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The Generalized Srivastava $H_B^{(n)}$ And $H_C^{(n)}$ Functions Of Matrix Arguments In Complex Case

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Abstract: *In this paper we define the Srivastava functions $H_B^{(n)}$ and $H_C^{(n)}$ of matrix arguments in complex case, which are the generalization of the H_B and H_C functions.*

I. Introduction

We have already discussed the Srivastava's triple hypergeometric functions H_B and H_C of matrix arguments. In this paper we define the Srivastava functions $H_B^{(n)}$ and $H_C^{(n)}$ of matrix arguments in complex case. All matrices used in this paper are hermitian positive definite. All the matrices appearing in this paper are p x p real Hermitian positive definite and meanings of all the other symbols used are the same as in the work of Mathai [1, 2].

Function Of Matrix Argument In The Complex Case:

We consider real valued scalar function of a single matrix argument of the type $\tilde{Z} = \tilde{X} + i\tilde{Y}$ where $\tilde{X}$ and $\tilde{Y}$ are p x p matrices with real elements and $i = \sqrt{-1}$ as well as scaler functions of many matrices $\tilde{Z}_j$, j = 1, 2, …….K where each $\tilde{Z}_j$ is of the type $\tilde{Z}$ above in the real case. We confined our discussion to the situation where the argument matrix was real symmetric positive definite. This was done so that the fractional power of matrices and functions of such matrices could be uniquely defined. Corresponding properties are of we restrict to the class of Hermitian positive definite matrices.

Definition: Hermitian positive definite matrix due to Mathai [3], We will denote the conjugate of $\tilde{Z}$ by $\tilde{\tilde{Z}}$ if $\tilde{Z}$ hermitian, then $\tilde{Z} = \tilde{Z}^*$, that is

$$\tilde{Z} = \tilde{Z}^* \quad \Rightarrow \quad \tilde{X} + i\tilde{Y} = (\tilde{X} + i\tilde{Y})^* = \tilde{X}' + i\tilde{Y}'$$

$$\Rightarrow \quad \tilde{X} = \tilde{X}' \text{ and } \tilde{Y} = \tilde{Y}'$$

Thus $\tilde{X}$ is the symmetric and $\tilde{Y}$ is skew symmetric. Further if $\tilde{Z}$ is hermitian positive definite, then all the eigen values of $\tilde{Z}$ are real and positive.

Further, matrix variate gamma in the complex case is

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}\Gamma(\alpha)\Gamma(\alpha-1)......\Gamma(\alpha-\pi+1)}$$

We will use the notation $\tilde{Z} > 0$ to indicate that $\tilde{Z}$ is hermitian positive definite. Constant matrices will be written without a tilde whether the elements are real or complex unless it has to be emphasized that the matrix involved has complex elements. Then in that case a constant matrix will also be written with a tilde.

1. Definitions

2.1 The Srivastava function $H_B^{(n)}$ of matrix arguments,

$H_B^{(n)} = H_B^{(n)}(\alpha_1, \dots, \alpha_n; \gamma_1, \dots, \gamma_n; -\tilde{X}_1, \dots, -\tilde{X}_n)$

is defined as that class of functions which has the following matrix transform:

$$M[H_B^{(n)}] = \int_{\tilde{X}_1>0} \dots \int_{\tilde{X}_n>0} |\tilde{X}_1|^{\rho_1-p} \dots |\tilde{X}_n|^{\rho_n-p} \times H_B^{(n)}(\alpha_1, \dots, \alpha_n; \gamma_1, \dots, \gamma_n; -\tilde{X}_1, \dots, -\tilde{X}_n)\, d\tilde{X}_1 \dots d\tilde{X}_n$$

$$= \frac{\tilde{\Gamma}_p(\alpha_1 - \rho_1 - \rho_n)\, \tilde{\Gamma}_p(\alpha_2 - \rho_1 - \rho_2) \dots \tilde{\Gamma}_p(\alpha_n - \rho_{n-1} - \rho_n)}{\tilde{\Gamma}_p(\alpha_1)\, \tilde{\Gamma}_p(\alpha_2) \dots \tilde{\Gamma}_p(\alpha_n)\, \tilde{\Gamma}_p(\gamma_1 - \rho_1) \dots \tilde{\Gamma}_p(\gamma_n - \rho_n)} \times \tilde{\Gamma}_p(\gamma_1) \dots \tilde{\Gamma}_p(\gamma_n)\, \tilde{\Gamma}_p(\rho_1) \dots \tilde{\Gamma}_p(\rho_n) \tag{2.1}$$

for $\mathrm{Re}(\alpha_1 - \rho_1 - \rho_n, \alpha_2 - \rho_1 - \rho_2, \dots \alpha_n - \rho_{n-1} - \rho_n, \gamma_i - \rho_i, \rho_i) > p - 1$, where $i = 1, \dots, n$.

2.2 The Srivastava function $H_C^{(n)}$ of matrix arguments,

$H_C^{(n)} = H_C^{(n)}(\alpha_1, \dots, \alpha_n; \gamma; -\tilde{X}_1, \dots, -\tilde{X}_n)$

is defined as that class of functions which has the following matrix transform:

$$M[H_C^{(n)}] = \int_{\tilde{X}_1>0} \dots \int_{\tilde{X}_n>0} |\tilde{X}_1|^{\rho_1-p} \dots |\tilde{X}_n|^{\rho_n-p} \times H_C^{(n)}(\alpha_1, \dots, \alpha_n; \gamma; -\tilde{X}_1, \dots, -\tilde{X}_n)\, d\tilde{X}_1 \dots d\tilde{X}_n$$

$$= \frac{\tilde{\Gamma}_p(\alpha_1 - \rho_1 - \rho_n)\, \tilde{\Gamma}_p(\alpha_2 - \rho_1 - \rho_2) \dots \tilde{\Gamma}_p(\alpha_n - \rho_{n-1} - \rho_n)}{\tilde{\Gamma}_p(\alpha_1)\, \tilde{\Gamma}_p(\alpha_2) \dots \tilde{\Gamma}_p(\alpha_n)\, \tilde{\Gamma}_p(\gamma - \rho_1 - \dots - \rho_n)} \times \tilde{\Gamma}_p(\gamma)\, \tilde{\Gamma}_p(\rho_1) \dots \tilde{\Gamma}_p(\rho_n) \tag{2.2}$$

for $\mathrm{Re}(\alpha_1 - \rho_1 - \rho_n, \alpha_2 - \rho_1 - \rho_2, \dots \alpha_n - \rho_{n-1} - \rho_n, \gamma - \rho_1 - \dots - \rho_n, \rho_i) > p - 1$, where $i = 1, \dots, n$.

2. In This Section Of The Paper We Will Prove Three Results – One For The Function $H_B^{(n)}$ And Two For TheFunction $H_C^{(n)}$ Of Matrix Arguments In Complex Case.

Theorem 3.1:

$$H_B^{(n)}(\alpha_1, \dots, \alpha_n; \gamma_1, \dots, \gamma_n; -\tilde{X}_1, \dots, -\tilde{X}_n) = \frac{1}{\tilde{\Gamma}_p(\alpha_1)\, \tilde{\Gamma}_p(\alpha_2) \dots \tilde{\Gamma}_p(\alpha_n)} \int_{\tilde{T}_1>0} \dots \int_{\tilde{T}_n>0} e^{-\mathrm{tr}(\tilde{T}_1 + \dots + \tilde{T}_n)}\, |\tilde{T}_1|^{\alpha_1-p} |\tilde{T}_2|^{\alpha_2-p} \dots |\tilde{T}_n|^{\alpha_n-p}$$

$$\times {}_0F_1\left(; \gamma_1; -\tilde{T}_2^{\frac{1}{2}}\, \tilde{T}_1^{\frac{1}{2}}\, \tilde{X}_1\, \tilde{T}_1^{\frac{1}{2}}\, \tilde{T}_2^{\frac{1}{2}}\right) \times {}_0F_1\left(; \gamma_2; -\tilde{T}_3^{\frac{1}{2}}\, \tilde{T}_2^{\frac{1}{2}}\, \tilde{X}_2\, \tilde{T}_2^{\frac{1}{2}}\, \tilde{T}_3^{\frac{1}{2}}\right) \dots \times {}_0F_1\left(; \gamma_n; -\tilde{T}_1^{\frac{1}{2}}\, \tilde{T}_n^{\frac{1}{2}}\, \tilde{X}_n\, \tilde{T}_n^{\frac{1}{2}}\, \tilde{T}_1^{\frac{1}{2}}\right) d\tilde{T}_1 \dots d\tilde{T}_n$$

for $\mathrm{Re}(\alpha_i) > p - 1$, where $i = 1, \dots, n$. (3.1)

Proof: Taking the M-transform of right side of eq. (3.1) with respect to the variable $\tilde{X}_1, \dots, \tilde{X}_n$ and the parameters $\rho_1, \dots, \rho_n$ respectively, we have, $\int_{\tilde{X}_1>0} \dots \int_{\tilde{X}_n>0} |\tilde{X}_1|^{\rho_1-p} \dots |\tilde{X}_n|^{\rho_n-p} \times {}_0F_1\left(; \gamma_1; -\tilde{T}_2^{\frac{1}{2}}\, \tilde{T}_1^{\frac{1}{2}}\, \tilde{X}_1\, \tilde{T}_1^{\frac{1}{2}}\, \tilde{T}_2^{\frac{1}{2}}\right)$

$$\times {}_0F_1\left(; \gamma_2; -\tilde{T}_3^{\frac{1}{2}}\, \tilde{T}_2^{\frac{1}{2}}\, \tilde{X}_2\, \tilde{T}_2^{\frac{1}{2}}\, \tilde{T}_3^{\frac{1}{2}}\right) \dots \times {}_0F_1\left(; \gamma_n; -\tilde{T}_1^{\frac{1}{2}}\, \tilde{T}_n^{\frac{1}{2}}\, \tilde{X}_n\, \tilde{T}_n^{\frac{1}{2}}\, \tilde{T}_1^{\frac{1}{2}}\right) d\tilde{X}_1 \dots d\tilde{X}_n \tag{3.2}$$

Making use of transformations

$\tilde{Y}_1 = \tilde{T}_2^{\frac{1}{2}}\, \tilde{T}_1^{\frac{1}{2}}\, \tilde{X}_1\, \tilde{T}_1^{\frac{1}{2}}\, \tilde{T}_2^{\frac{1}{2}},\ \tilde{Y}_2 = \tilde{T}_3^{\frac{1}{2}}\, \tilde{T}_2^{\frac{1}{2}}\, \tilde{X}_2\, \tilde{T}_2^{\frac{1}{2}}\, \tilde{T}_3^{\frac{1}{2}},\ \tilde{Y}_3 = \tilde{T}_1^{\frac{1}{2}}\, \tilde{T}_n^{\frac{1}{2}}\, \tilde{X}_n\, \tilde{T}_n^{\frac{1}{2}}\, \tilde{T}_1^{\frac{1}{2}};$

in the last expression and using the M-transform of a ${}_0F_1$ function, we get

$$|\tilde{T}_1|^{-\rho_1-\rho_n} |\tilde{T}_2|^{-\rho_1-\rho_2} \dots |\tilde{T}_n|^{-\rho_{n-1}-\rho_n} \times \frac{\tilde{\Gamma}_p(\gamma_1) \dots \tilde{\Gamma}_p(\gamma_n)}{\tilde{\Gamma}_p(\gamma_1-\rho_1) \dots \tilde{\Gamma}_p(\gamma_n-\rho_n)} \tilde{\Gamma}_p(\rho_1) \dots \tilde{\Gamma}_p(\rho_n) \tag{3.3}$$

Which is to be substituted on the right side of eq. (3.1), followed by integrating out of $\tilde{T}_1,\dots,\tilde{T}_n$ by using a Gamma integral to achieve $\mathrm{M}\left[\mathrm{H}_{\mathrm{B}}^{(\mathrm{n})}\right]$ as given by eq. (2.1).

Theorem 3.2:

$$H_C^{(n)}\left(\alpha_1,\dots,\alpha_n;\gamma;-\tilde{X}_1,\dots,-\tilde{X}_n\right)=\frac{1}{\tilde{\Gamma}_p(\alpha_1)\dots\tilde{\Gamma}_p(\alpha_n)}\int_{\tilde{T}_1>0}\dots\int_{\tilde{T}_n>0}e^{-tr\,(\tilde{T}_1+\dots+\tilde{T}_n)}\times\left|\tilde{T}_1\right|^{\alpha_1-p}\dots\left|\tilde{T}_n\right|^{\alpha_n-p}$$
$$\times\,{}_0F_1\left(;\gamma_1;-\tilde{T}_2^{\frac{1}{2}}\,\tilde{T}_1^{\frac{1}{2}}\,\tilde{X}_1\,\tilde{T}_1^{\frac{1}{2}}\,\tilde{T}_2^{\frac{1}{2}}-\tilde{T}_3^{\frac{1}{2}}\,\tilde{T}_2^{\frac{1}{2}}\,\tilde{X}_2\,\tilde{T}_2^{\frac{1}{2}}\,\tilde{T}_3^{\frac{1}{2}}-\dots-\tilde{T}_1^{\frac{1}{2}}\,\tilde{T}_n^{\frac{1}{2}}\,\tilde{X}_n\,\tilde{T}_n^{\frac{1}{2}}\,\tilde{T}_1^{\frac{1}{2}}\right)d\tilde{T}_1\dots d\tilde{T}_n \qquad (3.4)$$

$for\ Re(\alpha_i)>p-1, where\ i=1,\dots,n.$

Proof: This theorem follow in the same manner as the previous theorem, except that use of eq. (3.2) is to be made here.

Theorem 3.3:

$$H_C^{(2m)}\left(\alpha_1,\dots,\alpha_{2m};\gamma;-\tilde{X}_1,\dots,-\tilde{X}_{2m}\right)$$
$$=\frac{1}{\tilde{\Gamma}_p(\alpha_1)\,\tilde{\Gamma}_p(\alpha_3)\dots\tilde{\Gamma}_p(\alpha_{2m-1})}\int_{\tilde{T}_1>0}\dots\int_{\tilde{T}_m>0}e^{-tr\,(\tilde{T}_1+\dots+\tilde{T}_m)}$$
$$\times\left|\tilde{T}_1\right|^{\alpha_1-p}\left|\tilde{T}_2\right|^{\alpha_3-p}\dots\left|\tilde{T}_{m-1}\right|^{\alpha_{2m-3}-p}\left|\tilde{T}_m\right|^{\alpha_{2m-1}-p}\varphi_2^{(m)}\left(\alpha_2,\alpha_4,\dots,\alpha_{2m};\gamma;-\tilde{T}_1^{\frac{1}{2}}\,\tilde{X}_1\,\tilde{T}_1^{\frac{1}{2}}\right.$$
$$-\tilde{T}_2^{\frac{1}{2}}\,\tilde{X}_2\,\tilde{T}_2^{\frac{1}{2}},-\tilde{T}_2^{\frac{1}{2}}\,\tilde{X}_3\,\tilde{T}_2^{\frac{1}{2}}-\tilde{T}_3^{\frac{1}{2}}\,\tilde{X}_4\,\tilde{T}_3^{\frac{1}{2}},\dots,-\tilde{T}_{m-1}^{\frac{1}{2}}\,\tilde{X}_{2m-3}\,\tilde{T}_{m-1}^{\frac{1}{2}}$$
$$\left.-\tilde{T}_m^{\frac{1}{2}}\,\tilde{X}_{2m-2}\,\tilde{T}_m^{\frac{1}{2}},-\tilde{T}_m^{\frac{1}{2}}\,\tilde{X}_{2m-1}\,\tilde{T}_m^{\frac{1}{2}},-\tilde{T}_1^{\frac{1}{2}}\,\tilde{X}_{2m}\,\tilde{T}_1^{\frac{1}{2}}\right)d\tilde{T}_1\dots d\tilde{T}_m$$
$$(3.5)$$

$for\ Re(\alpha_1,\alpha_3,\dots,\alpha_{2m-1})>p-1.$

Proof: Taking the M-transform of the right side of eq. (3.5) with respect to the variables $\tilde{X}_1,\dots,\tilde{X}_{2m}$ and the parameters $\rho_1,\dots,\rho_{2m}$ respectively, we obtain

$$\int_{\tilde{X}_1>0}\dots\int_{\tilde{X}_{2m}>0}\left|\tilde{X}_1\right|^{\rho_1-p}\dots\left|\tilde{X}_{2m}\right|^{\rho_{2m}-p}$$
$$\times\varphi_2^{(m)}\left(\alpha_2,\alpha_4,\dots\alpha_{2m};\gamma;-\tilde{T}_1^{\frac{1}{2}}\,\tilde{X}_1\,\tilde{T}_1^{\frac{1}{2}}-\tilde{T}_2^{\frac{1}{2}}\,\tilde{X}_2\,\tilde{T}_2^{\frac{1}{2}},-\tilde{T}_2^{\frac{1}{2}}\,\tilde{X}_3\,\tilde{T}_2^{\frac{1}{2}}\right.$$
$$-\tilde{T}_3^{\frac{1}{2}}\,\tilde{X}_4\,\tilde{T}_3^{\frac{1}{2}},\dots,-\tilde{T}_{m-1}^{\frac{1}{2}}\,\tilde{X}_{2m-3}\,\tilde{T}_{m-1}^{\frac{1}{2}}$$
$$\left.-\tilde{T}_m^{\frac{1}{2}}\,\tilde{X}_{2m-2}\,\tilde{T}_m^{\frac{1}{2}},-\tilde{T}_m^{\frac{1}{2}}\,\tilde{X}_{2m-1}\,\tilde{T}_m^{\frac{1}{2}},-\tilde{T}_1^{\frac{1}{2}}\,\tilde{X}_{2m}\,\tilde{T}_1^{\frac{1}{2}}\right)d\tilde{X}_1\dots d\tilde{X}_{2m}$$
$$(3.6)$$

Applying the transformations

$$\tilde{Z}_1=\tilde{T}_1^{\frac{1}{2}}\,\tilde{X}_1\,\tilde{T}_1^{\frac{1}{2}},\tilde{Z}_2=\tilde{T}_2^{\frac{1}{2}}\,\tilde{X}_2\,\tilde{T}_2^{\frac{1}{2}},\tilde{Z}_3=\tilde{T}_2^{\frac{1}{2}}\,\tilde{X}_3\,\tilde{T}_2^{\frac{1}{2}},\tilde{Z}_4=\tilde{T}_3^{\frac{1}{2}}\,\tilde{X}_4\,\tilde{T}_3^{\frac{1}{2}},\dots$$
$$\tilde{Z}_{2m-3}=\tilde{T}_{m-1}^{\frac{1}{2}}\,\tilde{X}_{2m-3}\,\tilde{T}_{m-1}^{\frac{1}{2}},\tilde{Z}_{2m-2}=-\tilde{T}_m^{\frac{1}{2}}\,\tilde{X}_{2m-2}\,\tilde{T}_m^{\frac{1}{2}},\tilde{Z}_{2m-1}=-\tilde{T}_m^{\frac{1}{2}}\,\tilde{X}_{2m-1}\,\tilde{T}_m^{\frac{1}{2}},\tilde{Z}_{2m}=-\tilde{T}_1^{\frac{1}{2}}\,\tilde{X}_{2m}\,\tilde{T}_1^{\frac{1}{2}}$$

to the last expression followed by the use of another set of transformation in it,

$\tilde{U}_1=Z_1,\tilde{U}_2=\tilde{Z}_1+\tilde{Z}_2;\tilde{U}_3=\tilde{Z}_3,\tilde{U}_4=\tilde{Z}_3+\tilde{Z}_4;\dots;\tilde{U}_{2m-3}=Z_{2m-3},$

$\tilde{U}_{2m-2}=\tilde{Z}_{2m-3}+\tilde{Z}_{2m-2};\tilde{U}_{2m-1}=\tilde{Z}_{2m-1},\tilde{U}_{2m}=\tilde{Z}_{2m-1}+\tilde{Z}_{2m}$

$with, d\tilde{U}_1d\tilde{U}_2=d\tilde{Z}_1d\tilde{Z}_2,d\tilde{U}_3d\tilde{U}_4=d\tilde{Z}_3d\tilde{Z}_4,\dots,d\tilde{U}_{2m-3}d\tilde{U}_{2m-2}=d\tilde{Z}_{2m-3}d\tilde{Z}_{2m-2},$

$d\tilde{U}_{2m-1}d\tilde{U}_{2m}=d\tilde{Z}_{2m-1}d\tilde{Z}_{2m};$

$where, 0<\tilde{U}_1<\tilde{U}_2,0<\tilde{U}_3<\tilde{U}_4,\dots,0<\tilde{U}_{2m-3}<\tilde{U}_{2m-2},0<\tilde{U}_{2m-1}<\tilde{U}_{2m};$

then integrating out the m variables $\tilde{U}_1, \tilde{U}_3, \dots, \tilde{U}_{2m-3}, \tilde{U}_{2m-1}$ in the ensuing expression by employing a type-1 Beta integral leads us to

$$\left|\tilde{T}_1\right|^{-\rho_1-\rho_{2m}} \left|\tilde{T}_2\right|^{-\rho_2-\rho_3} \dots \left|\tilde{T}_m\right|^{-\rho_{2m-2}-\rho_{2m-1}} \times \frac{\tilde{\Gamma}_p(\alpha_1-\rho_1-\rho_2)\,\tilde{\Gamma}_p(\alpha_4-\rho_3-\rho_4)\dots\tilde{\Gamma}_p(\alpha_{2m}-\rho_{2m-1}-\rho_{2m})}{\tilde{\Gamma}_p(\alpha_2)\tilde{\Gamma}_p(\alpha_4)\dots\tilde{\Gamma}_p(\alpha_{2m})\,\tilde{\Gamma}_p(\gamma-\rho_1-\cdots-\rho_{2m})}$$
$$\times\ \tilde{\Gamma}_p(\gamma)\,\tilde{\Gamma}_p(\rho_1)\dots\tilde{\Gamma}_p(\rho_{2m})$$

3.7)

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