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A Geometrical Derivation of π (Pi)

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Abstract: *The aim of this research paper was to derive the value of, and analyze π using purely basic geometrical and trigonometric methods. The derivation has been proposed after studying regular polygons and observing the similarity of an infinite sided regular polygon with a circle. Several test cases with the formula derived have been checked to verify this derivation and apart from raw calculation of data, an alternate verification using calculus has also been shown.*
Keywords: *Pi, Regular Polygons, Geometrical Derivation*

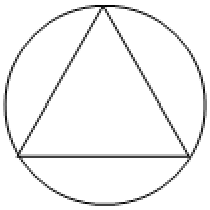
I. Introduction

Since time immemorial, the value of π has played a significant role in Mathematics. First roughly estimated by Archimedes, this irrational number today impacts our lives in more ways than we can imagine. We use this constant, π , to estimate the volumes and areas of a large number of 2 and 3 dimensional shapes and it has been the basis for a lot of groundbreaking mathematical work throughout history.
The value of π is defined to be the ratio between the circumference and the diameter of a circle.

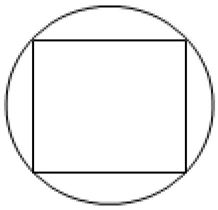
$$\pi = \frac{Circumference}{Diameter}$$

It is a rather arduous task to count exactly the number of points on the circumference of a circle and it is impossible to do so with a slide rule but if we observe the shape of a circle we see that it is nothing but an infinite sided regular polygon.

Every regular polygon can be circumscribed by a circle having a circumcentre in the middle of the polygon. During the course of the paper, we will be referring to polygons of different sizes having the same circumscribing circle. We have regular polygons of 3 sides, 4 sides, etc, and the greater the number of sides gets, the smaller becomes the size of each side.



Triangle - 3 sided regular polygon



Square - 4 sided regular polygon

Thus we can assume that a circle is nothing but an infinite sided regular polygon. Now, in terms of a polygon, π will be the ratio between the perimeter (circumference) and twice the distance between a vertex of the regular polygon and the circumcentre, or simply, the diameter of the circumscribing circle of the polygon(diameter). Therefore, we redefine the value of π as:

$$\pi = \frac{Perimeter}{Twice\ the\ distance\ between\ vertex\ and\ circumcentre}$$

II. Proposed equation

Using the basic geometric and trigonometric methods, we obtain this approximation of π:

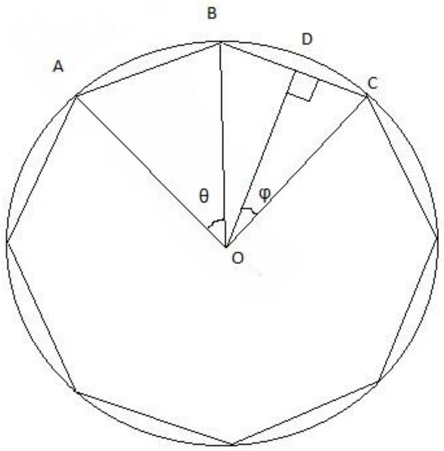
$$\pi = \lim_{n \to \infty} n \sin(\frac{180°}{n})$$

III. Proof

In order to establish the required ratio, we need to establish the general formula for the required ratio which will apply to all regular polygons. We will show one example of a regular polygon and use this to formulate the equation for all regular polygons, ie, the general formula.

Let n be the number of sides in a regular polygon.

Here we let n = 8.



Note:
By comparing this diagram with the previous diagrams of the triangle and square we can conclude that more the sides in the regular polygon, the closer the polygon is to resembling the circle.

Let the length of each side of the regular polygon be of size 1 unit.
Therefore,

$$Perimeter\ =\ n\,units$$

If we join the center O to every vertex of the n sided polygon (8 sided polygon), we obtain n congruent triangles. The angle each triangle makes at the vertex will be

$$\theta = \frac{360°}{n}$$

In triangle OBC,
Since triangle OBC is congruent to triangle AOB,

$$OB = OC$$

Therefore, triangle OBC is an isosceles triangle.
We now drop a perpendicular from O to BC. Since OD is perpendicular to BC,

$$Angle\ ODC\ =\ 90°$$

Therefore, Triangle ODC is a right angled triangle.
Since OBC is an isosceles triangle, the perpendicular OD bisects side BC.
Therefore,

$$DC = \frac{BC}{2} = \frac{1}{2}\,units$$

Again, the perpendicular OD also bisects Angle BOC (θ).
Therefore angle DOC (ø) is defined as

$$\emptyset = \frac{\theta}{2} = \frac{360°/n}{2} = \frac{180°}{n}$$

From Triangle ODC,

$$\sin\emptyset = \frac{DC}{CO}$$

$$\rightarrow CO = \frac{DC}{sin\emptyset}$$

Since CO is the radius (r) of the circumscribed circle and DC = 1/2,

$$r = \frac{1}{2sin\emptyset}$$

$$\rightarrow 2r = \frac{1}{sin\emptyset}$$

We know that 2r is nothing but the diameter of the circle.
Hence,

$$Diameter\ =\ Twice\ the\ distance\ between\ vertex\ and\ circumcentre = \frac{1}{sin\emptyset}$$

We have already defined π as the ratio between the circumference (perimeter) and diameter (twice the distance between the vertex and the circumcentre) .
Therefore,

$$\pi = \frac{n}{\frac{1}{sin\,\emptyset}}$$

$$\rightarrow\ \pi = n\ sin\ \emptyset$$

We know,

$$\emptyset\ =\ \frac{180°}{n}$$

Therefore,

$$\pi\ =\ n\ sin(\frac{180°}{n})$$

As a circle is an infinite sided regular polygon, we adjust the equation

$$\pi\ = \lim_{n\to\infty} n\ sin(\frac{180°}{n})$$

IV. Verification using Calculated Data

If we plug in various values of n into the equation we get the following results :

Serial Number	**Value of n**	**Value of π**	**Error**	**Number of accurate digits**
1.	3	2.598076211353316	0.5435164422364771...	0
2.	4	2.8284271274619	0.3131655884360327...	0
3.	5	2.938926261462366...	0.2026663921274272...	0
4.	6	2.9999999999999996..	0.14195265358979356...	0
5.	7	3.037186173822907...	0.10440647976688622...	1
6.	8	3.0614674589207183...	0.08012519466907486...	1
7.	10	3.090169943749474...	0.05142270984031905...	1
8.	12	3.105828541230249...	0.03576411235954424...	2
9.	20	3.1286893008046173...	0.012903352785175848...	2
10.	25	3.1333308391076065...	0.008261814482186658...	2
11.	50	3.1395259764656687	0.0020666771241244497...	2
12.	75	3.1406740296899716...	0.0009186238998215579...	3
13.	96	3.1410319508905093...	0.000560702699283766...	4
14.	100	3.141075907812829...	0.0005167457769639228...	4
15.	500	3.14171827794755...	0.000020670810317646726...	5
16.	1000	3.1415874858795636...	0.000005167710229514455...	5
17.	10000	3.141592601912665..	0.000000051677127910210174...	7
18.	100000	3.1415926530730216...	0.000000000516771514469383...	9
19.	1000000	3.1415926535846257...	0.0000000000005167422045815329...	12
20.	10000000	3.141592653589741...	0.000000000000051958437552457326...	14

V. Verification using Calculus

In the verification of this expression we will use π itself instead of 180° since this calculus verification must be done in radians, and

$$\pi\ (in\ radians)\ =\ 180°\ (in\ degrees)$$

Therefore, the expression will now stand as:

$$\pi = \lim_{n \to \infty} n \sin\left(\frac{\pi}{n}\right)$$

Evaluating the expression:

$$\lim_{n \to \infty} n \sin\left(\frac{\pi}{n}\right)$$
$$= \lim_{n \to \infty} \frac{\pi}{\pi} \times n \times \sin\left(\frac{\pi}{n}\right)$$
$$= \lim_{n \to \infty} \pi \times \frac{1}{\frac{\pi}{n}} \times \sin\left(\frac{\pi}{n}\right)$$
$$= \pi \times \lim_{n \to \infty} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}$$
$$\text{As } n \to \infty, \frac{\pi}{n} \to 0$$

Therefore the expression,

$$= \pi \times \lim_{\frac{\pi}{n} \to 0} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}$$
$$= \pi \times 1 = \pi$$

Thus, we have verified the derived expression.

VI. Alternate Equation

$$\boldsymbol{\pi = \lim_{n\to\infty} n \sin\left(\frac{180°}{n}\right) = \lim_{n\to\infty} n \cos\left(90° - \frac{180°}{n}\right) = \lim_{n\to\infty} n \cos\left(90° \times \frac{n-2}{n}\right)}$$

VII. Conclusion

Observing the data table we notice that the higher the value of n is, the closer we come to approximating π. Seeing that the values from the table corroborate our equation and also taking into account the alternate verification of the equation we come to the conclusion that our predicted equation is accurate and has been correctly verified.

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