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Analyticity Theorem and Operational Transform on Generalized Fractional Hilbert Transform

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Abstract: *The generalized fractional Hilbert transform plays an important role in signal processing, image reconstruction, etc. This paper generalizes the fractional Hilbert Transform to the spaces of generalized functions and proved analyticity theorem also obtained many operation formulae for the transform.*
Key words: *Hilbert transform, generalized fractional Hilbert Transform, Signal Processing.*

I. Introduction

The fractional integral transforms has become the focus of many research papers because of its recent applications in many fields such as image reconstruction, signal processing [5, 6]. Namias [2] introduced the concept of fractional Fourier transform. Bhosale and Chaudhary [1] had extended fractional Fourier transform to the distribution of compact support. The fractional Hilbert transform is a generalization of Hilbert transform. It is a powerful tool in signal processing, image reconstruction.

This paper is organized as follows. In section 2 the fractional Hilbert transform with parameter α is extended in the sense of generalized function. In section 3 analyticity theorem is proved and in section 4 fractional Hilbert transform of some functions are established. Lastly paper is concluded in section 5.

II. Generalized Fractional Hilbert Transform

2.1 The Test function Space $E(\mathbb{R}^n)$:

An infinitely differentiable complex valued function φ on $\mathbb{R}^n$ belongs to $E(\mathbb{R}^n)$ if for each compact set $K \subset S_\alpha$ where $S_\alpha = \{x \in \mathbb{R}^n, |x| \le a, a > 0\}$ and

$$\gamma_{E,K}(\varphi) = \sup_{x\in K} |D_x^k \varphi(x)| < \infty, \quad \text{for } k = 1,2,3,\dots\dots\dots\dots\dots$$

Clearly E is complete and so a Frechet space.

Let $E'(\mathbb{R}^n)$ denotes the dual space of $E(\mathbb{R}^n)$.

2.2 Proposition:

If $S_\alpha = \{x : x \in \mathbb{R}^n, |x| \le a, a > 0\}$. Let $t \in \mathbb{R}^n$ and $x \in S_\alpha$ and $0 \le \alpha \le \pi$ such that $\alpha \ne 0, \frac{\pi}{2}, \pi$, then $K_\alpha(x,t) \in E(\mathbb{R}^n)$

where $$K_\alpha(x,t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2} e^{i\frac{\cot\alpha}{2}x^2}}{\pi(t-x)} \tag{2.1}$$

and $E(\mathbb{R}^n)$ is the testing function space.

Proof: To show that $K_\alpha(x,t) \in E(\mathbb{R}^n)$

We show that $\gamma_{E,K}\{K_\alpha(x,t)\} = \sup_{x\in K} |D_x^n\{K_\alpha(x,t)\}| < \infty$

Consider

$$K_\alpha(x,t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2} e^{i\frac{\cot\alpha}{2}x^2}}{\pi(t-x)}$$

$$D_x^n\{K_\alpha(x,t)\} = D_x^n\left[\frac{C\, e^{i\frac{\cot\alpha}{2}x^2}}{(t-x)}\right] \quad \text{where } C = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}$$

Let $$f = \frac{e^{i\frac{\cot\alpha}{2}x^2}}{(t-x)}$$

Taking log of both sides

$$\log f = i\frac{\cot\alpha}{2}x^2 - \log(t-x)$$

$$f' = f\left[i\cot\alpha\, x + \frac{1}{t-x}\right]$$

$f^{''} = f\left[i\, cot\alpha + \frac{1}{(t-x)^2}\right] + \left[i\, cot\alpha\, x + \frac{1}{t-x}\right] f^{'}$

$\quad = f\left\{i\, cot\alpha + \frac{1}{(t-x)^2} + \left[i\, cot\alpha\, x + \frac{1}{t-x}\right]^2\right\}$

Continuing in this manner,

$f^{(n)} = f\left[C_1(x,t) + \left(i\, cot\alpha\, x + \frac{1}{t-x}\right)^n\right]$

where $C_1(x,t)$ is a function of x and t.

$\left|f^{(n)}(x)\right| \le |f| \left[|C_1(x,t)| + \left|\left(i\, cot\alpha\, x + \frac{1}{t-x}\right)^n\right|\right]$

$\quad \le \frac{\left|e^{i\frac{cot\alpha}{2}x^2}\right|}{|(t-x)|}\left[|C_1(x,t)| + \left|\left(i\, cot\alpha\, x + \frac{1}{t-x}\right)^n\right|\right]$

$\left|f^{(n)}\right| < \infty$

$D_x^n\{K_\alpha(x,t)\} < \infty$

$\gamma_{E,K}\{K_\alpha(x,t)\} = \sup_{x\in K}|D_x^n\{K_\alpha(x,t)\}| < \infty$

Hence $K_\alpha(x,t) \in E(\mathbb{R}^n)$.

2.3 The Generalized Fractional Hilbert Transform on $E'(\mathbb{R}^n)$:

The generalized fractional Hilbert transform of $f(x) \in E'(\mathbb{R}^n)$, where $E'(\mathbb{R}^n)$ is the dual of the testing function space $E(\mathbb{R}^n)$, can be defined as

$$H^\alpha[f(x)](t) = \langle f(x), K_\alpha(x,t)\rangle, \quad \text{for each} \quad t \in \mathbb{R}. \tag{2.2}$$

where $K_\alpha(x,t) = \frac{e^{-i\frac{cot\alpha}{2}(t^2-x^2)}}{\pi(t-x)}$ for $\alpha \ne 0, \frac{\pi}{2}, \pi$

The right hand side of (2.2) has meaning as the application of $f \in E'$ to $K_\alpha(x,t) \in E(\mathbb{R}^n)$. $H^\alpha[f(x)](t)$ is α^{th} order generalized fractional Hilbert transform of the function $f(t)$.
when the integral exists, where the integral is a Cauchy principal value.

III. Analyticity Theorem

In this section we have derived the analyticity theorem for the generalized fractional Hilbert transform.

Theorem: *Let* $f \in E'(\mathbb{R}^n)$ and its generalized fractional Hilbert transform is defined by $H^\alpha[f(x)](t) = \langle f(x), K_\alpha(x,t)\rangle$. Then $H^\alpha[f(x)](t)$ is analytic on $\mathbb{R}^n$ if $Supp\, f \subset S_\alpha = \{x: x \in \mathbb{R}^n, |x| \le \text{a},\ \text{a} > 0\}$ then $H^\alpha[f(x)](t)$ is differentiable and $D_t^k H^\alpha[f(x)](t) = \langle f(x), D_t^k K_\alpha(x,t)\rangle$.

Proof: Let $t: (t_1, t_2, \ldots\ldots, t_n) \in \mathbb{R}^n$,

We first prove that, $\frac{\partial}{\partial t_j} H^\alpha[f(x)](t) = \langle f(x), \frac{\partial}{\partial t_j} K_\alpha(x,t)\rangle$.

We prove the result of $k = 1$, the general result follows by induction.

For some $t_j \ne 0$, choose two concentric circles C and C[1] with centre at t_j and radii r and r_1 respectively, such that $0 < r < r_1 < |t_j|$. Let Δt_j be a increment satisfying $0 < |\Delta t_j| < r$.

Consider

$$\frac{H_\alpha(t_j + \Delta t_j) - H_\alpha(t_j)}{\Delta t_j} - \langle f(x), \frac{\partial}{\partial t_j} K_\alpha(x,t)\rangle = \langle f(x), \psi_{\Delta t_j}(x)\rangle \tag{2.3}$$

Where $\psi_{\Delta t_j}(x) = \frac{1}{\Delta t_j}\left[K_\alpha(x, t_1, t_2, \ldots\ldots, t_j + \Delta t_j, \ldots\ldots, t_j) - K_\alpha(x,t)\right] - \frac{\partial}{\partial t_j} K_\alpha(x,t)$

For any fixed $x \in \mathbb{R}^n$ and any fixed integer k

$D_x\, K_\alpha(x,t) = D_x\left[\frac{e^{-i\frac{cot\alpha}{2}t^2} e^{i\frac{cot\alpha}{2}x^2}}{\pi(t-x)}\right]$

$= C_{\alpha,t}\ D_x\left[\frac{e^{i\frac{cot\alpha}{2}x^2}}{(t-x)}\right]$

$= C_{\alpha,t}\ e^{i\frac{cot\alpha}{2}x^2}(t-x)^{-1}[(t-x)^{-1} + i\, cot\alpha\, x]$

$= [(t-x)^{-1} + i\, cot\alpha\, x]\, K_\alpha(x,t)$

where $C_{\alpha,t} = \frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}$

Since for any fixed $x \in \mathbb{R}^n$, fixed integer k and α ranging from 0 to π. $D_x\, K_\alpha(x,t)$ is analytic inside and on C^1, we have by Cauchy integral formula,

$$D_x \psi_{\Delta t_j}(x) = \frac{\Delta t_j}{2\pi i} \int_{C^1} \frac{D_x\, K_\alpha(x, \overline{t_j})}{(z - t_j - \Delta t_j)(z - t_j)^2}\, dz$$

$$= \frac{\Delta t_j}{2\pi i} \int_{C^1} \frac{M(x, \overline{t_j})}{(z - t_j - \Delta t_j)(z - t_j)^2}\, dz$$

where $\overline{t_j} = (t_1, t_2, \ldots\ldots, t_{j-1}, z, t_{j+1} \ldots., t_n)$.

But for all $z \in C^1$ and x restricted to a compact subset of $\mathbb{R}^n$, $0 \le \alpha \le \pi$,

$M(x, \overline{t_j}) = D_x\, K_\alpha(x, \overline{t_j})$ is bounded by a constant K.

Therefore we have,

$$\left|D_x \psi_{\Delta t_j}(x)\right| \le \left|\Delta t_j\right| \frac{K}{(r_1 - r) r_1}$$

Thus as $\left|\Delta t_j\right| \to 0$, $D_x \psi_{\Delta t_j}(x)$ tends to zero uniformly on the compact subset of $\mathbb{R}^n$ therefore it follows that $\psi_{\Delta t_j}(x)$ converges in $E(\mathbb{R}^n)$ to zero. Since $H_\alpha \in E'$ we conclude that (3.4.1) also tends to zero therefore $H_\alpha(t)$ is differentiable with respect to t_j. But this is true for all $j = 1, 2, \ldots\ldots, n$. Hence $H_\alpha(t)$ is analytic and

$D_t^k H_\alpha[f(x)](t) = \langle f(x), D_t^k K_\alpha(x,t) \rangle$.

IV. Fractional Hilbert Transform Of Some Functions

4.1 Result: $H^\alpha\left[\frac{1}{x}\right](t) = \frac{1}{t} H^\alpha[\,1\,](t)$

Proof: $$H^\alpha\left[\frac{1}{x}\right](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{x(t-x)}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx$$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{t}\left(\frac{1}{x} + \frac{1}{t-x}\right) e^{i\frac{\cot\alpha}{2}x^2}\, dx$$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{t\,\pi} \int_{-\infty}^{\infty} \frac{1}{x}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx + \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{t\,\pi} \int_{-\infty}^{\infty} \frac{1}{t-x}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx$$

$$H^\alpha\left[\frac{1}{x}\right](t) = \frac{1}{t} H^\alpha[\,1\,](t)$$

4.2 Result: $$H^\alpha\left[\frac{1}{x^2}\right](t) = \frac{1}{t}\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx + \frac{1}{t^2} H^\alpha[\,1\,](t)$$

Proof: $$H^\alpha\left[\frac{1}{x^2}\right](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2(t-x)}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx$$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2}\left(\frac{1}{x} + \frac{t}{x^2} + \frac{1}{t-x}\right) e^{i\frac{\cot\alpha}{2}x^2}\, dx$$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{t^2\,\pi} \int_{-\infty}^{\infty} \frac{1}{x}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx + \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{t\,\pi} \int_{-\infty}^{\infty} \frac{1}{x^2}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx + \frac{1}{t^2}\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{t-x}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx$$

$$H^\alpha\left[\frac{1}{x^2}\right](t) = \frac{1}{t}\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx + \frac{1}{t^2} H^\alpha[\,1\,](t)$$

4.3 Result: For any integer $n > 0$

$$H^\alpha\left[\frac{1}{x^n}\right](t) = \frac{1}{t^n}\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} \frac{1}{x^{k+1}}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx + \frac{1}{t^n} H^\alpha[\,1\,](t)$$

Proof: $H^{\alpha}\left[\frac{1}{x^n}\right](t) = \frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{x^n(t-x)}\, e^{i\frac{cot\alpha}{2}x^2}\, dx$

$$= \frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^n}\left(\frac{1}{x}+\frac{t}{x^2}+\frac{t^2}{x^3}+\ \dots\dots\dots\dots + \frac{t^{n-1}}{x^n}+\frac{1}{t-x}\right) e^{i\frac{cot\alpha}{2}x^2}\, dx$$

$$= \frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1} t^{k-n}\int_{-\infty}^{\infty}\frac{1}{x^{k+1}}\, e^{i\frac{cot\alpha}{2}x^2}\, dx + \frac{1}{t^n}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{t-x}\, e^{i\frac{cot\alpha}{2}x^2}\, dx$$

$$H^{\alpha}\left[\frac{1}{x^n}\right](t) = \frac{1}{t^n}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty}\frac{1}{x^{k+1}}\, e^{i\frac{cot\alpha}{2}x^2}\, dx + \frac{1}{t^n}H^{\alpha}[\,1\,](t)$$

4.4 Result: $H^{\alpha}\left[\frac{1}{(x+ia)}\right](t) = \frac{-ia}{(t+ia)}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\frac{cot\alpha}{2}x^2}}{x^2+a^2}\, dx$

$$+\frac{1}{(t+ia)}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+a^2}\, e^{i\frac{cot\alpha}{2}x^2}\, dx + \frac{1}{(x+ia)}H^{\alpha}[\,1\,](t)$$

Proof: $H^{\alpha}\left[\frac{1}{(x+ia)}\right](t) = \frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x+ia)(t-x)}\, e^{i\frac{cot\alpha}{2}x^2}\, dx$

$$= \frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{(x-ia)}{(x^2+a^2)(t-x)}\, e^{i\frac{cot\alpha}{2}x^2}\, dx$$

$$= \frac{e^{-i\frac{cot\alpha}{2}t^2}}{(t+ia)\,\pi}\int_{-\infty}^{\infty}\left(\frac{x}{x^2+a^2}-\frac{ia}{x^2+a^2}+\frac{1}{t-x}\right) e^{i\frac{cot\alpha}{2}x^2}\, dx$$

$$= \frac{e^{-i\frac{cot\alpha}{2}t^2}}{(t+ia)\,\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+a^2}\, e^{i\frac{cot\alpha}{2}x^2}\, dx - \frac{ia\, e^{-i\frac{cot\alpha}{2}t^2}}{(t+ia)\,\pi}\int_{-\infty}^{\infty}\frac{1}{x^2+a^2}\, e^{i\frac{cot\alpha}{2}x^2}\, dx$$

$$+\frac{1}{(t+ia)}H^{\alpha}[\,1\,](t)$$

$$H^{\alpha}\left[\frac{1}{(x+ia)}\right](t) = \frac{-ia}{(t+ia)}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\frac{cot\alpha}{2}x^2}}{x^2+a^2}\, dx$$

$$+\frac{1}{(t+ia)}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+a^2}\, e^{i\frac{cot\alpha}{2}x^2}\, dx\ \ + \frac{1}{(x+ia)}H^{\alpha}[\,1\,](t)$$

4.5 Result: $H^{\alpha}\left[e^{-i\frac{\cot\alpha}{2}x^2}\right](t) = 0$

Proof: $H_{\alpha}\left[e^{-i\frac{\cot\alpha}{2}x^2}\right](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\frac{\cot\alpha}{2}x^2}}{t-x}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{t-x}\, dx$$

$$H^{\alpha}\left[e^{-i\frac{\cot\alpha}{2}x^2}\right](t) = 0$$

Tabular form of fractional Hilbert transform of some functions

Sr. No.	Function	Fractional Hilbert Transform H[Function]
1	$\frac{1}{x}$	$\frac{1}{t} H^{\alpha}[1](t)$
2	$\frac{1}{x^2}$	$\frac{1}{t}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{x^2}e^{i\frac{cot\alpha}{2}x^2}dx+\frac{1}{t^2}H^{\alpha}[1](t)$
3	$\frac{1}{x^n}$	$\frac{1}{t^n}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty}\frac{1}{x^{k+1}}e^{i\frac{cot\alpha}{2}x^2}dx+\frac{1}{t^n}H^{\alpha}[1](t)$
4	$\frac{1}{(x+ia)}$	$\frac{-ia}{(t+ia)}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\frac{cot\alpha}{2}x^2}}{x^2+a^2}dx+\frac{1}{(t+ia)}\frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{xe^{i\frac{cot\alpha}{2}x^2}}{x^2+a^2}dx$ $+\frac{1}{(x+ia)}H^{\alpha}[1](t)$
5	$e^{-i\frac{\cot\alpha}{2}x^2}$	0

V. Conclusion

In this paper brief introduction of generalized fractional Hilbert transform is given and the analyticity theorem is proved. Fractional Hilbert transform of some functions are established which are useful in solving the differential equation occurring in the many branch of engineering.

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