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Multivariate Integral Representation Suggested By Laguerre And Jacobi Polynomials Of Matrix Argument In Complex Case

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Abstract: *In this paper we have introduced some new integrals involving multivariate polynomials suggested by Jacobi and Laguerre polynomials of real, symmetric and positive definite matrix (or complex symmetric matrix). All the results of this paper are new and have a wide range of application in the field of Mathematics Science.*

1. Introduction

In this paper some integrals associated with biorthogonal polynomial suggested by Jacobi and Laguerre polynomials of matrix argument have been evaluated with the help of the results given by Constantine [1] , James [2] and Subrahmaniam [3] .

The integral formulae, poised here are given in the form of theorems. A number of integrals have been deduced from these theorems. These integral formulae are applicable to various problems in multivariate distribution theory. It is also found useful in solving problems based on integral equations. The present study is in a way an extension to the line of approach intiated by Constantine [1] , Mathai [7] Subrahmaniam [3] . All matrices used in this paper are hermitian positive define.

Function Of Matrix Argument In The Complex Case:

We consider real valued scalar function of a single matrix argument of the type $\tilde{Z} = \tilde{X} + i\tilde{Y}$ where $\tilde{X}$ and $\tilde{Y}$ are p x p matrices with real elements and $i=\sqrt{-1}$ as well as scaler functions of many matrices $\tilde{Z}_j$, j = 1, 2, …….K where each $\tilde{Z}_j$ is of the type $\tilde{Z}$ above in the real case. We confined our discussion to the situation where the argument matrix was real symmetric positive definite. This was done so that the fractional power of matrices and functions of such matrices could be uniquely defined. Corresponding properties are of we restrict to the class of Hermitian positive definite matrices.

Definition : Hermitian positive definite matrix due to Mathai [19], We will denote the conjugate of $\tilde{Z}$ by $\tilde{\tilde{Z}}$ if $\tilde{Z}$ hermitian, then $\tilde{Z} = \tilde{Z}*$, that is

$$\tilde{Z} = \tilde{Z}* \quad \Rightarrow \quad \tilde{X} + i\tilde{Y} = (\tilde{X} + i\tilde{Y})* = \tilde{X}' + i\tilde{Y}'$$

$$\Rightarrow \quad \tilde{X} = \tilde{X}' \text{ and } \tilde{Y} = \tilde{Y}'$$

Thus $\tilde{X}$ is the symmetric and $\tilde{Y}$ is skew symmetric. Further if $\tilde{Z}$ is hermitian positive definite, then all the eigen values of $\tilde{Z}$ are real and positive. Further, matrix variate gamma in the complex case is

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}}\Gamma(\alpha)\Gamma(\alpha-1)......\Gamma(\alpha-\pi+1)$$

We will use the notation $\tilde{Z} > 0$ to indicate that $\tilde{Z}$ is hermitian positive definite. Constant matrices will be written without a tilde whether the elements are real or complex unless it has to be emphasized that the matrix involved has complex elements. Then in that case a constant matrix will also be written with a tilde.

Zonal Polynomial

Let $\tilde{V}_k$ be the vector space of homogeneous polynomial of degree k, then the Zonal polynomial $\tilde{C}_k(\tilde{X})$ is defined as the component of (tr $\tilde{X}^k$) in the subspace $\tilde{V}_k$. $\tilde{C}_k(\tilde{X})$ is also generalization of $\tilde{X}^k$. The exponential function has the following expansion

$$e^{tr(\tilde{X})} = \sum_{k=0}^{\infty}\frac{1}{k!}\left[tr(\tilde{X})\right]^k = \sum_{k=0}^{\infty}\sum_{K}\frac{\tilde{C}_K(\tilde{X})}{k!} \qquad ...(1.1)$$

The binomial expansion is the following for $I-\tilde{X}>0$ that is $\tilde{X}=\tilde{X}^*>0$ and all eigen values of $\tilde{X}$ are between 0 and 1.

$$\left|\det\left(I-\tilde{X}\right)\right|^{-\alpha} = \sum_{k=0}^{\infty}\sum_{K}\frac{(\alpha)_K}{k!}\tilde{C}_k(\tilde{X}), \qquad ...(1.2)$$

where

$$(\alpha)_K = \prod_{j=1}^{p}\left[\alpha-\frac{j-1}{2}\right]_{k_j},$$

with $K=(k_1,...,k_p), k_1+...+k_p = k$

$$\int_{O(\tilde{P})}\tilde{C}_K(\tilde{H}*\tilde{X}\tilde{H}\tilde{T})d\tilde{H} = \frac{\tilde{C}_K(\tilde{X})\tilde{C}_K(\tilde{T})}{\tilde{C}_K(I)} \qquad ...(1.3)$$

where I is the identity matrix, the integral is over the orthogonal group of p x p matrices and d$\tilde{H}$ is the invariate Harr measure. For detailed study consult Mathai [12].

2. Results Required In The Sequel:

The following results due to Subrahamaniam [3]

(i) $$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}...\int_{\tilde{S}^n>0}\prod_{i=1}^{n}e^{\sum tr\left(-\tilde{R}\tilde{S}^i\right)}\left|\tilde{S}^i\right|^{t^i-(p-1)}C_k\left(\tilde{S}^i\tilde{T}\right)d\tilde{S}^1 d\tilde{S}^2..d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\tilde{\Gamma}_p\left(t^i,K\right)\left|\tilde{R}\right|^{-t^i}\left\{C_k\left(\tilde{R}^{-1}\tilde{T}\right)\right\}^n \qquad ...(2.1)$$

where $\tilde{R}(t^i) > p-1$ and $\tilde{\Gamma}_p(t^i,k) = \prod^{\frac{p(p-1)}{2}}\prod_{f=1}^{m}\tilde{\Gamma}_p\left(t^i+k_j-\frac{j-1}{2}\right)$ ...(2.2)

The integration is over the space of p x p positive definite symmetric matrices. For $\tilde{Z}$ a complex symmetric matrix, and $\tilde{T}$ an arbitrary Complex Symmetric matrix.

(ii) $$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\cdots\int_{\tilde{S}^n>0}\prod_{i=1}^{n} e^{\sum tr\left(-\tilde{Z}\tilde{S}^i\right)}\left|\tilde{S}^i\right|^{t^i-(p-1)} C_k\left(\tilde{T}\left(\tilde{S}^i\right)^{-1}\right) d\tilde{S}^1 d\tilde{S}^2 \ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\tilde{\Gamma}_p\left(t^i, K\right)\left|\tilde{Z}\right|^{-t^i}\left\{C_k\left(\tilde{Z}\tilde{T}\right)\right\}^n \qquad \ldots(2.3)$$

where $\tilde{R}\left(t^i\right) > p-1+k_1;\ \tilde{Z}=\tilde{Z}'>0;\ \tilde{T}=\tilde{T}'>0;\quad R\left(\tilde{Z}\right)>0$

(iii) $$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\cdots\int_{\tilde{S}^n>0}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-(p-1)}\left|I+\tilde{S}^i\right|^{-\left(t^i+u\right)} C_k\left(\tilde{R}\tilde{S}^i\right) d\tilde{S}^1 d\tilde{S}^2 \ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\frac{\tilde{\Gamma}_p\left(t^i,k\right)\tilde{\Gamma}_p\left(u,-k\right)}{\tilde{\Gamma}_p\left(t^i+u\right)}\left\{C_k\left(\tilde{R}\right)\right\}^n \qquad \ldots(2.4)$$

where $\tilde{R}\left(t^i\right) > p-1;\tilde{R}\left(u^i\right) > p-1+k_1;\tilde{R}=\tilde{R}'>0;\ \tilde{R}\left(\tilde{R}\right)>0$

(iv) $$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\cdots\int_{\tilde{S}^n>0}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-(p-1)}\left|I+\tilde{S}^i\right|^{-\left(t^i+u\right)} C_k\left(\tilde{R}\left(\tilde{S}^i\right)^{-1}\right) d\tilde{S}^1 \ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\frac{\tilde{\Gamma}_p\left(t^i,-K\right)}{\tilde{\Gamma}_p\left(t^i+u\right)}\left\{C_k\left(\tilde{R}\right)\tilde{\Gamma}_p\left(u,k\right)\right\}^n \qquad \ldots(2.5)$$

A type-I beta integral can be given as follows:

(v) $$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\cdots\int_{\tilde{S}^n>0}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-(p-1)}\left|I+\tilde{S}^i\right|^{u-p} C_k\left(\tilde{R}\tilde{S}^i\right) d\tilde{S}^1 \ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\frac{\tilde{\Gamma}_p\left(t^i,K\right)}{\tilde{\Gamma}_p\left(t^i+u,K\right)}\left\{C_k\left(\tilde{R}\right)\tilde{\Gamma}_p\left(u\right)\right\}^n \qquad \ldots(2.6)$$

(vi) $$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\cdots\int_{\tilde{S}^n>0}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-(p-1)}\left|I+\tilde{S}^i\right|^{u-p} C_k\left(\tilde{R}\left(\tilde{S}^i\right)^{-1}\right) d\tilde{S}^1 \ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\frac{\tilde{\Gamma}_p\left(t^i,-K\right)}{\tilde{\Gamma}_p\left(t^i+u,K\right)}\left\{C_k\left(\tilde{R}\right)\tilde{\Gamma}_p\left(u\right)\right\}^n \qquad \ldots(2.7)$$

for $\tilde{R}\left(t^i\right) > (p-1)K_1;\tilde{R}\left(u\right) > (p-1);\tilde{R}\left(t^i+u\right) > (p-1)+K_1;k=k'>0$

follows:

$$\prod_{i=1}^{n}\frac{\tilde{\Gamma}_p\left(t^i,-K\right)}{\tilde{\Gamma}_p\left(t^i\right)} = \prod_{i=1}^{n}\frac{(-1)^k}{\left(p-t^i\right)_k} \qquad \ldots(2.8)$$

The result (2.8) is the direct generalization of the result [Ranville[13], p.p. 23, p.p. 32] to matrix variable case.

3. Main Results

Theorem 1: If $\tilde{R}$ and $\tilde{T}$ are arbitrary Complex symmetric matrices; then we have to prove that

$$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\cdots\int_{\tilde{S}^n>0} e^{\sum tr\left(\tilde{R}\tilde{S}^i\right)}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-(p-1)}\tilde{Z}_n^{\alpha^i}\left(\tilde{S}^i\tilde{T};l_i\right)d\tilde{S}^1\ldots d\tilde{S}^n$$

$$=\prod_{i=1}^{n}\tilde{\Gamma}_p\left(t^i,K\right)\left|\tilde{R}\right|^{-\sum t^i}\frac{\prod_p\left(\alpha^i+l_i n\right)}{\prod_p\left(\alpha^i\right)}F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}[+] \quad \ldots(3.1)$$

where $\tilde{R}\left(t^i\right)>p-1;\ \tilde{R}=\tilde{R}'>0;\ \tilde{T}=\tilde{T}'>0;\left\|\tilde{R}\right\|<1;\ \left\|\tilde{T}\right\|<1$

and throughout as and when i=1. 2. …n here, Also we know biorthogonal polynomial $\tilde{Z}_n^{\alpha}(\cdot)$ of matrix argument is given by

$$\tilde{Z}_n^{\alpha}\left(\tilde{S}\tilde{T};l\right)=\frac{\prod_p(\alpha+ln)}{\prod_p(\alpha)}\,{}_1F_{\alpha}[+] \quad \ldots(3.2)$$

Here; ${}_1F_1[+]={}_1F_1\left[-n;\frac{\alpha+p}{l},\ldots,\frac{\alpha+\frac{p(2l-1)}{2}}{l};\frac{1}{l}\left(\tilde{S}\tilde{T},l\right)\right]$

Also ${}_1F_1[+]=\sum_{k=0}^{n}\sum_{k}(-1)^k\binom{n}{k}\frac{C_k\left(\tilde{X}\right)}{\tilde{\Gamma}_p(lk+\alpha+1)} \quad \ldots(3.3)$

where ; K=l k and $\prod_p=\tilde{\Gamma}_p(\alpha+p)$

Again ; we have

$$\tilde{Z}_n^{\alpha_1}\left(\tilde{S}'\tilde{T};l_1\right)\ldots\tilde{Z}_n^{\alpha_n}\left(\tilde{S}^n\tilde{T};l_n\right)=\prod_{i=1}^{n}\frac{\prod_p\left(\alpha^i+l_i n\right)}{\prod_p\left(\alpha^i\right)}\times$$

$${}_1F_{l_1}\left[-n;\frac{\alpha_1+p}{l_1};\ldots,\frac{\alpha_1+\frac{(p+(2l_1-1))}{2}}{l_1};\frac{1}{l_1}\left(\tilde{S}^1\tilde{T},l_1\right)\right]\times$$

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$${}_1F_{l_n}\left[-n;\frac{\alpha_n+p}{l_n};\ldots,\frac{\alpha_n+\frac{(p+(2l_n-1))}{2}}{l_n};\frac{1}{l_n}\left(\tilde{S}^n\tilde{T},l_n\right)\right]$$

$$= \prod_{i=1}^{n} \frac{\prod_p \left(\alpha^i + l_i n\right)}{\prod_p \left(\alpha^i\right)} \text{ x}$$

$$F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\left[\begin{matrix} -;-n; & & -n; \\ -;\left(\frac{\alpha^1+p}{l_1}\right);\ldots,\left(\frac{\alpha^n+p}{l_n}\right);\left(\frac{1}{l_n}\left(\tilde{S}^n\tilde{T},l_n\right)\right) \end{matrix}\right]$$

where;

$$\left(\frac{\alpha^1+l_1}{l_1}\right)=\frac{\alpha^1+1}{l_1},\ldots,\frac{\alpha^1+l_1}{l_1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\frac{\alpha^n+l_n}{l_n}\right)=\frac{\alpha^n+1}{l_n},\ldots,\frac{\alpha^n+l_n}{l_n}$$

and $\left[\frac{1}{l_n}\left(\tilde{S}^n\tilde{T},l_n\right)\right]=\frac{1}{l_1}\left(\tilde{S}^1\tilde{T},l_1\right);\frac{1}{l_2}\left(\tilde{S}^2\tilde{T},l_2\right);\ldots;\frac{1}{l_n}\left(\tilde{S}^n\tilde{T},l_n\right)$

Proof: This integral identity is proved by considering the representation

$$C_k\left(\tilde{S}^1\right)=d_{k,k}\left|\tilde{S}^1_1\right|^{k_1-k_2},\ldots\left|\tilde{S}^1_p\right|^{k_p}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \ldots(3.4)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$C_k\left(\tilde{S}^n\right)=d_{k,k}\left|\tilde{S}^n_1\right|^{k_1-k_2},\ldots\left|\tilde{S}^n_m\right|^{k_m}$$

Where, $\tilde{S}^i_r=\left\{\begin{matrix}\tilde{S}^i_{ij} & ; i=1,2,3,\ldots r \\ & j=1,2,3,\ldots r\end{matrix}\right\}$

Also, for T-diagonal

$C_k(\tilde{T}) = d_{k.k}\ t_1^{k_1}t_2^{k_2}\ldots t_p^{k_p}$ …(3.5)

$C_k(\tilde{S}^1\tilde{T}) = d_{k.k}\ t_1^{k_1}\ldots t_p^{k_p}\left|\tilde{S}^1_1\right|^{k_1-k_2}\ldots\left|\tilde{S}^1_m\right|^{k_p}$ …(3.6)

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$C_k(\tilde{S}^n\tilde{T}) = d_{k.k}\ t_1^{k_1}\ldots t_p^{k_p}\left|\tilde{S}^n_1\right|^{k_1-k_2}\ldots\left|\tilde{S}^n_p\right|^{k_p}$

Starting with $\tilde{R}=I$ in the equation (3.1); we have

$$f(\tilde{T}) = \int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\dots\int_{\tilde{S}^n>0} e^{\sum tr(-\tilde{S}^i)}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-p}\tilde{Z}_n^{\alpha^i}\left(\tilde{S}^i\tilde{T};l_i\right)d\tilde{S}^1 d\tilde{S}^2\dots d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\frac{\prod_p\left(\alpha^i+l_i n\right)}{\prod_p\left(\alpha^i\right)}\sum_{k-0}^{n}\sum_{k}(-1)^k\binom{n}{k}$$

$$\frac{1}{\tilde{\Gamma}_p\left(l_i k+\alpha^i+1\right)}\int_{O(p)}\int_{O(p)}\dots\int_{O(p)}\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\dots\int_{\tilde{S}^n>0} e^{\sum tr(-\tilde{S}^i)}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-p}, C_k\left(\tilde{S}^i H^i\tilde{T}\tilde{H}^i\right)d\tilde{S}^1\dots d\tilde{S}^n d\tilde{H}^1\dots d\tilde{H}^n$$

$$f(\tilde{T}) = \eta\int_{O(p)}\int_{O(p)}\dots\int_{O(p)}\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\dots\int_{\tilde{S}^n>0} e^{\sum tr(-\tilde{S}^i)}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-p}, C_k\left(\tilde{S}^i H^i\tilde{T}\tilde{H}^i\right)d\tilde{S}^1\cdot d\tilde{S}^2\dots d\tilde{S}^n\cdot d\tilde{H}^1\dots d\tilde{H}^n$$

Here;

$$\sum_{k=0}^{n}\sum_{K} = \sum_{k=0}^{n}\dots\sum_{k=0}^{n}\sum_{k}\dots\sum_{k} \qquad \text{and}$$

$$\eta = \prod_{i=1}^{n}\frac{\prod_p\left(\alpha^i+l_i n\right)}{\prod_p\left(\alpha^i\right)}\sum_{k-0}^{n}\sum_{K}(-1)^k\binom{n}{k}\frac{1}{\tilde{\Gamma}_m\left(l_i k+\alpha^i+1\right)} \qquad \dots(3.7)$$

$$f(\tilde{T}) = \int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\dots\int_{\tilde{S}^n>0} e^{tr(-\tilde{S}^i)}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-p}\left\{\frac{C_k\left(\tilde{S}^i\right)C_k\left(\tilde{T}\right)}{C_k(I)}\right\}d\tilde{S}^1\dots d\tilde{S}^n$$

We have $f(\tilde{T}) = n\left\{\frac{f(I)C_k\left(\tilde{T}\right)}{C_k(I)}\right\}$ … (3.8)

Assuming that $\tilde{T}$ is diagonal; we can compare the term involving $t_1^{K_1}, t_2^{K_2}, \dots, t_m^{K_m}$;

$$\frac{f(I)}{C_k(I)} = \eta\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\dots\int_{\tilde{S}^n>0} e^{\sum tr(\tilde{R}\tilde{S}^i)}\prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-p}\chi_1\chi_2\dots\chi_n\, d\tilde{S}^1\dots d\tilde{S}^n \qquad \dots(3.9)$$

Where ;

$$\chi_1 = \left|S_1^i\right|^{k_1-k_2}$$

$$\chi_2 = \left|S_2^i\right|^{k_1-k_2}$$

....

....

$$\chi_n = \left|S_n^i\right|^{k_1-k_2}$$

Transforming $\tilde{S}^1$ to $\tilde{U}^{-1}\tilde{U}^1$; $\tilde{S}^2$ to $\tilde{U}^{-2}\tilde{U}^2$ and so on being upper triangular matrix;

We have

Since $J^i = 2^p\prod_{j=1}^{p}\tilde{U}_{jj}^i$ and $\left|\tilde{S}_i^i\right| = \prod_{j=1}^{i}\left(\tilde{U}_{jj}^i\right)^2$

The RHS of equation (3.8) changes to

$$\prod_{i=1}^{n}\prod_{j<i}\underbrace{\int\int\ldots\int}_{n\ times}\exp\left(\left(-\tilde{U}_{jj}^{i}\right)^{2}\right)d\tilde{U}_{jj}^{i}\prod_{j=1}^{p}\underbrace{\int\int\ldots\int}_{n\ times}\exp\left(\left(\tilde{U}_{jj}^{i}\right)^{2}\right)^{\left(\tilde{v}_{j}^{i}-1\right)}\times$$

$$\exp\left(\left(-\tilde{U}_{jj}^{i}\right)^{2}\right)d\left(\tilde{U}_{jj}^{i}\right)^{2}=\prod^{\frac{p(p-1)}{2}}\prod_{j=1}^{m}\tilde{\Gamma}_{p}\left(V_{j}^{i}\right)$$

where , $\tilde{V}_{j}=t^{i}+k_{j}-(j-1)/2$

This yields the result for $\tilde{R}$ =I in equation (3.1)

$$f\left(\tilde{T}\right)=\eta\prod_{i=1}^{n}\tilde{\Gamma}_{p}\left(t^{i},k\right)\left\{C_{k}\left(\tilde{T}\right)\right\}^{n} \qquad \ldots(3.10)$$

If $\tilde{R}\neq I;$ we can always reduce (3.1) to the form $\tilde{R}$ =I

Writing $\tilde{S}^{i}=\tilde{R}^{1/2}\tilde{S}^{i}\tilde{R}^{1/2}$

In the equation (3.1) Re calling that the Jacobian in $\left|\tilde{R}\right|^{-2p}$ and that

$$C_{k}\left(\tilde{R}^{-1/2}\tilde{S}^{i}\tilde{R}^{-1/2}\tilde{T}\right)=C_{k}\left(\tilde{S}^{i}\tilde{T}^{*}\right)$$

Where $\tilde{T}^{*}=\tilde{R}^{-1/2}\tilde{T}\tilde{R}^{-1/2};$ we have the result

$$f\left(\tilde{T}\right)=\eta\int_{\tilde{S}^{1}>0}\int_{\tilde{S}^{2}>0}\ldots\int_{\tilde{S}^{n}>0}e^{\sum tr\left(-\tilde{S}^{i}\right)}\prod_{i=1}^{n}\left|\tilde{S}^{i}\right|^{t^{i}-p}C_{k}\left(\tilde{S}^{i}\tilde{T}^{*}\right)d\tilde{S}^{1}\ldots d\tilde{S}^{n}$$

$$f\left(\tilde{T}\right)=\eta\,\frac{f(I)C_{k}\left(\tilde{T}^{*}\right)}{C_{k}(I)} \qquad \ldots(3.11)$$

which proves (3.1); if we recall that

$$C_{k}\left(\tilde{T}^{*}\right)=C_{k}\left(\tilde{R}^{-1/2}\tilde{T}^{i}\tilde{R}^{-1/2}\right)=C_{k}\left(\tilde{R}^{-1}\tilde{T}\right)$$

Now, the proof on equation (3.1) is given as

$$\int_{\tilde{S}^{1}>0}\int_{\tilde{S}^{2}>0}\ldots\int_{\tilde{S}^{n}>0}e^{\sum tr\left(\tilde{R}\tilde{S}^{i}\right)}\prod_{i=1}^{n}\left|\tilde{S}^{i}\right|^{t^{i}-(p-1)}\tilde{Z}_{n}^{\alpha^{i}}\left(\tilde{S}^{i}\tilde{T};l_{i}\right)d\tilde{S}^{1}\ldots d\tilde{S}^{n}$$

$$=\eta\prod_{i=1}^{n}\tilde{\Gamma}_{p}\left(t^{i},k\right)\left|\tilde{R}\right|^{-t^{i}}\left\{C_{k}\left(\tilde{R}^{-1}\tilde{T}\right)\right\}^{n}$$

On putting the value of η from equation (3.7); the above

$$=\prod_{i=1}^{n}\tilde{\Gamma}_{p}\left(t^{i},k\right)\left|\tilde{R}\right|^{-\sum t^{i}}\frac{\prod_{p}\left(\alpha^{i}l_{i}n\right)}{\prod_{p}\left(\alpha^{i}\right)}F_{0;l_{1};\ldots l_{n}}^{0;1;\ldots 1}[||]$$

Here ; $F_{0;l_{1};\ldots l_{n}}^{0;1;\ldots 1}[||]$

$$= F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\left[\begin{matrix} -;-n;-n; & & -n; & \\ -; \left(\dfrac{\alpha^1+p}{l_1}\right);\ldots, & \left(\dfrac{\alpha^n+p}{l_n}\right); & \left(\dfrac{1}{l_n}\left(\left(\tilde{R}^{-1}\tilde{T}, l_n\right)\right)\right) & \end{matrix}\right]$$

Theorem 2. For $\tilde{P}$ a Complex symmetric matrix $\tilde{R}(\tilde{P}) > 0$ and $\tilde{T}$ an arbitrary complex symmetric matrix $\tilde{R}(t^i) > (p-1)$ then we have

$$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\ldots\int_{\tilde{S}^n>0} e^{\sum \mathrm{tr}\left(\tilde{P}\tilde{S}^i\right)} \prod_{i=1}^{n}\left|\tilde{S}^i\right|^{t^i-p} \tilde{Z}_n^{\alpha^i}\left(\left(\tilde{S}^i\right)^{-1}\tilde{T}; l_n\right) d\tilde{S}^1\ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n}\tilde{\Gamma}_p\left(t^i, k\right)\left|\tilde{P}\right|^{-\sum t^i} \frac{\prod_p\left(\alpha^i + l_i n\right)}{\prod_p\left(\alpha^i\right)} F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\left[\begin{matrix}-\\-\end{matrix}\right] \qquad \ldots(3.12)$$

Here;

$$F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\left[\begin{matrix}-\\-\end{matrix}\right] = {}_1Fl_1\left[-n; \frac{\alpha^1+p}{l_1},\ldots,\frac{\alpha^1+l_1+p}{l_1}; \left(\frac{1}{l_1}\left(\left(\tilde{Z}^1\tilde{T}; l_1\right)\right)\right)\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\times {}_1Fl_n\left[-n; \frac{\alpha^n+p}{l_n},\ldots,\frac{\alpha^n+l_n+p}{l_n}; \left(\frac{1}{l_n}\left(\left(\tilde{Z}^n\tilde{T}; l_n\right)\right)\right)\right]$$

Also, $F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\left[\begin{matrix}-\\-\end{matrix}\right] = F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\left[\begin{matrix}-\\-\end{matrix}\right]\left[\begin{matrix} -;-n;-n;\ldots\ldots\ldots -n; \\ -;\left(\dfrac{\alpha^1+l_1+p}{l_1}\right);\ldots,\left(\dfrac{\alpha^n+l_n+p}{l_n}\right);\left(\dfrac{1}{l_n}\left(\left(\tilde{R}^n\tilde{T}, l_n\right)\right)\right)\end{matrix}\right]$

$$\left(\frac{\alpha^1+l_1+p}{l_1}\right) = \frac{\alpha^1+p-1}{l_1},\ldots\frac{\alpha^1+l_1+p-1}{l_1}$$

Here ,

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(\frac{\alpha^n+l_n+p}{l_n}\right) = \frac{\alpha^n+p-1}{l_n},\ldots\frac{\alpha^n+l_n+p-1}{l_n}$$

and $\left[\dfrac{1}{l_n}\left(\tilde{Z}^n\tilde{T}, l_n\right)\right] = \dfrac{1}{l_1}\left(\tilde{Z}^1\tilde{T}, l_1\right); \dfrac{1}{l_2}\left(\tilde{Z}^2\tilde{T}, l_2\right);\ldots\dfrac{1}{l_n}\left(\tilde{Z}^n\tilde{T}, l_n\right)$

4. Generalization Of Biothogonal Polynomial Suggested By Laguerre Polynomial Of Matrix Argument With Generalized Beta Integral:

Definition : Subramanium[3], Konhhauser, J.D.E [14,15] gives the generalization of the above integral, which uses the following integral

$$\int_{\sum L_r^1 = P^1} \int_{\sum L_r^2 = P^2} \cdots \int_{\sum L_r^n = P^n} \prod_{i=1}^{n} \varphi^{(i)}\left(\tilde{L}_1, \ldots, \tilde{L}_q\right) \prod_{r=1}^{q} d\tilde{L}_r^1 \ldots d\tilde{L}_r^n$$

$$= \prod_{i=1}^{n} \left\{ \prod_{r=1}^{q} \frac{\tilde{\Gamma}_p\left(a_r^i\right)}{\tilde{\Gamma}_p\left(a^i\right)} \right\} \left|\tilde{P}^i\right|^{a^i - p} \left|I - \tilde{P}^i\right|^{b^i - p} \qquad \ldots(4.1)$$

Here $\varphi^{(i)}\left(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_q\right) = \prod_{i=1}^{n} \left\{ \prod_{r=1}^{q} \left|L_r^i\right|^{a_r^i - p} \left|I - \sum_{r=1}^{q} \tilde{L}_r^i\right|^{b^i - p} \right\}$

The generalized biorthogonal polynomial suggested by Laguerre polynomial of matrix argument of positive symmetric matrix $\tilde{R}$ is

Here $\varphi^{(i)}\left(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_q\right) = \prod_{i=1}^{n} \left\{ \prod_{r=1}^{q} \left|L_r^i\right|^{a_r^i - p} \left|I - \sum_{r=1}^{q} \tilde{L}_r^i\right|^{b^i - p} \right\}$

The generalized biorthogonal polynomial suggested by Laguerre polynomial of matrix argument of positive symmetric matrix $\tilde{R}$ is

$$\prod_{i=1}^{n} \tilde{L}_p^i \tilde{Z}_n^{\alpha^i} \tilde{R} \sum_{r=1}^{q} \tilde{L}_p^i ; l_i = \prod_{i=1}^{n} \frac{\prod_p\left(\alpha^i + l_i n\right)}{\prod_p\left(\alpha^i\right)} \times F_{0;l_1;\ldots l_n}^{0;1;\ldots 1}\begin{bmatrix} * \\ - \end{bmatrix}$$

Here $F_{0;l_1;\ldots l_n}^{0;1;\ldots 1}\begin{bmatrix} * \\ - \end{bmatrix} = F_{0;l_1;\ldots l_n}^{0;1;\ldots 1}\left[\begin{matrix} -;-n;-n;\ldots \qquad -n; \\ -;\left(\frac{\alpha^1 + l_1 + p}{l_1}\right);\ldots;\left(\frac{\alpha^n + l_n + p}{l_n}\right);-;\left(\frac{1}{l_n}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^n, l_n\right)\right)\end{matrix}\right]$

...(4.2)

where $\left(\frac{1}{l_n}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^i ; l_n\right)\right) = \frac{1}{l_1}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^i ; l_1\right);\ldots;\frac{1}{l_n}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^i ; l_n\right)$

De waal's [16] result for any positive definite symmetric matrix $\tilde{R}$ is that

$$\int_{0<\sum_r \tilde{L}_r^1 < I} \int_{0<\sum_r \tilde{L}_r^2 < I} \cdots\cdots \int_{0<\sum_r \tilde{L}_r^n < I} \prod_{i=1}^{n} \varphi^{(i)}\left(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_q\right) C_K\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^i\right) \times$$

$$d\tilde{L}_1^1 \ldots d\tilde{L}_q^n = \prod_{i=1}^{n} \left\{ \frac{\tilde{\Gamma}_p\left(a^i\right) \prod_{r=1}^{q} \tilde{\Gamma}_p\left(a_r^i\right)}{\tilde{\Gamma}_p\left(a^i + b^i\right)} \right\} \times \left\{ \frac{\left(a^i\right)_k}{\left(a^i + b^i\right)_k} \right\} \times \left\{ C_k\left(\tilde{R}\right) \right\}^n \qquad \ldots(4.3)$$

Replace Zonal polynomial $C_k\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^n\right)$ by generalized biorthogonal polynomial suggested by Laguerre polynomial $\tilde{Z}_n^{\alpha^i}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^n, l_n\right)$ of matrix argument; then L.H.S. of equation (4.3) becomes

$$\int_{0<\sum_r \tilde{L}_r^1 < I} \int_{0<\sum_r \tilde{L}_r^2 < I} \cdots\cdots \int_{0<\sum_r \tilde{L}_r^n < I} \prod_{i=1}^{n} \varphi^{(i)}\left(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_q\right) \tilde{Z}_n^{\alpha^i}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^n, l_n\right) \times d\tilde{L}_1^1 \ldots d\tilde{L}_q^n$$

$$= \int_{0<\sum_r \tilde{L}_r^1 < I} \int_{0<\sum_r \tilde{L}_r^2 < I} \cdots\cdots \int_{0<\sum_r \tilde{L}_r^n < I} \prod_{i=1}^{n} \varphi^{i}\left(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_q\right) \frac{\prod_p \left(\alpha^i + l_i n\right)}{\prod_p \left(\alpha^i\right)} \times$$

$$\sum_{k=0}^{n} \sum_{k} (-1)^k \binom{n}{k} \frac{C_k\left(\tilde{R}\sum_r \tilde{L}_r^n\right)}{\tilde{\Gamma}_p\left(l_i k + \alpha^i + 1\right)} dL_1^1 \ldots dL_q^n$$

on putting the value of $\tilde{Z}_n^{\alpha^i}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^i, l_n\right)$ from equation (4.2) ; with yields.

$$= \eta \prod_{i=1}^{n} \left\{ \frac{\tilde{\Gamma}_p\left(a^i\right) \prod_{r=1}^{q} \tilde{\Gamma}_p\left(a_r^i\right)}{\tilde{\Gamma}_p\left(a^i + b^i\right)} \right\} \times \left\{ \frac{\left(a^i\right)_k}{\left(a^i + b^i\right)_k} \right\} \times \left\{ C_k\left(\tilde{R}\right) \right\}^n$$

where η is defined by equation (3.7).

Theorem 3 : For any positive definite symmetric matrix $\tilde{R}$ and for $\tilde{R}\left(a^i\right) > p-1; \tilde{R}\left(b^i\right) > p-1; \tilde{R}\left(a^i + b^i\right) > p-1;$ we have

$$\int_{0<\sum_r \tilde{L}_r^1 < I} \int_{0<\sum_r \tilde{L}_r^2 < I} \cdots\cdots \int_{0<\sum_r \tilde{L}_r^n < I} \prod_{i=1}^{n} \varphi^{(i)}\left(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_q\right) \tilde{Z}_n^{\alpha^i}\left(\tilde{R}\sum_{r=1}^{q}\tilde{L}_r^i, l_i\right) \times d\tilde{L}_1^1 \ldots d\tilde{L}_q^n$$

$$= \prod_{i=1}^{n} \left\{ \frac{\tilde{\Gamma}_p\left(\alpha^i + l_i n\right)}{\tilde{\Gamma}_p\left(\alpha^i\right)} \right\} \cdot \prod_{r=1}^{q} \tilde{\Gamma}_p\left(a_r^i\right) \times \left\{ \frac{\tilde{\Gamma}_p\left(a^i\right)_k}{\tilde{\Gamma}_p\left(a^i + b^i\right)_k} \right\} F \begin{matrix} 0;1;\ldots 1 \\ 0; l_1; \ldots l_n \end{matrix} \begin{bmatrix} * \\ - \end{bmatrix} \qquad \ldots(4.4)$$

Proof : To prove theorem 9; we note that left hand side can be written as

$$\int_0^1 \int_0^1 \underset{n\,\text{times}}{\cdots} \int_0^1 \prod_{i=1}^{n} f\left(\tilde{P}^{(i)}\right) \tilde{Z}_n^{\alpha^i}(RP^i, l_i)\, d\tilde{P}^i \ldots d\tilde{P}^n$$

Where, $f(\tilde{P}^i)$ is given by the R. H. S. in (4.4) this can be written as the R.H.S. of (4.4) by using the equation (3.20); the quantity on the R.H.S. can be written as

$$\underbrace{\int_0^1\int_0^1 \cdots \int_0^1}_{n\,\text{times}} \prod_{i=1}^{n} f\left(\tilde{P}^{(i)}\right) \tilde{Z}_n^{\alpha^i}(\tilde{R}\tilde{P}^i, l_i)\, d\tilde{P}^i \ldots d\tilde{P}^n$$

$$= \eta \prod_{i=1}^{n}\left\{\prod_{r=1}^{q}\tilde{\Gamma}_p(a_r^i)\left[\frac{\tilde{\Gamma}_p(a^i,k)}{\tilde{\Gamma}_p(a^i+b^i,k)}\right]\right\}\left\{C_k(\tilde{R})\right\}^n$$

$$= \prod_{i=1}^{n}\prod_{r=1}^{q}\left\{\frac{\prod_p(\alpha^i+l_i n)}{\prod_p(\alpha^i)}\right\}\tilde{\Gamma}_p(a_r^i)\left[\frac{\tilde{\Gamma}_p(a^i,k)}{\tilde{\Gamma}_p(a^i+b^i,k)}\right]\sum_{k=o}^{n}\sum_{K}(-1)^k\binom{n}{k}\frac{C_k(\tilde{R})}{\tilde{\Gamma}_p(1+\alpha^i+l_i n)}$$

$$= \prod_{i=1}^{n}\prod_{r=1}^{q}\left\{\frac{\prod_p(\alpha^i+l_i n)}{\prod_p(\alpha^i)}\right\}\tilde{\Gamma}_p(a_r^i)\left[\frac{\tilde{\Gamma}_p(a^i,k)}{\tilde{\Gamma}_p(a^i+b^i,k)}\right]F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\begin{bmatrix} * \\ + \end{bmatrix}$$

Here; $F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\begin{bmatrix} * \\ - \end{bmatrix} = F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\begin{bmatrix} -;-n;-n;\ldots;-n \\ -;\left(\frac{\alpha^1+p}{l_1}\right);\ldots;\left(\frac{\alpha^n+p}{l_n}\right);\left(\frac{1}{l_n}(\tilde{R},l_n)\right)\end{bmatrix}$

Since $(a^i)_k = \dfrac{\tilde{\Gamma}_p(a^i,k)}{\tilde{\Gamma}_p(a^i)}$

Hence ; the result proved.

Theorem 4: For any positive definite symmetric $\tilde{R}$ and $\tilde{R}(a^i) > p-1$; $\tilde{R}(b^i) > p-1$; $\tilde{R}(a^i+b^i) > p-1$; $\tilde{R}(a_r^i) > p-1$; where $r = 1,2,\ldots,q$ and $i = 1,2,\ldots,n$; we have

$$\int_{0<\sum_r L_r^1<1}\int_{0<\sum_r L_r^2<1}\cdots\int_{0<\sum_r L_r^n<1}\prod_{i=1}^{n}\varphi^i(\tilde{L}_1,\tilde{L}_2,\ldots,\tilde{L}_q)\, Z_n^{\alpha^i}\left(\tilde{R}\left[\sum_{r=1}^{q}\tilde{L}_r^i\right]^{-1}\tilde{L}_r^i; l_i\right)d\tilde{L}_1^1\ldots d\tilde{L}_q^n$$

$$= \prod_{i=1}^{n}\prod_{r=1}^{q}\left\{\frac{\prod_p(\alpha^i+lin)}{\prod_p(\alpha^i)}\right\}\tilde{\Gamma}_p(a_r^i)\left[\frac{\tilde{\Gamma}_p(a^i,k)}{\tilde{\Gamma}_p(a^i+b^i,-k)}\right]F^{0;1;\ldots 1}_{0;l_1;\ldots l_n}\begin{bmatrix} * \\ + \end{bmatrix} \quad \ldots(4.5)$$

5. Integral Involving Biorthogonal Polynomial Suggested By Jacobi Polynomial Of Positive Symmetric Definite MatrixWith Zonal Polynomial

Theorem 5 :

If $\tilde{R}$ is complex symmetric and $\tilde{T}$ is an arbitary complex symmetric $p \times p$ matrices, then

$$\int_{\tilde{S}^1>0}\int_{\tilde{S}^2>0}\cdots\int_{\tilde{S}^n>0}\prod_{i=1}^{n} e^{\sum tr(\tilde{R}\tilde{S}^i)}\left|\tilde{S}^i\right|^{t^i-p}\left|\tilde{W}_\gamma^i\right|^{(\alpha,\beta)}\left(I-2\tilde{S}^i\tilde{T}, l_i\right)d\tilde{S}^1\ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n} \tilde{\Gamma}_p(t^i, k) |\tilde{R}|^{-t^i} \frac{\prod_p(\alpha^i + \gamma)}{\prod_p(\alpha^i)} \times F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}[\$\$] \qquad \ldots(5.1)$$

Where, $F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}[\$\$]$

$$= F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}\left[\begin{matrix} -;-\gamma^1,-\gamma^2,\ldots, \qquad -\gamma^n\ ; \\ -;\Delta(l_i,-\alpha^i+\beta^1+\gamma^1+p;\ldots;\Delta(l_n,-\alpha^n+p);) \\ ((\tilde{S}^i\tilde{T})^{l_i}) \end{matrix}\right]$$

Here Biorthogonal polynomial suggested by Jacobi polynomial of matrix argument is given by

$$\tilde{W}_{\gamma}^{(\alpha^i,\beta)}(\tilde{S}^i\tilde{T}, l_i) = \frac{\prod_p(\alpha^i + \gamma)}{\prod_p(\alpha^i)} \times F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}[\$] \qquad \ldots(5.2)$$

Here $F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}[\$]$

$$= F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}\left[\begin{matrix} -;-\gamma^1,-\gamma^2,\ldots, \qquad -\gamma^n; \\ -;\Delta(l_i,-\alpha^i+\beta^1+\gamma^1+p;\ldots;\Delta(l_n,-\alpha^n+p);) \\ ((\tilde{S}^i\tilde{T})^{l_i}) \end{matrix}\right] \qquad \ldots(5.3)$$

and $\prod_p(\alpha^i) = \prod_p(\alpha^i + p)$

Proof : The proof of this theorem is quite obvious as given in theorem 1.

Theorem 6 : For $\tilde{P}$ a Complex Symmetric $R(\tilde{P}) > 0$ and $\tilde{T}$ an arbitrary complex symmetric matrix; than for ; $\tilde{R}(t^i) > p$;

$$\int_{\tilde{S}^1>0} \int_{\tilde{S}^2>0} \cdots \int_{\tilde{S}^n>0} \prod_{i=1}^{n} e^{\sum tr(-\tilde{P}\tilde{S}^i)} |\tilde{S}^i|^{t^i-p} |\tilde{W}_{\gamma}^{i}|^{(\alpha^i,\beta)} (I - 2(\tilde{S}^i)^{-1}\tilde{T}, l_i) d\tilde{S}^1 \ldots d\tilde{S}^n$$

$$= \prod_{i=1}^{n} \tilde{\Gamma}_p(t^i, -k) |\tilde{P}|^{-t^i} \frac{\prod_p(\alpha^i + \gamma)}{\prod_p(\alpha^i)} \times F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}\begin{bmatrix} \wedge \\ \wedge \end{bmatrix} \qquad \ldots(5.4)$$

where, $F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}\begin{bmatrix} \wedge \\ \wedge \end{bmatrix} =$

$$= F^{0;l_1 \quad ;\ldots l_n}_{0;l_1+1;\ldots l_n+1}\left[\begin{matrix} -;-\gamma^1,-\gamma^2,\ldots, \qquad -\gamma^n \\ -;\Delta(l_i,-\alpha^i+\beta^1+\gamma^1+p;\ldots;\Delta(l_n,-\alpha^n+p);) \\ ((\tilde{P}^i\tilde{T})^{l_i}) \end{matrix}\right]$$

Proof : The proof follow exactly along the lines of the one for (3.1); we use the identity

$$C_k\left(\left(\tilde{S}^1\right)^{-1}\tilde{T}\right)=d_{k,k}t_1^{k_1},\ldots t_p^{k_p}\left|\tilde{S}^1{}_1\right|^{-(k_1+k_2)}\ldots\left|\tilde{S}^1{}_p\right|^{-k_p}$$

………………………………………………

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$$C_k\left(\left(\tilde{S}^n\right)^{-1}\tilde{T}\right)=d_{k,k}t_1^{k_1},\ldots t_p^{k_p}\left|\tilde{S}^n{}_p\right|^{-(k_1+k_2)}\ldots\left|\tilde{S}^n{}_p\right|^{-k_p}$$

The rest of the proof is quite obvious.

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