm{R}$ be a measurable function such that $\int_\Omega f d\mu$ exists. Define

$$\lambda(A) = \int_A f d\mu$$

for all $A \in \mathrm{F}$. Then

(1) λ is a fuzzy signed measure

(2) If $f \ge 0$, then λ is a measure

(3) $\lambda^+(A) = \int_A f^+ d\mu$, $\lambda^-(A) = \int_A f^- d\mu$, $|\lambda|(A) = \int_A |f| d\mu$.

Proof :

(1) $\int_A f \, d\mu = \sup_{0 \le \alpha \le \infty} \{\min\{\alpha, \mu(A \cap F_\alpha)\}\}$

Since $\lambda(A) = \int_A f \, d\mu = \sup_{0 \le \alpha \le \infty} \{\min\{\alpha, \mu(A \cap F_\alpha)\}\}$

$\lambda(\phi) = \int_\phi f \, d\mu = \sup_{0 \le \alpha \le \infty} \{\min\{\alpha, \mu(\phi \cap F_\alpha)\}\} = 0$

Let $A, B \in \mathrm{F}$ such that $A \subseteq B \Rightarrow \int_A f \, d\mu \le \int_B f \, d\mu \Rightarrow \lambda(A) \le \lambda(B)$

$\Rightarrow$ λ is signed measure

(2) since $f \ge 0 \Rightarrow \int_A f \, d\mu \ge 0 \Rightarrow \lambda(A) \ge 0 \Rightarrow \lambda$ is a signed measure

Finally since $f = f^+ - f^-$ be arbitrary measurable function. Then $\lambda(A) = \int_A f^+ d\mu - \int_A f^- d\mu$

Since $\int_A f^+ d\mu < \infty$ or $\int_A f^- d\mu < \infty$, the result follows.

From the above proof we have λ is the difference of two measures λ^+ and λ^-, where $\lambda^+(A) = \int_A f^+ d\mu$, $\lambda^-(A) = \int_A f^- d\mu$, at least one of the measures λ^+ and λ^- must be finite.

Definition(3.3)

Let λ be a fuzzy signed measure on the measurable space (Ω, F). A set $A \in \mathrm{F}$ is said to be a fuzzy positive set (with respect to λ) if $\lambda(B) \ge 0$ for every measurable subset B of A.

Similarly, a set A is called a fuzzy negative set (with respect to λ) if $\lambda(B) \le 0$ for every measurable subset B of A.

- A set that is both fuzzy positive and fuzzy negative (with respect to λ) is called fuzzy null set, i.e. a measurable set is called a fuzzy null set iff every measurable subset of it has λ measure zero.

Remark

The distinction between a null set and a set of measure zero, while every null set must have measure zero, a set of measure zero may well be a union of two sets whose measure are not zero but are negative of each ether.

Theorem(3.4)

Let λ be a fuzzy signed measure on the measurable space (Ω, F), and let A be a measurable set.

(1) A is fuzzy positive iff for every measurable set B , $A \cap B$ is measurable and $\lambda(A \cap B) \geq 0$

(2) A is fuzzy negative iff for every measurable set B , $A \cap B$ is measurable and $\lambda(A \cap B) \leq 0$

Proof :

(1) Assume A is fuzzy positive and let B is a measurable set is measurable set

Since A is measurable set $\Rightarrow$ $A \cap B$ is measurable set.

Since A is positive set, $A \cap B \subseteq A$ and $A \cap B$ measurable $\Rightarrow$ $\lambda(A \cap B) \geq 0$

Conversely, let $A \cap B$ is measurable and $\lambda(A \cap B) \geq 0$ for every measurable set B

Let C be a measurable and $C \subseteq A \Rightarrow C = A \cap C \Rightarrow \lambda(C) = \lambda(A \cap C) \geq 0$

Theorem(3.5)

Let λ be a fuzzy signed measure on the measurable space (Ω, F)

(1) Every measurable subset of a fuzzy positive (rsp. fuzzy negative) set is fuzzy positive (rsp. fuzzy negative)

(2) The union of a countable of fuzzy positive (rsp. fuzzy negative) sets is fuzzy positive (rsp. fuzzy negative)

Proof :

(1) Let A be a measurable subset of a fuzzy positive set B , and let C be a measurable subset of $A \Rightarrow C \subseteq B$

Since B is fuzzy positive $\Rightarrow \lambda(C) \geq 0 \Rightarrow A$ is fuzzy positive

(2) Let $\{A_n\}$ be a sequence of fuzzy positive sets and let $A = \bigcup_{n=1}^{\infty} A_n$

Let B be a measurable subset of A . Put $B_n + B \cap A_n \cap A_{n-1}^c \cap \cdots A_1^c$

$\Rightarrow B_n$ is measurable subset of A_n and so $\lambda(B_n) \geq 0$.

Since the B_n are disjoint and $B = \bigcup_{n=1}^{\infty} B_n \supseteq B_n$, we have $\lambda(B) = \sum_{n=1}^{\infty} \lambda(B_n) \geq 0$

$\Rightarrow A$ is fuzzy positive.

Theorem(3.6) Hahn- Decomposition

Let λ be a fuzzy signed measure on the measurable space (Ω, F) . There is a fuzzy positive set A and a fuzzy negative set B with $A \cap B = \phi, \quad A \cup B = \Omega$

Proof :

Assume λ dose not take ∞

$v = \sup\{\lambda(A) : A$ is fuzzy positive set with respect $\lambda\}$

Since ϕ is fuzzy positive , then $v \geq 0$

Let $\{A_n\}$ be a sequence of fuzzy positive sets such that $v = \lim_{n \to \infty} \lambda(A_n)$

Set $A = \bigcup_{n=1}^{\infty} A_n$, by using part(2) of theorem (3.5), we have A is fuzzy positive , also $\lambda(A) \leq v$

Since $A \mid A_n \subset A \Rightarrow \lambda(A \mid A_n) \geq 0$ and $\lambda(A) \geq \lambda(A_n)$, so $\lambda(A) \geq v$

$\Rightarrow 0 \leq \lambda(A) = v < \infty \Rightarrow \lambda(A) \geq 0$

Let $B = A^c$. To prove B is fuzzy negative

Let C be fuzzy positive set and $C \subseteq B$, then $A \cap C = \phi$ and $A \cup C$ fuzzy positive set

$\Rightarrow \quad v \geq \lambda(A \cup C) \Rightarrow \quad \lambda(C) = 0$

Since $0 \leq v < \infty$, and therefore B does not contain a fuzzy positive subsets with a fuzzy positive measurements. And therefore does not positively measurements subsets, so B is fuzzy negative set.

Remarks

(1) The Hahn decomposition is not unique.

(2) The Hahn decomposition A, B give two measures λ^+ and λ^- defined by

$\lambda^+(C) = \lambda(A \cap C), \quad \lambda^-(C) = -\lambda(B \cap C)$

Notice that $\lambda^+(B) = 0$ and $\lambda^-(A) = 0$. Clearly $\lambda = \lambda^+ - \lambda^-$

IV. Relation Between Measures

In this section, we introduce the definition of singular set function and we discuss the relation between the types of fuzzy measures. Finally we prove the Radon-Nikodym theorem on fuzzy measure space.

Definition(4.1)

Let μ_1, μ_2 be two fuzzy measures on measurable space (Ω, F). We say that μ_1 is fuzzy singular with respect to μ_2 (written $\lambda \perp \mu$) if there are $A, B \in \mathrm{F}$ with $A \cap B = \phi, \quad A \cup B = \Omega$ and

$\mu_1(A) = 0, \quad \mu_2(B) = 0$

Remarks

(1) If μ_1, μ_2 are two fuzzy measures on measurable space (Ω, F), then $\mu_1 \perp \mu_2$ if there is a set $A \in \mathrm{F}$ such that $\mu_1(A) = 0, \quad \mu_2(A^c) = 0$

(2) μ_1 is fuzzy singular with respect to μ_2 iff μ_2 is singular with respect to μ_1, so we may say that μ_1 and μ_2 are mutually singular

Theorem(4.2) Jordan - Decomposition

Let λ be a fuzzy signed measure on the measurable space (Ω, F). There are two mutually singular measures λ^+ and λ^- such that $\lambda = \lambda^+ - \lambda^-$. This decomposition is unique.

Proof :

Since λ be a fuzzy signed measure on the measurable space (Ω, F).

By using Hahn- decomposition, there is a fuzzy positive set A and a fuzzy negative set B with $A \cap B = \phi, \quad A \cup B = \Omega$

Defined λ^+ and λ^- by $\lambda^+(C) = \lambda(A \cap C), \quad \lambda^-(C) = -\lambda(B \cap C)$ for all $C \in \mathrm{F}$

$\lambda^+(B) = \lambda(A \cap B) = \lambda(\phi) = 0, \quad \lambda^-(A) = -\lambda(B \cap A) = -\lambda(\phi) = 0 \quad \Rightarrow \quad \lambda^+ \perp \lambda^-$

Clearly $\lambda = \lambda^+ - \lambda^-$

Definition(4.3)

Let μ_1, μ_2 be two fuzzy measures on measurable space (Ω, F). We say that μ_1 is absolute continuous with respect to μ_2 (written $\mu_1 \ll \mu_2$) if $\mu_1(A) = 0$ implies $\mu_2(A) = 0$ for every $A \in \mathrm{F}$

Lemma(4.4)

Let $\{A_\alpha\}_{\alpha \in D}$ where D countable set of real numbers. Suppose $A_\alpha \subset A_\beta$ whenever $\alpha < \beta$. Then there is a measurable function f such that $f(x) \leq \alpha$ on A_α and $f(x) \geq \alpha$ on A_α^c

Proof :

For $x \in \Omega$, set $f(x) =$ first α such that $x \in A_\alpha \Rightarrow \quad f(x) = \inf\{\alpha \in D : x \in A_\alpha\}, \inf\{\phi\} = \infty$

- If $x \notin A_\alpha$, $x \notin A_\beta$ for any $\beta < \alpha$ and so, $f(x) \geq \alpha$
- If $x \in A_\alpha$, then $f(x) \leq \alpha$ provided we show that f is measurable

Claim: $\forall \gamma$ real , $\{x : f(x) < \gamma\} = \bigcup_{\beta < \gamma, \beta \in D} A_\beta$

If $f(x) < \gamma$, then $x \in D_\beta$ save $\beta < \gamma$. If $x \in A_\beta$, $\beta < \gamma \quad \Rightarrow \quad f(\alpha) < \gamma$

Remarks

(1) Suppose $\{A_\alpha\}_{\alpha \in D}$ as in lemma (4.4)but this time $\alpha < \beta$ implies only $\mu\{D_\alpha \mid A_\alpha\} = 0$.

Then there exists a measurable function f on Ω such that $f(x) \le \alpha$ a.e. on A_α and $f(x) \ge \alpha$ a.e. on A_α^c

(2) Suppose D is dense. Then the function in lemma (3.7) is unique and the function in (1) is unique μ a.e.

Theorem(4.5) Radon-Nikodym theorem

Let μ be a σ -finite fuzzy measure on the measurable space (Ω, F) , and λ be a fuzzy measure on (Ω, F) . Assume $\lambda << \mu$, then there is a nonnegative measurable function $f : \Omega \to \mathbb{R}^*$ such that

$$\lambda(A) = \int_A f d\mu$$

for all $A \in \mathrm{F}$. The function f is unique a.e. $[\mu]$. We call f is called the Radon-Nikodym derivative of λ with respect to μ and write $f = \dfrac{d\lambda}{d\mu}$

Proof :

Assume $\mu(\Omega) = 1$. Let $\lambda_\alpha = \lambda - \alpha\mu$, $\alpha \in \mathbb{R}$ $\Rightarrow$ λ_α is a fuzzy signed measure .

By using Hahn- Decomposition, There is a fuzzy positive set A_α and a fuzzy negative set B_α with

$A_\alpha \cap B_\alpha = \phi, \quad A_\alpha \cup B_\alpha = \Omega$

$\Omega = B_\alpha, \quad B_\alpha = \phi$, if $\alpha \le 0$

$B_\alpha \mid B_\beta = B_\alpha \cap B_\beta^c = B_\alpha \cap A_\beta$

Since A_α is fuzzy positive and B_α is fuzzy negative $\Rightarrow$ $\lambda_\beta(B_\alpha \mid B_\beta) \ge 0$ and $\lambda_\alpha(B_\alpha \mid B_\beta) \ge 0$

Since $\lambda_\alpha = \lambda - \alpha\mu$, where $\alpha \in \mathbb{R}$

$\Rightarrow$ $\lambda(B_\alpha \mid B_\beta) - \beta\mu(B_\alpha \mid B_\beta) \ge 0$ and $\lambda(B_\alpha \mid B_\beta) - \alpha\mu(B_\alpha \mid B_\beta) \le 0$

Thus $\beta\mu(B_\alpha \mid B_\beta) \le \lambda(B_\alpha \mid B_\beta) \le \alpha\mu(B_\alpha \mid B_\beta)$

Thus, if $\alpha < \beta$, we have $\mu(B_\alpha \mid B_\beta) = 0$

Thus, there exists a measurable function such that for all $\alpha \in \mathbb{R}$, $f(x) \ge \alpha$ a.e. on A_α and $f(x) \le \alpha$ a.e. on B_α .Since $B_0 = 0$, $f \ge 0$ a.e.

Let N be very large . Put $E_k = A \cap (\dfrac{B_{k+1}}{N} \mid B_{\frac{k}{N}})$, $k = 0,1,2,\cdots$, $E_\infty = \Omega \mid \bigcup_{k=0}^{\infty} B_{\frac{k}{N}}$

Then $E_0, E_1, \cdots, E_\infty$ are disjoint and $A = \bigcup_{k=0}^{\infty} E_k \cup E_\infty$.

We have, $\dfrac{k}{N} \le f(x) \le \dfrac{k+1}{N}$ a.e. and so, $\dfrac{k}{N}\mu(E_k) \le \int_{E_k} f(x)d\mu \le \dfrac{k+1}{N}\mu(E_k)$ (1)

Also $E_k \subset A_{\frac{k}{N}} \Rightarrow \dfrac{k}{N}\mu(E_k) \le \lambda(E_k)$ (2) and

$E_k \subset \dfrac{B_{k+1}}{N} \Rightarrow \lambda(E_k) \le \dfrac{k+1}{N}\mu(E_k)$ (3)

Thus

$\lambda(E_k) - \frac{1}{N}\mu(E_k) \le \frac{k}{N}\mu(E_k) \le \int_{E_k} f(x)dx \le \frac{k}{N}\mu(E_k) + \frac{1}{N}\mu(E_k) \le \lambda(E_k) + \frac{1}{N}\mu(E_k)$ on $E_\infty, f \equiv \infty$ a.e.

If $\mu(E_\infty) > 0$, then $\lambda(E_\infty) = 0$ since $(\lambda - \alpha\mu)(E_\infty) \ge 0$ for all α

If $\mu(E_\infty) = 0$, then $\lambda(E_\infty) = 0$, so either way: $\lambda(E_\infty) = \int_{E_\infty} f \, d\mu$

Add $\lambda(A) - \frac{1}{N}\mu(A) \le \int_A f \, d\mu \le \lambda(A) + \frac{1}{N}\mu(A) \Rightarrow \lambda(A) = \int_A f d\mu$ as $N \to \infty$

Since N is arbitrary, we are done

Uniqueness : If $\lambda(A) = \int_A g \, d\mu$ for all $A \in \mathrm{F}$

$\Rightarrow \lambda(A) - \alpha\mu(A) = \int_A (g - \alpha) d\mu$ for all α for all $A \subset A_\alpha$

Since $0 \le \lambda(A) - \alpha\mu(A) = \int_A (g - \alpha) d\mu$, we have $g - \alpha \ge 0$ a.e.$[\mu]$ on A_α

$\Rightarrow g \ge \alpha$ a.e.$[\mu]$ on A_α . similarly $\Rightarrow g \le \alpha$ a.e.$[\mu]$ on $B_\alpha \Rightarrow f = g$ a.e.

Suppose μ is σ - finite : $\lambda << \mu$. Let Ω_i be such that $X_i \cap X_j = \phi$, $\bigcup_i X_i = \Omega$, $\mu(X_i) < \infty$

Put $\mu_i(A) = \mu(A \cap X_i)$ and $\lambda_i(A) = \lambda(A \cap X_i)$. Then $\lambda_i << \mu_i$

$\Rightarrow$ there is $f_i \ge 0$ such that $\lambda_i(A) = \int_A f_i \, d\mu_i \Rightarrow \lambda(A \cap X_i) = \int_{A \cap X_i} f_i \, d\mu = \int_A dX_i \, d\mu$

Remark

The space needs to be σ -finite

For example : let $\Omega = [0,1]$, $\mathrm{F} = \beta([0,1])$, μ is counting measure

$\Rightarrow \mu$ is a measure on F , but not σ -finite

$\Rightarrow \lambda << \mu$, where λ is Lebesgue measure

If $\lambda(A) = \int_A f d\mu \Rightarrow f(x) = 0$ for all $x \in \Omega \Rightarrow \lambda = 0$. This contradiction

Theorem(4.6) Lebesgue Decomposition Theorem

Let μ and λ be a σ -finite fuzzy measures on the measurable space (Ω, F) .Then λ has a unique decomposition as $\lambda_1 + \lambda_2$, where λ_1 and λ_2 are fuzzy measures on (Ω, F) such that $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$.

Proof :

Let $v = \mu + \lambda \Rightarrow v$ is σ -finite fuzzy measure

Since $v(A) = 0 \Rightarrow \mu(A) = \lambda(A) = 0$

$\Rightarrow \mu \ll v$ and $\lambda << v$, by using Radon-Nikodym theorem there are a nonnegative measurable functions f, g such that $\mu(A) = \int_A f dv$ and $\lambda(A) = \int_A g dv$ for all $A \in \mathrm{F}$.

Let $A = \{f > 0\}$, $B = \{f = 0\} \Rightarrow A, B \in \mathrm{F}$ and $A \cap B = \phi$, $A \cup B = \Omega$ and $\mu(B) = 0$

Let $\lambda_1(C) = \lambda(B \cap C)$, then $\lambda_1(A) = v(B \cap A) = v(\phi) = 0$, so $\lambda_1 \perp \mu$

Set $\lambda_2(C) = \lambda(A \cap C) \Rightarrow \lambda_2(C) = \int_{A \cap C} g dv$

Clearly $\lambda_1 + \lambda_2 = \lambda$ and it only remains to show that $\lambda_1 << \mu$. Assume $\mu(A) = 0$

Then $\mu(A) = \int_A f dv = 0 \quad \Rightarrow \quad f = 0$ a.e. $[\lambda]$. $(f \geq 0)$ on A

Since $f > 0$ on $A \cap C \quad \Rightarrow \quad v(A \cap C) = 0$. Thus $\lambda_2(C) = \int_{A \cap C} g dv = 0 \Rightarrow \quad \lambda_2 \perp \mu$

To prove uniqueness, if $\lambda = \lambda_1 + \lambda_2 = \eta_1 + \eta_2$, where $\lambda_1, \eta_1 << \mu$ and $\lambda_2, \eta_2 \perp \mu$, then $\lambda_1 - \eta_1 = \eta_2 - \lambda_2$ is both absolutely continuous and singular with respect to μ, hence it is identically 0.

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