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Uniform Boundedness of Shift Operators

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Abstract.*We show, if thenormsof* S_k *areuniformly boundedon* $l^{\{p_n\}}$ *for a bounded* $\{p_n\}$ *if and onlyifthereexists* $r, 1 \leq r < \infty$*,such thatthenormsin* $l^{\{p_n\}}$ *andtheclassical space* l^r *are equivalent. A "pointwise-bounded" family of continuous linear operators from a Banach space to a normed space is "Uniformly bounded."*
Stated another way, letX be a Banach space and Y be a normed space. If $\mathcal{A}$ *is a collection of bounded linear mappings of X into Y such that for each* $x \epsilon X$*,* $sup\{\|Ax\|; A \in \mathcal{A}\} < \infty$*, then* $sup\{\|A\|: A \in \mathcal{A}\} < \infty$*.*
Keywords:*Banach space, characteristicfunction, linear mappings,Uniformly bounded.*

I. Introduction

A crucial difference between $L^{p(x)}$ and the classical Lebesgue space is that $L^{p(x)}$ is not, in general, invariant under translation [1]. Moreover, there is a function $f \in L^{p(x)}$ which is not $p(x)-$ mean continuous provided p is continuous and non-constant.

Consider a discrete analogue $l^{\{p_n\}}$ of $L^{p(x)}$. In[2] it is proved that under certain assumptions on $\{p_n\}$ the norms of shift operators given by

$$S_k a = \{(S_k a)_n\}, \quad (S_k a)_n = a_{n-k}, \quad a = \{a\},$$

areuniformlyboundedon $l^{\{p_n\}}$.Recallthat $\{p_n\}$ neednotbeconstant.Asan immediateconsequenceit is shownthatthenormsofaveragingoperatorsgivenby

$$(T_k a)_n = \frac{1}{k}(a_n + a_{n+1} + \cdots + a_{n+k-1}), a = \{a_n\} \in \ell^{\{p_n\}},$$

areuniformlyboundedon $l^{\{p_n\}}$,too.

In this paper we prove the following assertion: Thenormsof S_k areuniformly boundedon $l^{\{p_n\}}$ for a bounded $\{p_n\}$ if and onlyifthereexists $r, 1 \leq r < \infty$,such thatthenormsin $l^{\{p_n\}}$ andtheclassical space l^r are equivalent [3].

II. Preliminaries

Let $\mathbb{Z}$ denotethesetofallintegersandlet μ denotethesetofallmappings $a: \mathbb{Z} \to \mathbb{R}$. Wewill alsodenoteelementsof μ by $= \{a_n\}$.Let

$$\varepsilon = \{p \in \mu; 1 \leq p_n \ for\ all\ n \in \mathbb{Z}\}.$$

Denoteby $p^* = sup\{p_n; n \in \mathbb{Z}\}$ forany $p \in \varepsilon$ *and*

$$\mathfrak{B} = \{p \in \varepsilon; p^* < \infty\}.$$

Letthesymbol χ^k standforthecharacteristicfunctionoftheset $\{n \in \mathbb{Z}; -k \leq n \leq k\}$. Let $a^k,\ a \in \mu$.Saythat $a \geq 0\ if\ a_n \geq 0$ for each $n \in \mathbb{Z}$ *and* $a^k \nearrow a$ *if* $(a^k)_n \nearrow a_n$ foreach $n \in \mathbb{Z}$.

We recall thedefinitionofaBanachfunctionspace.

DEFINITION2.1. Alinearspace $X, X \subset \mu$, iscalled a Banach functionspace if there existsa functional $\|.\|_X: \mu \to [0, \infty]$ withthe normpropertysatisfying :

(i) $a \in X$ *if and only if* $\|a\|_X < \infty$;
(ii) $\|a\|_X = \||a|\|_X$ *for all* $a \in \mu$;
(iii) *if* $0 \leq a^k \nearrow a$ *then* $\|a^k\|_X \nearrow \|a\|_X$;
(iv) $\|a\chi^k\|_X < \infty$ *for any* $k \in \mathbb{N}$;
(v) *for any* $k \in \mathbb{N}$ *there is a positive constant* c_k *such that*

$$\sum_{|n| \leq k} |a_n| \leq c_k \|a\|_X \text{ for all } a \in X .$$

DEFINITION2.2.Let $p \in \varepsilon$ .Denotefor $a \in \mu$ the Luxemburgnormby

$$\|a\|_{\{p_n\}} = inf\left\{\lambda > 0; \sum_{n\in\mathbb{Z}} \left|\frac{a_n}{\lambda}\right|^{p_n} \leq 1\right\}.$$

Definethespace $l^{\{p_n\}}$ by:

$$.l^{\{p_n\}} = \{a; \|a\|_{\{p_n\}} < \infty\}.$$

Remarkthatwewillusetheusualsymbols l^r and $\|a\|_r$ inthecaseofconstantmapping$r \in \varepsilon$ .Recall that $\|a\|_r = \left(\sum_{n\in\mathbb{Z}} |a_n|^r\right)^{1/r}$ in this case .

LEMMA2.3.Thespace $l^{\{p_n\}}$ is aBanachfunctionspace.

DEFINITION2.4.Let$p, q \in \varepsilon$, andlet$T$bealinearmappingon μ .Wewillsaythat$T$is boundedfrom $l^{\{p_n\}}$ into $l^{\{q_n\}}$ if

$$\|T\|_{\{p_n \to q_n\}} \coloneqq sup\{\|T_a\|_{\{q_n\}}; \|a\|_{\{p_n\}} \leq 1\} < \infty .$$

LEMMA2.5.Let$p \in \mathfrak{B}$.Then

$$l^{\{p_n\}} = \left\{a; \sum_{n\in\mathbb{Z}} |a_n|^{p_n} < \infty\right\}.$$

LEMMA2.6.Let$p, q \in \mathcal{B}$,andlet$T$bealinear mappingwhichmaps μ into itself.Let$c$bea positiveconstantsuchthat:

$$\sum_{n\in\mathbb{Z}} |a_n|^{p_n} \leq 1 \Rightarrow \sum_{n\in\mathbb{Z}} |(T_a)_n|^{q_n} \leq c .$$

Then

$$\|T\|_{\{p_{n\to} q_n\}} \leq \max\ (1, c).$$

*Proof.*Assume$\|a\|_{\{p_n\}} \leq 1$.Thenitiseasyto verify that $\sum_{n\in Z} |a_n|^{p_n} \leq 1$ and accordingtothe assumptionswe have

$$\sum_{n\in Z} |(Ta)_n|^{q_n} \leq \max\ (1, c)$$

Then

$$\sum_{n\in Z} \left|\left(T\left(\frac{a}{\max\ (1, c)}\right)\right)_n\right|^{q_n} \leq \sum_{n\in Z} \left|\frac{(Ta)_n}{\max\ (1, c)}\right|^{q_n}$$

$$\leq \frac{1}{\max\ (1, c)} \sum_{n\in Z} |(Ta)_n|^{q_n} \leq \frac{c}{\max\ (1, c)} \leq 1 .$$

Thisgives $\|T\|_{\{p_n\to q_n\}} \leq \max(1, c)$andthe resultfollows.

COROLLARY2.7. Let $p, q \in B$ and $\{T_m\}_{m=1}^{\infty}$ a sequence of linear mappings such that $T_m : \mu \to \mu$. Let $\tilde{c} \geq 0$, a constant such that, if $\sum_{n\in Z} |a_n|^{p_n} \leq 1$ and, $\sum_{n\in Z} \sum_{m=1}^{\infty} |(T_{m_a})|^{q_n} \leq \tilde{c}$, , then $\sum_{m=1}^{\infty} \|T_m\|_E \leq \max\ (1, \tilde{c})$ where $E = \{p_n \to q_n\}$.

*Proof.*Lemma 2.6 implies that

$$\sum_{n\in Z} \sum_{m=1}^{\infty} |(T_{m_a})_n|^{q_n} \leq \max\ (1, \tilde{c})$$

and

$$\sum_{n\in Z} \sum_{m=1}^{\infty} \left|\left(T_m\left(\frac{a}{\max\ (1, \tilde{c})}\right)\right)_n\right|^{q_n} \leq \sum_{n\in Z} \sum_{m=1}^{\infty} \left|\frac{(T_{m_a})_n}{\max\ (1, \tilde{c})}\right|^{q_n}$$

$$\leq \frac{1}{\max\ (1, \tilde{c}\ )} \sum_{n \in Z} \sum_{m=1}^{\infty} \left| \left(T_{m_a}\right)_n \right|^{q_n} \leq M \leq 1,$$

Where

$$M = \frac{\tilde{c}}{\max\ (1, \tilde{c}\ )}$$

which gives the result.

LEMMA2.8.Let$p,\ q \in \mathfrak{B}$andlet$T$bealinear mappingwhichmaps μ into itself.Let$c > 1$bea positivenumbersuchthat$\|T\|_{\{p_n \to q_n\}} \geq c$ [5] .Thenthereexistsan $a \in \mu$ suchthat

$$\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leq 1\ and \sum_{n \in \mathbb{Z}} |(T_a)_n|^{q_n} \geq c\ .$$

Proof. Since $\|T\|_{\{p_n \to q_n\}} \geq c$wehavean$a \in \mu$suchthatforany$\lambda \in c$it is

$$\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leq 1\ and \sum_{n \in \mathbb{Z}} \left|\frac{(T_a)_n}{\lambda}\right|^{q_n} > 1\ .$$

Consideringonly$1 \leq \lambda < c$, wecanwrite

$$1 < \sum_{n \in \mathbb{Z}} \left|\frac{(T_a)_n}{\lambda}\right|^{q_n} \leq \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} |(T_a)_n|^{q_n}\ ,$$

from which it follows that

$$\sum_{n \in \mathbb{Z}} |(T_a)_n|^{q_n} \geq c\ ,$$

andtheproofisfinished [4].

COROLLARY2.9. If $\{T_m\}$ is a sequence of linear mappings, and $T_m : \mu \to \mu$. Let $\tilde{c}_j > 1$ such that $E = \{p_n \to q_n\}$.

For $a_j \in \mu$ we have $\sum_{n \in Z} \sum_{j=1}^{\infty} \left|\left(a_j\right)_n\right|^{p_n} \leq 1$ and $\sum_{n \in Z} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \left|\left(T_{m_{a_j}}\right)_n\right|^{p_n} \geq \tilde{c}_j$.

*Proof.*For any $1 \leq \lambda_j < \tilde{c}_j$ we have $\sum_{n \in Z} \sum_{j=1}^{\infty} \left|\left(a_j\right)_n\right|^{p_n} \leq 1$ and

$$\sum_{n \in Z} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \left| \frac{\left(T_{m_{a_j}}\right)_n}{\lambda_j} \right|^{q_n} > 1\ .$$

Then

$$1 < \sum_{n \in Z} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \left| \frac{\left(T_{m_{a_j}}\right)_n}{\lambda_j} \right|^{q_n}$$

$$\leq \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \sum_{n \in Z} \sum_{m=1}^{\infty} \left|\left(T_{m_{a_j}}\right)_n\right|^{q_n}.$$

Therefore

$$\sum_{n \in Z} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \left|\left(T_{m_{a_j}}\right)_n\right|^{q_n} \geq \tilde{c}_j\ .$$

LEMMA2.10.Let$p,\ q \in \mathfrak{B}$andlet$T$bealinearmapping fromμ intoitself. Assumethatthere exists anumber$c > 1$and$a \in \mu$suchthat

$$\sum_{n\in\mathbb{Z}} |a_n|^{p_n} \le 1 \ and \ \sum_{n\in\mathbb{Z}} |(T_a)_n|^{q_n} \ge c\,.$$

Then $\|T\|_{\{p_n\to q_n\}} \ge c^{1/q^*}$.

*Proof.*Clearly,$\|a\|_{\{p_n\}} \le 1$.Further

$$\|T\|_{\{p_n\to q_n\}} \ge \|T_a\|_{\{q_n\}} = inf\left\{\lambda > 0; \sum_{n\in\mathbb{Z}} \left|\frac{(T_a)_n}{\lambda}\right|^{q_n} \ge 1\right\}.$$

Take$\lambda < c^{1/q^*}$.Then

$$\sum_{n\in\mathbb{Z}} \left|\frac{(T_a)_n}{\lambda}\right|^{q_n} > \sum_{n\in\mathbb{Z}} \frac{|(T_a)_n|^{q_n}}{c^{q_n/q^*}} \ge \sum_{n\in\mathbb{Z}} \frac{|(T_a)_n|^{q_n}}{c} \ge 1\,.$$

Consequently, $\|T\|_{\{p_n\to q_n\}} \ge c^{1/q^*}$.

III. Key assertions

Given$\varepsilon \in \mu$ weadoptthenotation $\mathbb{P}(\varepsilon) = \{n \in \mathbb{Z}: \varepsilon_n > 0\}$.

DEFINITION3.1.Let $\varepsilon \in \mu$.Wesaythat$\varepsilon \in \mathfrak{S}\varepsilon \in \mathfrak{S}$ifthere exists arealnumber $c > 0$ suchthat

$$\sum_{n\epsilon\mathbb{P}(\varepsilon)} \varepsilon_n c^{1/\varepsilon_n} < \infty\,. \tag{1}$$

Set

$$v(\varepsilon) = inf\left\{\frac{1}{c}\left(1 + \sum_{n\epsilon\mathbb{P}(\varepsilon)} \varepsilon_n\, c^{1/\varepsilon_n}\right); c > 0\right\}.$$

REMARK3.2.Itiseasytoseethat$\varepsilon \in \mathfrak{S}$ifandonly if$v(\varepsilon) < \infty$and$|\varepsilon| \in \mathfrak{S}$ if andonlyif$\varepsilon \in \mathfrak{S}and-\varepsilon \in \mathfrak{S}$.

LEMMA3.3. Let$k > 0$ and $\propto\in \mu$be such that$0 < \propto_n \le k$ for $n \in \mathbb{Z}$.Let$\varepsilon \in \mathfrak{S}$.Then$\alpha\varepsilon \in \mathfrak{S}$.

Proof. Letcsatisfy (1). Withoutlossofgeneralitywecanassume $c \le 1$ .Set $d = c^k$ .Letus estimate

$$\sum_{n\in\mathbb{P}(\propto\varepsilon)} \alpha_n \varepsilon_n d^{1/(\alpha_n\varepsilon_n)}.$$

Since$0 < a_n \le k$,wehave$d = c^k \le c^{a_n}$andusingthesimplefactthat $\mathbb{P}(\alpha\varepsilon) = \mathbb{P}(\varepsilon)$we obtain

$$\sum_{n\in\mathbb{P}(\propto\varepsilon)} \alpha_n \varepsilon_n d^{1/(\alpha_n\varepsilon_n)} = \sum_{n\in\mathbb{P}(\varepsilon)} \alpha_n\, \varepsilon_n (c^k)^{1/(\propto_n\varepsilon_n)}$$
$$\le k \sum_{n\in\mathbb{P}(\varepsilon)} \varepsilon_n (c^{\alpha_n})^{1/(\alpha_n\varepsilon_n)} = k \sum_{n\epsilon\mathbb{P}(\varepsilon)} \varepsilon_n c^{1/\varepsilon_n} < \infty\,,$$

whichfinishestheproof.

LEMMA3.4.Let $\in \mathfrak{S}$, $b \in \mu$satisfy$\varepsilon < 1$, $0 \le b$. Then

$$\sum_{n\in\mathbb{Z}} b_n \le 1 \Rightarrow \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} \le 1 + e^{1/e} v(\varepsilon).$$

Proof. Let c satisfy (1) and assume $\sum_{n\in\mathbb{Z}} \mathrm{b_n} \le 1$. set

$$\mathbb{Z}_1 = \{n \in \mathbb{Z}; \varepsilon_n \le 0\},$$
$$\mathbb{Z}_2 = \left\{n \in \mathbb{P}(\varepsilon); b_n > \varepsilon_n c^{1/\varepsilon_n}\right\},$$
$$\mathbb{Z}_3 = \left\{n \in \mathbb{P}(\varepsilon); b_n \le \varepsilon_n c^{1/\varepsilon_n}\right\}.$$

Since$\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$ arepairwisedisjointand$\mathbb{Z}_1 \cup \mathbb{Z}_2 \cup \mathbb{Z}_3 = \mathbb{Z}$, we canwrite

$$\sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \sum_{n\in\mathbb{Z}_1} b_n^{1-\varepsilon_n} + \sum_{n\in\mathbb{Z}_2} b_n^{1-\varepsilon_n} + \sum_{n\in\mathbb{Z}_3} b_n^{1-\varepsilon_n} = I_1 + I_2 + I_3 \tag{2}$$

Notethat,accordingto theassumptions,$b_n \le 1$ forall$n \in \mathbb{Z}$.

Let$n \in \mathbb{Z}_1$.Then$1 - \varepsilon_n \ge 1$and$b_n^{1-\varepsilon_n} \le b_n$ .Thus

$$I_1 \le \sum_{n\in\mathbb{Z}_1} b_n \le 1\,. \tag{3}$$

Let $n \in \mathbb{Z}_2$.Then $b_n > \varepsilon_n c^{1/\varepsilon_n}$ and,consequently, $b_n^{-\varepsilon_n} < \left(\varepsilon_n c^{1/\varepsilon_n}\right)^{-\varepsilon_n}$.Since $1 > \varepsilon_n > 0$, then $\varepsilon_n^{-\varepsilon_n} \leq e^{1/e}$ and $b_n^{1-\varepsilon_n} \leq \frac{1}{c} e^{1/e} b_n$ .Thus

$$I_2 \leq \frac{1}{c} e^{1/e} \sum_{n \in z_2} b_n \leq \frac{1}{c} e^{1/e}. \qquad (4)$$

Let $n \in \mathbb{Z}_3$.Then $0 \leq b_n \leq \varepsilon_n c^{1/\varepsilon_n}$,whichgives

$$b_n^{1-\varepsilon_n} \leq \varepsilon_n c^{1/\varepsilon_n} \left(\varepsilon_n c^{1/\varepsilon_n}\right)^{-\varepsilon_n} \leq \frac{1}{c} e^{1/e} \varepsilon_n c^{1/\varepsilon_n}$$

And

$$I_3 \leq \frac{1}{c} e^{1/e} \sum_{n \in \mathbb{Z}_3} \varepsilon_n c^{1/\varepsilon_n} .$$

This yields with (2),(3) and (4)

$$\sum_{n \in \mathbb{Z}} b_n^{1-\varepsilon_n} \leq 1 + \frac{1}{c} e^{1/e} \left(1 + \sum_{n \in \mathbb{Z}_3} \varepsilon_n c^{1/\varepsilon_n}\right).$$

Consequently,

$$\sum_{n \in \mathbb{Z}} b_n^{1-\varepsilon_n} \leq 1 + e^{1/e} v(\varepsilon).$$

LEMMA3.5.Let $\varepsilon \notin \mathcal{S}$, $\varepsilon < 1$.Thenthereexists $b \in \mu$, $0 \leq b$,*such that*

$$\sum_{n \in \mathbb{Z}} b_n \leq 1 \text{ and } \sum_{n \in \mathbb{Z}} b_n^{1-\varepsilon_n} = \infty .$$

*Proof.*Assumefirst

$$0 < \varepsilon_n < 1 \quad for\ all \quad n \in \mathbb{Z} \qquad (5)$$

Set $N_0 = -1$. Wewillconstructsequences $\{N_k\}_{k \in \mathbb{N}}$, $N_k \in \mathbb{N}$,and $\{c_k\}_{k \in \mathbb{N}}$, $c_k \in (0, \infty)$, satisfyingforany $k \in \mathbb{N}$

$$0 < c_k \leq \frac{1}{2^k} \text{ and } \sum_{N_{k-1} \leq |n| \leq N_k} \varepsilon_n c_k^{1/\varepsilon_{n\varepsilon}} = 1 . \qquad (6)$$

Accordingto theassumptionon $\{\varepsilon_n\}$, wehave

$$\sum_{n \in \mathbb{Z}} \varepsilon_n c^{1/\varepsilon_n} = \infty \qquad \text{for all } c > 0 . \qquad (7)$$

Thus, wecan find $N_1 \in \mathbb{N}$ such that

$$\sum_{|n| \leq N_1} \varepsilon_n \left(\frac{1}{2}\right)^{1/\varepsilon_n} \geq 1 .$$

Then there exists a number $0 < c_1 \leq \frac{1}{2}$such that

$$\sum_{|n| \leq N_1} \varepsilon_n c_1^{1/\varepsilon_n} = \sum_{N_0 < |n| \leq N_1} \varepsilon_n c_1^{1/\varepsilon_n} = 1 .$$

Assumethatwehaveconstructedpositiveintegers $N_1 < N_2 < \cdots < N_k$ andreal numbers $c_1, c_2, \dots, c_k$ such that

$$0 < c_r \leq \frac{1}{2^r} \text{ and } \sum_{N_{r-1} < |n| \leq N_r} \varepsilon_n c_r^{1/\varepsilon_n} = 1 .$$

For $r = 1, 2, \dots, k$. Accordingto (7),we canfind N_{k+1} suchthat

$$\sum_{N_k < |n| \leq N_{k+1}} \varepsilon_n \left(\frac{1}{2^{k+1}}\right)^{1/\varepsilon_n} \geq 1 .$$

Thenwe cantake c_{k+1} suchthat

$$0 < c_{k+1} \leq \frac{1}{2^{k+1}} \text{ and } \sum_{N_k < |n| \leq N_{k+1}} \varepsilon_n \left(c_{k+1}\right)^{1/\varepsilon_n} = 1$$

whichproves (6).

Define $b \in \mu$ by

$$b_n = \left(\varepsilon_n c_k^{1/\varepsilon_n}\right)^{1/(1-\varepsilon_n)} \text{ if } N_{k-1} < |n| \leq N_k .$$

Using (6) wehave

$$\sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \sum_{k=1}^{\infty} \sum_{N_{k-1}<|n|\le N_k} \varepsilon_n c_k^{1/\varepsilon_n} = \sum_{k=1}^{\infty} 1 = \infty\,.$$

Let us estimate $\sum_{n\in Z} b_n$.Clearly , by (5) it is $0 < \varepsilon_n c_k^{1/\varepsilon_n} \le 1$ for $n \in \mathbb{Z}$and $k \in \mathbb{N}$. Since $1-\varepsilon_n^2 \le 1$ we obtain

$$b_n = \left(\varepsilon_n c_k^{1/\varepsilon_n}\right)^{1/(1-\varepsilon_n)} \le \left(\varepsilon_n c_k^{1/\varepsilon_n}\right)^{1+\varepsilon_n}$$

whichimplieswith (6)

$$\sum_{n\in\mathbb{Z}} b_n \le \sum_{k=1}^{\infty} \sum_{N_{k-1}<|n|\le N_k} \left(\varepsilon_n c_k^{1/\varepsilon_n}\right)^{1+\varepsilon_n}$$
$$\le \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{N_{k-1}<|n|\le N_k} \left(\varepsilon_n c_k^{1/\varepsilon_n}\right) \varepsilon_n^{\varepsilon_n} c_k$$
$$\le \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{N_{k-1}<|n|\le N_k} \varepsilon_n c_k^{1/\varepsilon_n} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1\,.$$

Assumethat (5) isnotsatisfied. Since$\varepsilon \notin \mathfrak{S}$,theset $\mathbb{P}(\varepsilon)$must be infinite. Thenthereexistsaone-to-onemapping$\pi : \mathbb{P}(\varepsilon) \to \mathbb{Z}$. Set$\delta_n = \varepsilon_{\pi^-(n)}$, $n \in \mathbb{Z}$ .Then$\delta \notin \mathfrak{S}$ andsatisfies (5).Thus,there exists $a \in \mu, a > 0$, such that

$$\sum_{n\in\mathbb{Z}} a_n \le 1 \ and \ \sum_{n\in\mathbb{Z}} (a_n)^{1-\delta_n} = \infty\,.$$

Define

$$b_n = \begin{cases} a_{\pi(n)} & if \ \ n \in P(\varepsilon) \\ 0 & if \ \ n \notin P(\varepsilon) \end{cases}$$

Now,it is easy tosee that

$$\sum_{n\in\mathbb{Z}} b_n = \sum_{n\in\mathbb{P}(\varepsilon)} a_{\pi(n)} = \sum_{k\in\mathbb{Z}} a_k \le 1$$

and

$$\sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \sum_{n\in\mathbb{P}(\varepsilon)} a_{\pi(n)}^{1-\delta_{\pi(n)}} = \sum_{k\in\mathbb{Z}} a_k^{1-\delta_k} \le \infty\,.$$

Thus,$b$satisfies the desiredproperties,whichcompletestheproof.

IV. Equivalence of $l^{\{p_n\}}$ norms

Letusdenoteby Idtheidentityoperatoronμandby $l^{\{p_n\}} \hookrightarrow l^{\{q_n\}}$the imbeddingof $l^{\{p_n\}}$into $l^{\{q\}}$.Recallthat$l^{\{p_n\}} \hookrightarrow l^{\{q_n\}}$if $\|\mathrm{Id}\|_{\{p_n\to q_n\}} < \infty$.

THEOREM4.1.Let$p, q \in \mathfrak{B}$, $p - q \in \mathfrak{S}$ .Then

$$l^{\{p_n\}} \hookrightarrow l^{\{q_n\}}$$

*Proof.*Let$\sum_{n\in\mathbb{Z}} |a_n|^{p_n} \le 1$.. For each $n \in \mathbb{Z}$ set$b_n = |a_n|^{p_n}$, , $\varepsilon_n = \frac{p_n - q_n}{p_n}$. Then $\sum_{n\in\mathbb{Z}} b_n \le 1$and,accordingtoLemma3.3, $\{\varepsilon_n\} \in \mathfrak{S}$ .By Lemma3.4,wehave

$$\sum_{n\in\mathbb{Z}} |a_n|^{q_n} = \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} \le 1 + e^{-1/e} v(\varepsilon)$$

andconsequently,usingLemma2.6

$$\|\mathrm{Id}\|_{l^{\{p_n\}} \hookrightarrow l^{\{q_n\}}} \le 1 + e^{-1/e} v(\varepsilon) < \infty$$

whichprovesthelemma [3].

THEOREM4.2.Let$p, q \in \mathfrak{B}$ and let

$$l^{\{p_n\}} \hookrightarrow l^{\{q_n\}}.$$

Then$p - q \in \mathfrak{S}$.

*Proof.*Assume$p - q \notin \mathfrak{S}$.*Set* $\varepsilon_n = \frac{p_n - q_n}{p_n}forn \in \mathbb{Z}$ .According toLemma 3.3, $\{\varepsilon_n\} \notin \mathfrak{S}$. Moreover,$\varepsilon_n < 1$ for$n \in \mathbb{Z}$.Lemma3.5givestheexistenceof$b \in \mu$, $0 \le b$,suchthat

$$\sum_{n\in\mathbb{Z}} b_n \le 1 \ \text{ and} \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \infty.$$

Set $a_n = b_n^{1/p_n}$, $n \in \mathbb{Z}$ .Then

$$\sum_{n\in\mathbb{Z}} a_n^{p_n} = \sum_{n\in\mathbb{Z}} b_n \le 1$$

and

$$\sum_{n\in\mathbb{Z}} a_n^{q_n} = \sum_{n\in\mathbb{Z}} b_n^{1-\varepsilon_n} = \infty$$

Which yields with Lemma 2.5.$l^{\{p_n\}} \nrightarrow l^{\{q_n\}}$and the proof is complete [1].

THEOREM4.3.Let $p, q \in \mathfrak{B}$. Thenthenormsinspaces l^{p_n} and l^{q_n} are equivalentif andonlyif $|p-q| \in \mathcal{S}$.

V. Shift operators

In this section we showthattheuniformboundednessofshiftoperatorsisequivalent totheexistenceofareal$r \ge 1$suchthatthenormsinthespaces $l^{\{p_n\}}$ and l^r are equivalent.

Let$p \in \mathfrak{B}$befixedinthis part.

DEFINITION 5.1. For each $k \in \mathbb{Z}$ defineashiftoperator$S_k$from μintoitselfby

$$(S_k a)_n = a_{n-k}, a \in \mu , n \in \mathbb{Z} .$$

Set

$$D = sup\{\|S_k\|_{\{p_n \to p_n\}}; k \in \mathbb{Z}\}.$$

LEMMA5.2.Let $r \in [1, \infty)$be such that the norms in the spaces $l^{\{p_n\}}$and l^r are equivalent . Then $D < \infty$.

*Proof.*Let c satisfy $c^{-1}\|a\|_{\{p_n\}} \le \|a\|_r \le c\|a\|_{\{p_n\}}$ for all $a \in \mu$. Let $k \in \mathbb{Z}$bearbitrary.Since$\|S_k\|_{\{r\to r\}}= 1$,we immediatelyobtain

$$\|S_k\|_{\{p_n\to p_n\}} \le \ \|Id\|_{\{p_n\to r\}}\|S_k\|_{\{r\to r\}}\|Id\|_{\{r\to p_n\}} \ \le c^2.$$

Thus,$D\le c^2$, whichfinishes theproof [1].

LEMMA5.3. Assume that

$$\lim_{n\to\infty}|p_{n+1} - p_n| \ne 0 .$$

Theneither S_1 *or* S_{-1}isunboundedon $l^{\{p_n\}}$.

*Proof.*Accordingtotheassumptions,thereexists$\alpha > 0$suchthat $|p_{n+1} - p_n| \ge \alpha$forinfinitelymanypositiveintegers$n_1 < n_2 < \ \dots$ .Set

$$\mathbb{P} = \{n \in \mathbb{N}; p_n - p_{n+1} \ge \alpha\} \ \ and \ \ \mathbb{Z}_- = \{n \in \mathbb{N}; p_n - p_{n+1} \le -\alpha\}.$$

Theneither $\mathbb{P}$ or$\mathbb{Z}_-$is infinite.

Assumefirst that $\mathbb{P}$ is infinite.Choose $y \in \mathbb{R}$suchthat

$$y\left(1-\frac{\alpha}{p^*}\right) \le 1 < y . \qquad (8)$$

Let$\pi: \mathbb{P} \to \mathbb{N}$ beone-to-onemappingandlet$a \in \ \mu$begivenby

$$a_n = \begin{cases} (\pi(n))^{-y/p_n}, & n \in \mathbb{P}, \\ 0, & n \notin \mathbb{P}. \end{cases}$$

By (8) we have

$$\sum_{n\in\mathbb{Z}} (a_n)^{p_n} = \sum_{n\in\mathbb{P}} (\pi(n))^{-y} = \sum_{k=1}^{\infty} k^{-y} < \infty$$

and

$$\sum_{n\in Z}((S_1 a)_n)^{p_n} = \sum_{n\in Z}(a_{n-1})^{p_n} = \sum_{n\in Z}(a_n)^{p_{n+1}} = \sum_{n\in P}(\pi(n))^{-yp_{n+1}/p_n}$$
$$\ge \sum_{n\in\mathbb{P}}\left((\pi(n))^{-y(p_n-\alpha)/p_n}\right) \ge \sum_{n\in\mathbb{P}}(\pi(n))^{-y\left(1-\alpha/p^*\right)} \ge \sum_{k=1} k^{-y\left(1-\alpha/p^*\right)} = \infty.$$

T

hus,$S_1$is unbounded.

If $\mathbb{Z}_-$ is infinitethenanalogously$S_{-1}$is unbounded,whichprovesthelemma.

As aneasyconsequencewe obtainthefollowinglemma.

LEMMA5.4.Let$D < \infty$.Then

$$\lim_{n\to\infty} |p_{n+1} - p_n| = \lim_{n\to -\infty} |p_{n+1} - p_n| = 0 .$$

LEMMA5.5.*Let* $\lim_{n\to\infty} |p_{n+1} - p_n| = 0$.*Denote*$\underline{p} = \lim_{n\to\infty} \inf p_n$,$\overline{p} = \lim_{n\to\infty} \sup p_n$.*Let* $\underline{p} < \overline{p}$. *Thenfor any*$c > 1$ *thereexists* $m \in \mathbb{Z}$ *suchthat*$\|S_m\|_{\{p_n \to p_n\}} \geq c$.

*Proof.*Let $c > 1$. Assume $\underline{p} < \overline{p}$.Let $\delta = \frac{1}{3}\left(\overline{p} - \underline{p}\right)$and, $\{b_n\}_{n\in\mathbb{N}}, b_n > 0$, be a sequencesatisfying

$$\sum_{n=1}^{\infty} (b_n)^{\overline{p}-\delta} \leq 1 \ and \ \sum_{n=1}^{\infty} (b_n)^{\underline{p}+\delta} = \infty. \quad (9)$$

Thenthere exist$N \in \mathbb{N}$ such that

$$\sum_{n=1}^{N} (b_n)^{\underline{p}+\delta} \geq c^{p^*}. \quad (10)$$

According to the assumption $\lim_{n\to\infty} |p_{n+1} - p_n| = 0$, there are $n_1, n_2 \in \mathbb{N}$, $n_2 > n_1 + N$, such that for any $1 \leq s \leq N$ it is$p_{(n_1+s)} > \overline{p} - \delta$and $p_{(n_2+s)} < \underline{p} + \delta$. Let $a \in \mu$ be given by

$$a_n = \begin{cases} b_{n-n_1} & if\ n \in \{n_1 + s; 1 \leq s \leq N\}, \\ 0 & if\ n \notin \{n_1 + s; 1 \leq s \leq N\}. \end{cases}$$

Set $m = n_2 - n_1$ .By (9) , we have $b_n \leq 1$ and consequently ,

$$\sum_{n\in\mathbb{Z}} (a_n)^{p_n} = \sum_{s=1}^{N} (b_s)^{p(n_1+s)} \leq \sum_{s=1}^{N} (b_s)^{\overline{p}-\delta} \leq 1 .$$

Using (10) ,we obtain

$$\sum_{n\in\mathbb{Z}} ((S_m a)_n)^{p_n} = \sum_{n\in\mathbb{Z}} (a_{n-m})^{p_n} = \sum_{n\in\mathbb{Z}} (a_n)^{p_{n+m}} =$$

$$\sum_{s=1}^{N} (b_s)^{p(n_1+s+m)} = \sum_{s=1}^{N} (b_s)^{p(n_2+s)} \geq \sum_{s=1}^{N} (b_s)^{\underline{p}+\delta} \geq c^{p^*}.$$

ByLemma 2.10 wehave$\|S_m\|_{\{p_n\to p_n\}} \geq c$,, whichprovesthe lemma.

As aneasyconsequencewe obtainthefollowing lemma.

LEMMA5.6. Let $D < \infty$. Thenexistlimits$\lim_{n\to\infty} p_n$and$\lim_{n\to-\infty} p_n$.

LEMMA5.7.Let $D < \infty$. Then $\lim_{n\to\infty} p_n = \lim_{n\to-\infty} p_n$

Proof. Set

$$p_l = \lim_{n\to-\infty} p_n\ ,\ \ p_r = \lim_{n\to\infty} p_n \quad (11)$$

Let $p_l \neq p_r$.Without lossofthe generalitywecan assume$p_l > p_r$. Let $c > 1$bean arbitraryrealnumber.Set$\delta = \frac{1}{3}(p_1 - p_r)$. Let $0 < b_k$ satisfy

$$\sum_{n=1}^{\infty} (b_n)^{p_l-\delta} \leq 1 \text{ and} \sum_{n=1}^{\infty} (b_n)^{p_r+\delta} = \infty .$$

According to (11)and $p_l > p_r$thereis$N_1 \in \mathbb{N}$suchthat$p_n \geq p_l - \delta$for$n \leq -N_1$and $p_n \leq p_r + \delta$for $n \geq N_1$. Take $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and

$$\sum_{n=N_1}^{N_2} (b_n)^{p_r+\varepsilon} \geq c^{p^*}.$$

Let $a \in \mu$be given by

$$a_n = \begin{cases} b_{-n} & if - N_2 \leq n \leq -N_1, \\ 0 & other\ wise\ . \end{cases}$$

Set$m = N_1 + N_2$.Since$0 \leq a_n \leq 1$for$n \in \mathbb{Z}$we obtain

$$\sum_{n\in\mathbb{Z}} (a_n)^{p_n} = \sum_{n=-N_2}^{-N_1} (b_{-n})^{p_n} = \sum_{n=N_1}^{N_2} (b_n)^{p_{-n}} \leq \sum_{n=N_1}^{N_2} (b_n)^{p_1-\delta} \leq 1$$

and

$$\sum_{n\in\mathbb{Z}} ((S_m a)_n)^{p_n} = \sum_{n\in\mathbb{Z}} (a_{n-m})^{p_n} = \sum_{n\in\mathbb{Z}} (a_n)^{p_{n+m}} =$$

$$\sum_{n=-N_2}^{-N_1}(b_{-n})^{p_{n+m}} = \sum_{n=N_1}^{N_2}(b_n)^{p_{m-n}} \geq \sum_{n=N_1}^{N_2}(b_n)^{p_r+\delta} \geq c^{p^*}$$

Thus,byLemma2.10, $\|S_m\|_{\{p_n\to p_n\}} \geq c$.

Givena realnumber$r$we keep thesame symbolforthe constantmapping$r:\mathbb{Z}\to\mathbb{R}$givenby $r_k = r$ forall$k\in\mathbb{Z}$.

LEMMA5.8.Let $r = lim_{n\to-\infty} p_n = lim_{n\to\infty} p_n$ and let $\{p_n - r\} \notin \mathcal{S}$ .Then for any $c > 1$there is$m\in\mathbb{Z}$such that $\|S_m\|_{\{p_n\to p_n\}} \geq c$.

Proof. Since $\{p_n - r\} \notin \mathcal{S}$wehavebyLemma3.3that$\left\{1-\frac{r}{p_n}\right\} \notin \mathcal{S}$ set $\delta_n = 1-\frac{r}{p_n}$, $n \in \mathbb{Z}$. ByLemma3.5thereis $b\in\mu$, $0\leq b_n$,suchthat

$$\sum_{k\in\mathbb{Z}} b_n \leq 1 \ and \sum_{n=\mathbb{Z}} b_n^{1-\delta_n} = \infty \quad (12)$$

Given $N\in\mathbb{N}$denote

$$\mathbb{Z}(N) = \{n \in \mathbb{Z}\, ; -N \leq n \leq N\}. \quad (13)$$

Let$c > 1$beanarbitraryrealnumber.Fix$N\in\mathbb{N}$suchthat

$$\sum_{n\in\mathbb{Z}(N)} b_n^{1-\delta_n} \geq 2e^{p^*}.$$

By (1 2) wehave$0 \leq b_n \leq 1$andduetothefactthattheset $\mathbb{Z}(N)$isfinitewecan choose$\varepsilon > 0$ with

$$\sum_{n\in\mathbb{Z}(N)} b_n^{\,1-\delta_n+\varepsilon/p_n} = \sum_{n\in\mathbb{Z}(N)} b_n^{\,(r+\varepsilon)/p_n} \geq c^{p^*}. \quad (14)$$

Takingthisεwecanfind $n_1\in\mathbb{Z}$suchthat $p_n < r+\varepsilon$forall $n \geq n_1$.Set$m = n_1 + N$. Then$p_{n+m} < r+\varepsilon$ forall$n\in\mathbb{Z}(N)$ and,by (14),

$$\sum_{n\in\mathbb{Z}(N)} b_n^{\,p_{(n+m)}/p_n} \geq \sum_{n\in\mathbb{Z}(N)} b_n^{\,(r+\varepsilon)/p_n} \geq c^{p^*}. \quad (15)$$

Let$a\in\mu$begivenby

$$a_n = \begin{cases} (b_n)^{1/p_n} & if\ n\in\mathbb{Z}(N) \\ 0 & if\ n\notin\mathbb{Z}(N) \end{cases}$$

Thenby (12)and(15)we obtain

$$\sum_{n\in\mathbb{Z}}(a_n)^{p_n} = \sum_{n\in\mathbb{Z}(N)} b_n \leq 1$$

And

$$\sum_{n\in\mathbb{Z}}((S_m a)_n)^{p_n} = \sum_{n\in\mathbb{Z}}(a_{n-m})^{p_n} = \sum_{n\in\mathbb{Z}}(a_n)^{p_{n+m}} = \sum_{n\in\mathbb{Z}(N)}(b_n)^{p_{(n+m)}/p_n} \geq c^{p^*}.$$

Thus , due to Lemma 2.10, We have $\|S_m\|_{\{p_n\to p_n\}} \geq c$, which proves the lemma.

LEMMA 5.9. Let $r = lim_{n\to-\infty} p_n = \lim_{n\to\infty} p_n$ and$\{r - p_n\} \notin \mathbb{P}$.Then for any $c > 1$there is $m\in\mathbb{Z}$such that$\|S_m\|_{\{p_n\to p_n\}} \geq c$.

Proof. The proof is analogous to that of Lemma 5.8 and therefore we will proceed faster. Given $c > 1$ set$\delta_n = 1-\frac{p_n}{r}$. Since$\{\delta_n\} \notin \mathcal{S}$, there is $b_n\in\mu$and $N\in\mathbb{N}$ such that

$$\sum_{n\in\mathbb{Z}} b_n \leq 1 \ and \sum_{n\in\mathbb{Z}(N)} b_n^{1-\delta_n} \geq 2c^{p^*} \quad (16)$$

Where$\mathbb{Z}(N)$is givenby (13).Take$\varepsilon > 0$ suchthat

$$\sum_{n\in\mathbb{Z}(n)} b_n^{(1-\delta_n)/(r-\varepsilon)} = \sum_{n\in\mathbb{Z}(N)} b_n^{p_n/(r-\varepsilon)} \geq c^{p^*}. \quad (17)$$

Find$n_1\in\mathbb{Z}$ satisfying$p_n \geq r-\varepsilon\ if\ n \geq n_1$.Set$m = n_1 + N$. Define$a\in\mu$ By

$$a_n = \begin{cases} (b_{n-m})^{1/p_n} & if\ \ n-m\in\mathbb{Z}(N) \\ 0 & if\ \ n-m\notin\mathbb{Z}(N). \end{cases}$$

Thenby (16) and (17) we obtain

$$\sum_{n\in\mathbb{Z}}(a_n)^{p_n} \le \sum_{n\in\mathbb{Z}} b_n \le 1$$

and

$$\sum_{n\in\mathbb{Z}}((S_{-m}a)_n)^{p_n} = \sum_{n\in\mathbb{Z}}(a_{n+m})^{p_n} = \sum_{n\in\mathbb{Z}(N)}(b_n)^{p_{(n)}/p_{(n+m)}} \ge c^{p^*}.$$

Thus,dueto Lemma2.10, we have$\|S_{-m}\|_{\{p_n\to p_n\}} \ge c$,whichprovestheLemma.

LEMMA5.10.LetD $< \infty$. Then there exists $r \in [1,\infty)$ such that the norms in $l^{\{p_n\}}$ and in l^r are equivalent.

This Lemma with Lemma 5.2 immediately gives the following theorem.

THEOREM5.11. The following statements are equivalent:

(i) D $< \infty$;

(ii) there is $r \in [1,\infty)$ such that the norms in$l^{\{p_n\}}$and in l^r are equivalent.

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