IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 5 Ver. III (Sep. - Oct. 2015), PP 60-63
www.iosrjournals.org

Left Multiplicative Generalized Derivations on Right Ideal in Semi Prime Rings

Dr. C. Jaya Subba Reddy[1], V.Vijay kumar[2], S.MallikarjunaRao[3],
[1,2,3]*Department of Mathematics, S.V.University, Tirupati-517502,AndhraPradesh,India.*

Abstract: *Let R be a ring. A map $F: R \to R$ is called left multiplicative generalized derivation if $F(xy) = d(x)y + xF(y)$ is fulfilled for all x, y in R.Where $d: R \to R$ is any map (not necessarily derivative or an additive mapping).The following results are proved:*

(i) $F(xy) \pm xy \in Z,$
(ii) $F(xy) \pm yx \in Z,$
(iii) $F(x)F(y) \pm xy \in Z,$
(iv) $F(x)F(y) \pm yx \in Z$ *, for all* $x, y \in S.$

Key words: *Semi prime ring, Multiplicative generalized derivation, Left multiplicative generalized derivation.*

I. Introduction

Let R be an associative ring. The center of R is denoted by Z. For $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anticommutator $xy + yx$. we shall make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is prime if for $a, b \in R, aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime if for $a \in R, aRa = (0)$ implies $a = 0$. An additive map d from R to R is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let $F: R \to R$ be a map associated with another map $g: R \to R$ so that $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$. If F is additive and g is a derivation of R, then F is said to be a generalized derivation of R that was introduced by Breasar [3]. In [7], Hvala gave the algebraic study of generalized derivations of prime rings. The concept of multiplicative derivations appears for the first time in the work of Daif [4] and it was motivated by the work of Martindale [8]. The notion of multiplicative derivation was extended in Daifas follows. A map $F: R \to R$ is called a multiplicative generalized derivations if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In this definition we consider that d is any map (not necessarily a additive). To give its precise definition, we make a slight generalization of Daif and Tammam-El-Sayiad's definition for multiplicative generalized derivation was extended Daif(see [5]). A map $F: R \to R$ (not necessarily additive) is called multiplicative generalized derivation if $F(xy) = F(x)\, y + x\, g(y)$ for all $x, y \in R$, where g is any map (not necessarily derivation or additive map). A map $F: R \to R$ (not necessarily additive) is called left multiplicative generalized derivation if $F(xy) = g(x)\, y + x\, F(y)$ forall $x, y \in R$ where g is any map (not necessarily derivation or additive map).BasudebDhara and Shakir Ali [2] have studied multiplicative generalized derivation in prime rings and semi prime rings. In this paper we extended some results left multiplicative generalized derivation in semi prime rings.

II. Main result

Theorem 1: Let R be a semiprime ring , S be a non zero right ideal of R and $F: R \to R$ be aleft multiplicative generalized derivation associated with the map $g: R \to R$. If $F(xy) \pm xy \in Z$ for all $x, y \in S$, then $[g(x), x]S = (0)$ for all $x \in S$.

Proof: First we assume that $F(xy) - xy \in Z$ for all $, y \in S$. (1)

Now we replace x with zx in (1), where $z \in S$ and then we get

$F(zxy) - zxy = g(z)xy + zF(xy) - zxy = g(z)xy + z(F(xy) - xy) \in Z$ for all $x, y, z \in S$. (2)

From (1) & (2) we have

$g(z)xy \in Z \Rightarrow [g(z)xy, z] = 0$ for all $x, y, z \in S$. (3)

Substitutes yr for y in (3), where $r \in R$,we obtain

$0 = [g(z)xyr, z] = g(z)xy[r, z] + [g(z)xy, z]r = g(z)xy[r, z]$ (4)

Taking $y = yg(z)$, we get

$g(z)xyg(z)[r, z] = 0$,(5)

Which implies $g(z)xyRg(z)[r, z] = 0$ for all $x, y, z \in S$and $r \in R$.

Interchanging x and y and then subtracting one from the other, we get

$g(z)[x, y]Rg(z)[r, z] = 0$for all $x, y, z \in S$.

In particular, $g(z)[x,y]Rg(z)[x,z] = 0$ for all $x,y,z \in S$.
The semi primeness of R yields that $g(z)[x,z] = 0$ for all $x,z \in S$.
Thus we have $g(z)[x,z] = 0$ for all $x,z \in S$.(6)
Left multiplying (6) by z, we get
$zg(z)[x,z] = 0$ for all $x,z \in S$. (7)
Replace x by zx in (6) to get

$$g(z)[zx,z] = g(z)z[x,z] = 0$$

$\Rightarrow g(z)z[x,z] = 0$ for all $x,z \in S$. (8)
Now (7) and (8) together imply that
$[g(z),z][x,z] = 0$ for all $x,z \in S$. (9)
Replacing x by xg(z) in the last expression, we obtain

$$[g(z),z][xg(z),z] = 0$$

$$[g(z),z]x[g(z),z] + [g(z),z][x,z]g(z) = 0$$

$\Rightarrow [g(z),z]x[g(z),z] = 0$ for all $x,z \in S$.
$[g(z),z]SR[g(z),z]S = 0$ for all $z \in S$.
Hence, the semi primeness of R forces that $[g(z),z]S = 0$ for all $z \in S$.
In a similar manner we can prove by assuming $F(xy) + xy \in Z$ for all $x,y \in S$, that $[g(x),x]S = (0)$ for all $x \in S$.

Corollary 1: Let R be a semiprime ring and $F: R \to R$ be left multiplicative generalized derivation associated with the map $g: R \to R$. If $F(xy) \pm xy \in Z$ for all $x,y \in R$, then $[g(x),x] = (0)$ for all $x \in R$.

Theorem 2: Let R be a semiprime ring, S be a nonzero right ideal of R and $F: R \to R$ be a left multiplicative generalized derivation associated with the map $g: R \to R$. If $F(xy) \pm yx \in Z$ for all $x,y \in S$, then $[x,S]x \subseteq Z$ for all $x \in S$ and $[g(x),x]S = (0)$ for all $x \in S$. Moreover, if R is 3-torsion free, then $[S,S]S = (0)$

Proof: First we consider that
$F(xy) - yx \in Z$ for all $x,y \in S$. (10)
In the above relation replacing y with xy and x with x^2, respectively and then subtracting one from the other, we obtain
$(F(x^2y) - xyx) - (F(x^2y) - yx^2) \in Z$. (11)
This implies that $[y,x]x \in Z$ for all $x,y \in S$. Thus for all $x \in S$, $[x,S]x \subseteq Z$.
Now substituting zx for x in (10), where $z \in S$, we get
$F(zxy) - yzx = g(z)xy + zF(xy) - yzx = z(F(xy - yx) + [z,y]x + g(z)xy \in Z$ (12)
Commuting both sides of (12) with z and then using (10), we obtain
$[[z,y]x,z] + [g(z)xy,z] = 0$ for all $x,y,z \in S$.(13)
Replacing y with zy in (13), we get

$$\big[[z,zy]x,z\big] + [g(z)xzy,z] = 0$$

$$[z[z,y]x,z] + [g(z)xzy,z] = 0$$

$z\big[[z,y]x,z\big] + [g(z)xzy,z]0 = 0$ for all $x,y,z \in S$.(14)
Left multiplying (13) by z and then subtracting it from (14), we get

$$z\big[[z,y]x,z\big] + z[g(z)xy,z] - (z\big[[z,y]x,z\big] + [g(z)xzy,z]) = 0$$

$\big[[g(z)x,z]y,z\big] = 0$ for all $x,y,z \in S$. (15)
Replacing y with yr, $r \in R$ in the above relation and then using(15), we have

$$\big[[g(z)x,z]yr,z\big] = \big[[g(z)x,z]y,z\big]r + [g(z)x,z]y[r,z] = [g(z)x,z]y[r;z] = 0$$

$[g(z)x,z]y[r,z] = 0$ (16)
In particular, for $r = g(z)x$, we have
$[g(z)x,z]y[g(z)x,z] = 0$ for all $x,y,z \in S$.
Since S is right ideal of R, it follows that
$[g(z)x,z]yR[g(z)x,z]y = (0)$ for all $x,y,z \in S$.
Since R is semiprime ring,
$[g(z)x,z]y = 0$ for all $x,y,z \in S$.
Now replacing x with $xg(z)$, we get
$[g(z)xg(z),z]y = 0$(17)
That is, $\big(g(z)xg(z)z - zg(z)xg(z)\big)y = 0$ for all $x,y,z \in S$. (18)
Now we put $x = xg(z)u$ where $u \in S$, and then obtain
$\big(g(z)xg(z)ug(z)z - zg(z)xg(z)ug(z)\big)y = 0$ for all $x,y,z \in S$. (19)
$\big(g(z)xzg(z)ug(z) - g(z)xzg(z)ug(z)\big)y = 0$ (20)
$\big(g(z)x[g(z),z]ug(z)\big)y = 0$ for all $x,y,z \in S$. (21)
This implies $[g(z),z]x[g(z),z]u[g(z),z]y = 0$ for all $x,y,z \in S$, and so
$[g(z),z]^3S = (0)$ for all $z \in S$.

Since a semiprime ring contains no nonzero nilpotent right ideals, see[6] it follows that$[g(z),z]S=(0)$ for all$z\in S$,as we desired.
Next assume that R is 3-torsion free. Then $[x,S]x\subseteq Z$ for all $x\in S$ yields$[y,z]x+[x,z]y\in Z$ for all $x,y,z\in S$.
Replacing y with x^2, it reduces to
$[x^2,z]x+[x,z]x^2\in Z$for all $x,z\in S$.
This implies, by using the fact $[x,S]x\subseteq z$ for all $x\in R$, that $3[x,S]x^2\subseteq Z$.
Since R is 3- torsion free,$[x,S]x^2\subseteq Z$ for all $x\in S$.
Commuting both sides with S, we obtain $\left[[x,S]x^2,S\right]=(0)$, that gives
$[x,S]x[x,S]=(0)$for all $x\in S$ and so
$(0)=([x,S]x)^2$for all $x\in S$.
Since the center of a semiprime rings contains no nonzero nilpotent elements,
$(0)=[x,S]x$for all $x\in S$
This yields$(0)=[x,S^2]x=[x,S]Sx$ for all $x\in S$
Since S is right ideal of R, it follows that
$(0)=[x,S]SR[x,S]S$for all $x\in S$.
The semi primeness of R yields $(0)=[x,S]S$ for all$x\in S$,as we desired.
The same argument can be adapted if $F(xy)+yx\in Z$ for all $x,y\in S$.

Corollary 2: Let R be semiprime ring and $F:R\to R$ be a left multiplicative generalized derivation associated with the map $g:R\to R$. If $F(xy)\pm yx\in Z$ for all $x,y\in R$, then $[x,R]x\subseteq Z$ for all $x\in R$ and $[g(x),x]=0$ for all $x\in R$. Moreover, if$R$ is 3- torsion free , then R is commutative.

Theorem 3: Let R be semi prime ring, S be a nonzero right ideal of R and $F:R\to R$ be a left multiplicative generalized derivation associated with the map $g:R\to R$. If $F(x)F(y)\pm xy\in Z$ for all $x,y\in S$, then $[g(x),x]S=(0)$ for all $x\in S$.

Proof: We begin with the situation
$F(x)F(y)-xy\in Z$for all $x,y\in S$. (22)
Replacing x with the $zx,z\in S$,we have
$F(zx)F(y)-zxy\in S$ (23)
Which gives$\big(g(z)x+zF(x)\big)F(y)-zxy\in Z$ for all $,y,z\in S$. (24)
Commuting both sides of (24) with z and using the (22), we get
$[g(z)xF(y),z]=0$for all $x,y,z\in S$. (25)
Now putting $x=xz$ in the above relation we obtain,
$[g(z)xzF(y),z]=0$for all $x,y,z\in S$. (26)
Now putting y$=zy$ in (25), we get

$$[g(z)xF(zy),z]=0$$

$\big[g(z)x\big(g(z)y+zF(y)\big),z\big]=0$for all $x,y,z\in S$.
$[g(z)xg(z)y,z]+[g(z)xzF(y),z]=0$for all $x,y,z\in S$ (27)
Using of (26), the above relation reduces to
$[g(z)xg(z)y,z]=0$for all $x,y,z\in S$. (28)
In (28),we replace y with $yg(z)$ and then using(28), wee obtain
$g(z)xg(z)y[g(z),z]=0$for all $x,y,z\in S$. (29)
This implies $[g(z),z]x\{g(z),z]y[g(z),z]=0$ for all $x,y,z\in S$.
That is,$([g(z),z]S)^3=(0)$ for all$z\in S$.
Since R is semiprime, it contains nononzero nilpotent right ideals, implying $[g(z),z]S=0$ for all $z\in S$, as we desired.
By the same argument, we may obtain the same conclusion when $F(x)F(y)+xy\in Z$ for all $x,y\in S$.

Corollary 3: Let R be a semiprime ring and $F:R\to R$ be a left multiplicative generalized derivation associated with the map$g:R\to R$ If $F(x)F(y)\pm xy\in Z$ for all$x,y\in R$,then $[g(x),x]=0$ for all $x\in R$.

Theorem 4: Let R be a semiprimering ,S be a non-zero right ideal of R and $F:R\to R$ be a left multiplicative generalized derivation associated with the map $g:R\to R$. If $F(x)F(y)\pm yx\in Z$ for all $x,y\in Z$, then $[g(x),x]S=0$ for all $x\in S$.

Proof: First weconsider the case
$F(x)F(y)-yx\in Z$for all $x,y\in S$. (30)
Replacing $x$with zx ,we get

$$F(zx)F(y)-yzx\in Z$$

$\big(g(z)x+zF(x)\big)F(y)-yzx\in Z$for all $x,y,z\in S$. (31)
This gives $z(F(x)F(y)-yx)+[z,y]x+g(z)xF(y)\in Z$ for all $x,y,z\in S$. (32)
Commuting both sides of (32) with z and then using (30), it reduces to

$[g(z)xF(y),z]+\big[[z,y]x,z\big]=0$for all $x,y,z\in S$. (33)
Putting $y=zy$ in the above relation we get
$\big[g(z)x\big(g(z)y+zF(y)\big),z\big]+z\big[[z,y]x,z\big]=0$for all $x,y\in S$.
$[g(z)xg(z)y,z]+[g(z)xzF(y),z]+z\big[[z,y]x,z\big]=0$for all $x,y,z\in S$ (34)
Putting $x=xz$ in (33), we get
$[g(z)xzF(y),z]+\big[[z,y]x,z\big]z=0$for all $x,y,z\in S$ (35)
Subtracting (35) from (34), we have

$$[g(z)xzF(y),z]+\big[[z,y]x,z\big]z-[g(z)xg(z)y,z]-[g(z)xzF(y),z]-z\big[[z,y]x,z\big]=0$$

$\Big[\big[[z,y]x,z\big],z\Big]+[g(z)xg(z)y,z]=0$for all $x,y,z\in S$. (36)
Putting $y=zy$,the above relation yields
$z\big[[z,y]x,z\big],z]+[g(z)xg(z)zy,z]=0$for all $x,y,z\in S$. (37)
Left multiplying (36) by z and then subtracting it from (37),we get

$$z\big[[z,y]x,z\big],z]+[g(z)xg(z)zy,z]-z\Big[\big[[z,y]x,z\big],z\Big]-z[g(z)xg(z)y,z]=0$$
$$[g(z)xg(z)zy,z]-z[g(z)xg(z)y,z]=0,$$
$$[g(z)xg(z)zy,z]-[zg(z)xg(z)y,z]=0,$$

$\big[[g(z)xg(z),z]y,z\big]=0$for all $x,y,z\in S$. (38)
Now we substitute $yg(z)xg(z)\,for\ x$ in (38) and get

$$[[g(z)xg(z),z],z]yg(z)xg(z),z]=0$$

$\big[[g(z)xg(z),z]y,z\big]+[g(z)xg(z),z]y[g(z)xg(z),z]=0$for all $x,y,z\in S$. (39)
Using (38), it reduce to
$[g(z)xg(z),z]y[g(z)xg(z),z]=0$for all $x,y,z\in S$.(40)
Since S is right ideal , it follows that
$[g(z)xg(z),z]yR[g(z)xg(z),z]y=0$and hence
$[g(z)xg(z),z]y=0$for all $x,y,z\in S$.

This is same as (17) in the proof of the theorem 2. Thus using the same arguments as we used in the last paragraph of the proof of theorem2, we get the required result.

In the same manner the conclusion can be obtained when $F(x)F(y)+yx\in Z$ for all $x,y,z\in S$. Hence, the theorem is now proved.

Corollary 4: Let R be a semiprime ring and $F:R\rightarrow R$ be a left multiplicative generalized derivation associated with the map $g:R\rightarrow R$. If $F(x)F(y)\pm yx\in Z$ for all $x,y\in R$, then $[g(x),x]=0$ for all $x\in R$.

References

[1] Anderson, F.w.:Letures on noncommutative rings. University of Oregon, Oregon (2002).
[2] BasudebDhara and Shakir Ali: On multiplicative (generalized)-derivations in prime and semiprime rings .Aequat. Math.86 (2013).
[3] Bresar, M.: On the distance of the composition of two derivations to the generalized derivations.Glasgowmath.J.(33), 89-93 (1991).
[4] Daif, M.N.:When is a multiplicative derivation additive. Int. J. Math. Math. Sci.14(3),615-618(1991).
[5] Daif, M.N.,Tammam EI-Sayiad, M.S.: Mulplicative generalized derivations which are additive.East-West J.Math.9(1), 31-37(1997).
[6] Herstein,I.N.:Rings with involutiopn. The University of Chicago Press ,Chicago(1976).
[7] Hvala,B.:Generalized derivations in rings.Comm.Alegbra 26(4),1147-116(1998).
[8] Martindale,W.S.III.:When are multiplicative maps additive .Proc.Amer.Math.Soc.21,695- 698(1969).